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Univalent functions with univalent Gelfond-Leontev derivatives

V. SRINIVAS

RIASSUNTO: *Si studiano funzioni della forma* $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) = z + \sum_{n=2}^{\infty} a_n z^n$ *che sono analitiche nel cerchio unitario* U. *Si determinano condizioni necessarie e sufficienti in termini delle* $\{b_n\}$ *per* f *trinomiali nonché i limiti delle* b_n, $1 \le n \le 4$, *per una forma particolare di* f *in certe sottoclassi di funzioni univalenti con derivate di Gelfond-Leontev univalenti in* U.

ABSTRACT: *We investigate functions of the form* $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) = z + \sum_{n=2}^{\infty} a_n z^n$ *that are analytic in the unitary circle* U. *Necessary and sufficient conditions in terms of* $\{b_n\}$ *for a trinomial* f *and the bounds of* b_n, $1 \le n \le 4$ *for a particular form of* f *in certain subclasses of univalent functions with univalent Gelfond-Leontev derivatives in* U *are determined.*

KEY WORDS: *Univalent – Convex – Gelfond-Leontev derivatives.*

A.M.S. CLASSIFICATION: 30C50

1 – Introduction

Let S be the set of all univalent analytic functions in $U = \{z\colon |z| < 1\}$ such that $f(0) = 0 = f'(0) - 1$. We define

$$T \equiv \left\{ f \in S\colon f(z) = z - \sum_{n=2}^{\infty} a_n z^n \text{ with } a_n \ge 0 \text{ for } n \ge 2 \right\}.$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in U, $\{d_n\}_{n=1}^{\infty}$ denote a non-decreasing sequence of positive numbers and D be the operator which transforms the function $f(z)$ into

$$Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}.$$

The operator D, called the Gelfond-Leontev derivative was introduced and studied by GELFOND-LEONTEV [1] in connection with the generalization of Fourier series.

Now we define

$T_1(D) \equiv \{f \in T\colon Df(z)$ is analytic and univalent in $U\}$,

$C \equiv \{f \in T\colon f(z)$ is convex in $U\}$,

$C_1(D) \equiv \{f \in C\colon Df(z)$ is analytic, univalent and convex in $U\}$

and

$T_1 \equiv T_1(D)$ with $d_n \equiv n$.

The classes $T_1(D)$ and $C_1(D)$ were introduced by PATEL [3].

A function f in S may be expressed as

$$\Psi(g) = \frac{z}{g(z)}, \quad \text{where} \quad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in U.$$

MITRINOVIČ [2] obtained a sufficient condition for $\Psi(g)$ to be in the class S as

$$|b_1| + \sum_{n=2}^{\infty}(n-1)|b_n| \le 1. \tag{1}$$

READE et al. [4] proved that inequality (1) is also sufficient condition for $\Psi(g)$ to be starlike and, more generally, that $\Psi(g)$ is in$S^*(\alpha)$, the subclass of S consisting of starlike functions of order α, $0 \le \alpha \le 1$, if

$$\sum_{n=2}^{\infty}(n-1+\alpha)|b_n| \le \begin{cases} (1-\alpha) - (1-\alpha)|b_1|, & 0 \le \alpha \le 1/2 \\ (1-\alpha) - \alpha|b_1|, & 1/2 \le \alpha \le 1. \end{cases}$$

Recently SILVERMAN et al. [5] investigated the converse problem. They determined necessary conditions in terms of $\{b_n\}$ for $\Psi(g)$ to be in the class $T^*(\alpha) \equiv S^*(\alpha) \cap T$, as well as necessary and sufficient conditions in terms of $\{b_n\}$, b_1 and b_2 real, for f to be either a typically real or a univalent cubic polynomial.

The purpose of the present paper is to determine (i) necessary and sufficient conditions in terms of $\{b_n\}$ for a trinomial $\Psi(g)$ to be in the classes $T_1(D)$ and $C_1(D)$ respectively, (ii) the bounds of b_n, $1 \le n \le 4$, when a particular form of $\Psi(g) \in T_1(D)$ for a particular D.

2 – A division operator on trinomials

We begin with characterizing the coefficient sequence $\{b_n\}$ with b_1, b_2 real for which $\Psi(g)$ is in $T_1(D)$ and is a cubic polynomial. For that we need the following lemma.

