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# Continuous dependence and differentiability of solutions of impulsive systems of integro-differential equations with respect to initial data and parameter

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RIASSUNTO: *Si studia un sistema impulsivo di equazioni integro-diffrenziali del tipo di Volterra. Si stabiliscono alcuni teoremi di continuità e differenziabilità delle soluzioni, si determina l'equazione variazionale corrispondente, si dimostra una formula analoga a quella di Alekssev.*

ABSTRACT: *In the present paper theorems of continuous dependence and differentiability of the solutions of impulsive systems of integro-differential equations of Volterra type with fixed moments of impulse effect are proved. The corresponding variational equations and an analogue of Alekssev's formula for such systems are obtained.*

KEY WORDS: *Continuous dependence – Differentiability – Impulsive systems – Initial data parameter.*

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## 1 – Introduction

It is characteristic for the evolution of many real processes that at certain moments they change their state by jumps. An adequate mathematical model of such processes are the impulsive systems of differential

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and integro-differential equations. That is why in the recent years these systems are an object of intensive research by many authors [1] - [7].

The elaboration of the qualitative theory of the impulsive systems of differential and integro-differential equations requires the use of theorems of the fundamental theory of these systems. Results related to the existence, uniqueness, continuity and differentiability of the solutions of impulsive systems of differential equations are obtained in [4] - [7].

In the present paper impulsive systems of integro-differential equations of Volterra type with fixed moments of impulse effect are investigated. Theorems of continuity (Theorem 1,2) and differentiability (Theorem 4) of the solutions of such systems with respect to the initial conditions and a parameter are proved. The corresponding variational equations (Theorem 4), variation of parameters formula and an analogue of Alekseev's formula (Theorem 5) are obtained.

## 2 – Preliminary notes and notation

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with norm $|\cdot|$ and let $\mathbb{R}_+ = [0,\infty)$. Consider the impulsive system of integro-differential equations

$$(1)\qquad \begin{cases} x'(t) = f(t,x(t),\lambda) + \int\limits_{t_0}^{t} g(t,s,x(s),\lambda)ds, \ t \neq \tau_k\,; \\ \Delta x|_{t=\tau_k} = I_k(x(\tau_k),\lambda)\,, \\ x(t_0^+) = x_0\,, \end{cases}$$

where $t \in \mathbb{R}_+$, $x\colon \mathbb{R}_+ \longrightarrow \Omega$, $f\colon \mathbb{R}_+\times\Omega\times\Lambda \longrightarrow \mathbb{R}^n$, $g\colon \mathbb{R}_+^2\times\Omega\times\Lambda \longrightarrow \mathbb{R}^n$, $I_k\colon \Omega\times\Lambda \longrightarrow \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^m$, $\Omega$ and $\Lambda$ are domains,

$$0 = \tau_0 < \tau_1 < \tau_2 \ldots < \tau_k < \ldots\,, \lim_{k\longrightarrow\infty} \tau_k = \infty, \Delta x|_{t=\tau_k} = x(\tau_k^+) - x(\tau_k^-)\,.$$

The impulsive systems of integro-differential equations of the form (1) are characterized in the following way:

1. For $t \neq \tau_k$, $k = 1,2,\ldots$ the mapping point $(t,x(t))$ moves along some of the integral curves of the system

$$x'(t) = f(t,x(t),\lambda) + \int\limits_{t_0}^{t} g(t,s,x(s),\lambda)ds$$

2. At the moment $t = \tau_k$ the system is subject to an impulse effect, as a result of which the mapping point is transferred "momentarily" from the position $(\tau_k, x(\tau_k))$ into the position $(\tau_k, x(\tau_k)+I_k(x(\tau_k), \lambda))$. Afterwards, for $\tau_k < t \leq \tau_{k+1}$ the solution $x(t)$ of system (1) coincides with the solution $y(t)$ of the system

$$\begin{cases} y'(t) = f(t, y(t), \lambda) + \int\limits_{\tau_k}^{t} g(t, s, x(s), \lambda)ds\,, \\ y(\tau_k) = x(\tau_k) + I_k(x(\tau_k), \lambda)\,. \end{cases}$$

At the moment $t = \tau_{k+1}$ the system is subject to a new impulse effect.

3. At the moments $t = \tau_k$, $k = 1, 2, \ldots$ of impulse effect the solution $x(t)$ of system (1) is continuous from the left, i.e. for

$$x(\tau_k^-) = x(\tau_k)\,;\ x(\tau_k^+) = x(\tau_k) + I_k(x(\tau_k), \lambda)\,.$$

In the further considerations we shall use the following notation: $x(t; t_0, x_0, \lambda)$-solution of system (1) for which $x(t_0^+; t_0, x_0, \lambda) = x_0$; $f_x = (\partial f_i/\partial x_j)$-the Jacobi matrix of the function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$;
$E$-the unit $n \times n$-matrix;
$0_{nm}$-the zero $n \times m$-matrix.

## 3 – Main results

### 3.1 – Continuous dependence and differentiability of the solutions with respect to the initial conditions and a parameter

THEOREM 1. *Let the following conditions hold:*

1. *The function $f\colon \mathbb{R}_+ \times \Omega \times \Lambda \longrightarrow \mathbb{R}^n$ is continuous in $(\tau_{k-1}, \tau_k] \times \Omega \times \Lambda$, $k = 1, 2, \ldots$ and for each $k = 1, 2, \ldots$ and $(x_0, \lambda_0) \in \Omega \times \Lambda$ there exists a finite limit of $f(t, x, \lambda)$ as $(t, x, \lambda) \longrightarrow (\tau_k, x_0, \lambda_0), t > \tau_