LEMMA 1. (PATEL [3]). *Let* $f(z) = z - a_2 z^2 - a_{p+1} z^{p+1}$, $p \ge 2 (a_2 > 0, a_{p+1} \ge 0)$. *Then* $f \in T_1(D)$ *if and only if*

$$a_{p+1} \le \min \begin{cases} (1 - 2a_2)/(p+1) \\ d_2 a_2 / p d_{p+1} \, . \end{cases}$$

Define the region $\mathcal{D}_1$ by

$$\mathcal{D}_1 = \Big\{ (x, y) \in \mathbb{R}^2 \colon y - 2/9 \le (x - 1/3)^2, x^2 \le y \le \\ \le (x + d_2/4d_3)^2 - (d_2/4d_3)^2 \text{ and } 0 < x \Big\} \, .$$

THEOREM 1. *For the function* $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *with* b_1, b_2 *real the function* $\Psi(g) = z - a_2 z^2 - a_3 z^3 (a_2 > 0, a_3 \ge 0)$ *is in* $T_1(D)$ *if and only if*

(2) $$b_1 b_{n-1} - b_n = (b_1^2 - b_2) b_{n-2} \quad \textit{for} \quad n = 3, 4, \dots ,$$

and

$$(b_1, b_2) \in \mathcal{D}_1 \, .$$

PROOF. Let $\Psi(g) = z - a_2z^2 - a_3z^3 \in T_1(D)$ with $g(z) = 1 + \sum_{n=1}^{\infty} b_nz^n$. Now $b_n = \sum_{k=0}^{n-1} b_ka_{n+1-k}$ for $n \geq 1$ with $a_p = 0$ for $p \geq 4$. Hence $b_1 = a_2$, and $b_2 = a_3 + b_1^2$. For $n \geq 3$,

$$b_n = a_2b_{n-1} + a_3b_{n-2} = b_1b_{n-1} + (b_2 - b_1^2)b_{n-2}.$$

This gives Equation (2).

Applying the necessity part of Lemma 1 for$\Psi(g) = z - a_2z^2 - a_3z^3 \in T_1(D)$, we have

$$b_2 - b_1^2 \leq \min \begin{cases} (1-2b_1)/3 \\ d_2b_1/2d_3. \end{cases}$$

This implies

$$b_2 - 2/9 \leq (b_1 - 1/3)^2 \quad \text{and} \quad b_2 \leq (b_1 + d_2/4d_3)^2 - (d_2/4d_3)^2,$$

which in turn implies that $(b_1, b_2) \in \mathcal{D}_1$.

Conversely, let Equation (2) hold and $(b_1, b_2) \in \mathcal{D}_1$, for $f(z) = \Psi(g) = z - \sum_{n=2}^{\infty} a_nz^n$. Equation (2) gives that $f(z)$ is a cubic polynomial. Hence $f(z) = z - a_2z^2 - a_3z^3$. Since $(b_1, b_2) \in \mathcal{D}_1$, we have

$$a_2 = b_1 > 0, \quad a_3 = b_2 - b_1^2 \geq 0$$

and

$$a_3 \leq \min \begin{cases} (1-2a_2)/3 \\ d_2a_2/2d_3. \end{cases}$$

Now, the sufficiency part of Lemma 1 gives that $f \in T_1(D)$. Proof of the theorem is therefore complete.

Next, we characterize the coefficient sequence $\{b_n\}, b_k$ real, $1 \leq k \leq p$, for which $\Psi(g)$ is in $T_1(D)$ and is of the form $\Psi(g) = z - a_2z^2 - a_{p+1}z^{p+1}$ with $p \geq 3$.

Define the region $\mathcal{D}_2$ by

$$\mathcal{D}_2 = \Big\{(x_1, x_2, \dots, x_p) \in \mathbb{R}^p : 0 < x_1, \quad x_k = x_1^k \\ \text{for } 2 \le k \le p-1, \quad x_p - \frac{1}{p+1} \le x_1^p - \frac{2x_1}{p+1} \\ \text{and} \quad x_1^p \le x_p \le x_1\Big(x_1^{p-1} + \frac{d_2}{pd_{p+1}}\Big)\Big\}$$

where $p \ge 3$.

THEOREM 2. *For the function $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ with b_n real for $1 \le n \le p$, the function $\Psi(g) = z - a_2 z^2 - a_{p+1} z^{p+1} \in T_1(D)$ if and only if*

$$(3) \qquad b_1 b_{n-1} - b_n = (b_1^p - b_p) b_{n-p} \quad \textit{for} \quad n \ge p+1,$$

and

$$(b_1, b_2, \dots, b_p) \in \mathcal{D}_2$$

where $p \ge 3$.

PROOF. Let $\Psi(g) = z - \sum_{n=2}^{\infty} a_n z^n \in T_1(D)$ with $a_k = a_l = 0$ for $3 \le k \le p$, $p + 2 \le l$. Now $b_n = \sum_{k=0}^{n-1} b_k a_{n+1-k}$ for $n \ge 1$. Hence $b_1 = a_2$ and $b_2 = b_1^2$. Assume that $b_m = b_1^m$ for $1 \le m \le q \le p-2$ where $q \in \mathbb{N}$ and $q \le p-2$. Now $b_{q+1} = b_1^{q+1}$. So $b_i = b_1^i$ for $1 \le i \le p-1$ and $b_p = b_1^p + a_{p+1}$. For $n \ge p+1$, we have $b_n = b_1 b_{n-1} + a_{p+1} b_{n-p}$. This gives Equation (3).

Applying the necessity part of lemma 1 for $\Psi(g) = z - a_2 z^2 - a_{p+1} z^{p+1} \in T_1(D)$, we have

$$b_p - b_1^p \mathcal{D} \le \min \begin{bmatrix} (1 - 2b_1)/(p+1) \\ d_2 b_1 / p d_{p+1} \end{bmatrix}$$

which implies

$$b_p - 1/(p+1) \le b_1^p - 2b_1/(p+1)$$

and

$$b_p \le b_1(b_1^{p-1} + d_2/pd_{p+1}).$$

Thus we have $(b_1, b_2, \dots, b_p) \in \mathcal{D}_2$.

Conservely, let Equation (3) hold and $(b_1, b_2, \dots, b_p) \in \mathcal{D}_2$, for $\Psi(g) = z - \sum_{n=2}^{\infty} a_n z^n$. For $n \ge 1$, we have $a_{n+1} = b_n - \sum_{k=2}^{n} a_k b_{n-k+1}$. Hence $a_2 = b_1$ and $a_3 = 0$. Assume that $a_l = 0$ for $3 \le l \le j \le p-1$ where $3 \le j \le p-1$ and $j \in \mathbb{N}$. Now $a_{j+1} = b_j - b_1^j = 0$. Therefore $a_m = 0$ for $4 \le m \le p$. Now $a_{p+1} = b_p - b_1^p$. Making use of Equation (3) and induction we get that $\Psi(g) = z - a_2 z^2 - a_{p+1} z^{p+1}$.

The point $(b_1, b_2, \dots, b_p) \in \mathcal{D}_2$ and the sufficiency part of Lemma 1 give that $\Psi(g) \in T_1(D)$. Hence proof of the theorem is complete.

Next, we characterize the coefficient sequence $\{b_n\}, b_1, b_2$ real for which $\Psi(g)$ is in $C_1(D)$ and is a cubic polynomial. For this we need the following lemma.

LEMMA 2. (PATEL [3]). *Let* $f(z) = z - a_2 z^2 - a_{p+1} z^{p+1}, p \ge 2 (a_2 > 0, a_{p+1} \ge 0)$. *Then* $f \in C_1(D)$, *if and only if*

$$a_{p+1} \le \min \begin{bmatrix} (1 - 4a_2)/(p+1)^2 \\ d_2 a_2 / p^2 d_{p+1} \end{bmatrix}$$

Define the region $\mathcal{D}_3$ by

$$\mathcal{D}_3 = \Big\{ (x, y) \in \mathbb{R}^2 \colon x^2 \le y \le \Big(x - \frac{2}{9}\Big)^2 + \frac{5}{81}, \\ y + \Big(\frac{d_2}{8d_3}\Big)^2 \le \Big(x + \frac{d_2}{8d_3}\Big)^2 \text{ and } 0 < x \Big\}.$$

THEOREM 3. *For the function* $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *with* b_1, b_2 *real* $\Psi(g) = z - a_2 z^2 - a_3 z^3 \in C_1(D)$ *if and only if*

Equation (2) *holds and*

$$(b_1, b_2) \in \mathcal{D}_3.$$

PROOF. Follow the method of proof of Theorem 1 using Lemma 2 in place of Lemma 1.

Next, we characterize the coefficient sequence $\{b_n\}$, b_k real, $1 \le k \le p$, for which $\Psi(g)$ is in $C_1(D)$ and is of the form $\Psi(g) = z - a_2 z^2 - a_{p+1} z^{p+1}$ with $p \ge 3$.

Define the region $\mathcal{D}_4$ by

$$\mathcal{D}_4 = \Big\{(x_1, x_2, \dots, x_p) \in \mathbb{R}^p : 0 < x_1, \quad x_k = x_1^k \\ \text{for } 2 \le k \le p-1, \quad x_p - (p+1)^{-2} \le x_1\Big(x_1^{p-1} - 4(p+1)^{-2}\Big) \\ \text{and } x_1^p \le x_p \le x_1\Big(x_1^{p-1} + d_2/p^2 d_{p+1}\Big)\Big\}$$

where $p \ge 3$.

THEOREM 4. *For the function $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ with b_k real for $1 \le k \le p$, the function $\Psi(g) = z - a_2 z^2 - a_{p+1} z^{p+1} \in C_1(D)$ if and only if Equation* (3) *holds and*

$$(b_1, b_2, \dots, b_p) \in \mathcal{D}_4 .$$

PROOF. Follow the method of proof of Theorem 2 using Lemma 2 in place of Lemma 1.

3 – Bounds of $b_n, 1 \le n \le 4$

Now we study the bounds of $b_n, 1 \le n \le 4$. To begin with we find the bounds of b_1 in

THEOREM 5. *If $\Psi(g) \in T_1(D)$, then $0 < b_1 \le 1/2$.*

Equality holds in the upper inequality only for $t(z) = z/(1 + \sum_{n=1}^{\infty} z^n/2^n)$.

PROOF. It is known that for $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$, $a_2 \leq 1/2$. We have $b_1 = a_2$ and $T_1(D) \subseteq T$. Now the assertion of the theorem is immediate.

Next we consider cubic polynomials in $T_1(D)$ for a special D and find the bounds on b_2 in

THEOREM 6. *For the function $\Psi(g) = z - a_2 z^2 - a_3 z^3 \in T_1(D)$ where $3d_2 \leq 2d_3$, we have $0 < b_2 \leq 1/4$.*

The function $t(z)$ of Theorem 5 gives sharpness in the upper inequality.

PROOF. By Lemma 1 we have,

$$b_2 - b_1^2 \leq \min \begin{bmatrix} (1-2b_1)/3 \\ d_2 b_1/2d_3 \end{bmatrix}$$

When $0 < b_1 < 1/3$ we have

$$d_2 b_1/2d_3 < 1/9 < (1-2b_1)/3 .$$

And

$$b_2 - b_1^2 \leq d_2 b_1/2d_3, \quad b_2 \leq b_1^2 + d_2 b_1/2d_3 < 1/4 .$$

When $1/3 \leq b_1 \leq 2d_3/(3d_2 + 4d_3)$, we have

$$b_2 - b_1^2 \leq d_2 b_1/2d_3, \quad b_2 \leq b_1^2 + d_2 b_1/2d_3 < 1/4 .$$

Finally, when $2d_3/(3d_2 + 4d_3) \leq b_1 \leq 1/2$, we have

$$(1-2b_1)/3 \leq d_2 b_1/2d_3 .$$

And

$$b_2 - b_1^2 \leq (1-2b_1)/3, \quad b_2 \leq b_1^2 + 2b_1/3 + 1/3 < 1/4 .$$

Hence the theorem is proved in full.

Next we consider a trinomial in $T_1(D)$ for a special D and find the bounds on b_3 in

THEOREM 7. *For the function* $\Psi(g) = z - a_2z^2 - a_4z^4 \in T_1(D)$ *where* $4d_2 \le 3(\sqrt{6}-2)d_4$, *we have* $0 < b_3 \le 1/8$.

The $t(z)$ *of Theorem* 5 *gives sharpness in the upper inequality.*

PROOF. By Lemma 1, we have

$$b_3 - b_1^3 \le \min \begin{bmatrix} (1-2b_1)/4 \\ d_2b_1/3d_4 \end{bmatrix}$$

When $0 < b_1 < 1/\sqrt{6}$, we have

$$d_2b_1/3d_4 \le (1-2b_1)/4.$$

And

$$b_3 - b_1^3 \le d_2b_1/3d_4, \quad b_3 \le b_1^3 + d_2b_1/3d_4 \le 1/8$$

When $1/\sqrt{6} \le b_1 \le 3d_4/2(2d_2+3d_4)$, we have

$$d_2b_1/3d_4 \le (1-2b_1)/4.$$

And

$$b_3 \le b_1^3 + d_2b_1/3d_4 < 1/8.$$

Finally, when $3d_4/2(2d_2+3d_4) < b_1 \le 1/2$, we have

$$(1-2b_1)/4 < d_2b_1/3d_4.$$

And

$$b_3 - b_1^3 \le (1-2b_1)/4, \quad b_3 \le b_1^3 - b_1/2 + 1/4 \le 1/8.$$

Thus proof of the theorem is complete.

Next we consider a trinomial in T_1 and find the bounds on b_4 which suggest that for $\Psi(g) \in T_1, b_n \le \frac{1}{2^n}$ is not true in general.

THEOREM 8. *For the function* $\Psi(g) = z - a_2z^2 - a_5z^5 \in T_1$ *we have* $0 < b_4 \le 41/625$. *The function* $\Psi(g) = z - \frac{2z^2}{5} - \frac{z^5}{25} = z/(1 + \frac{2}{5}z + \frac{4}{25}z^2 + \frac{8}{125}z^3 + \frac{41}{625}z^4 + \dots)$ *gives the sharpness.*

PROOF. By Lemma 1, we have

$$b_4 - b_1^4 \leq \min \begin{cases} (1-2b_1)/5 \\ b_1/10 . \end{cases}$$

When $0 < b_1 \leq 2/5$, we have, $b_1/10 \leq (1-2b_1)/5$ and

$$b_4 - b_1^4 \leq b_1/10, \quad b_4 \leq b_1^4 + b_1/10 \leq 41/625 .$$

When $2/5 \leq b_1 \leq 10^{-1/3}$ we have $(1-2b_1)/5 \leq b_1/10$ and

$$b_4 \leq b_1^4 - 2b_1/5 + 1/5 \leq 41/625$$

Finally, when $10^{-1/3} \leq b_1 \leq 1/2$, we have $(1-2b_1)/5 \leq b_1/10$ and

$$b_4 \leq b_1^4 - 2b_1/5 + 1/5 \leq 1/16 < 41/625 .$$

Hence proof of the theorem is complete.

Finally, we find the bounds on b_2 when $\Psi(g) \in T_1$ is of general form in

THEOREM 9. *Let $\Psi(g) \in T_1(D)$ and $\beta_0 \equiv \sup_{n\geq 2}\{d_n/n\}$. If $\beta_0 < \infty$, and $0 < b_1 \leq \beta_0/d_2$, then*

(4)
$$0 < b_2 < (2\beta_0 + d_3)/4d_3 .$$

Set $\tilde{\beta} = \sup\{b_2 \colon \Psi(g) \in T_1(D)\}$. If in addition to the above conditions, $2d_3 \geq 3d_2$, then

(5)
$$1/4 \leq \tilde{\beta} \leq (2\beta_0 + d_3)/4d_3 .$$

PROOF. We have $b_2 = a_3 + b_1^2$.

It is known (PATEL [3]) that, if $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T_1(D)$, $\beta_0 \equiv \sup_{n\geq 2}\{d_n/n\} < \infty$ and $0 < a_2 \leq \beta_0/d_2$, then $a_3 < \beta_0/2d_3$.

Since $\Psi(g) \in T_1(D)$, we have

$$b_2 < \beta_0/2d_3 + 1/4\,,$$

by using Theorem 5. This gives Inequalities (4).

By using Theorem 6, we prove Inequalities (5) and the theorem in full.

COROLLARY. *If* $\Psi(g) \in T_1$ *then* $0 < b_2 < 5/12$.
Set $\beta = \sup\{b_2 : \psi(g) \in T_1\}$. *Then* $1/4 \leq \beta \leq 5/12$.

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INDIRIZZO DELL'AUTORE:

V. Srinivas - Department of Mathematics - Indian Institute of Technology - Kapur - 208016 - India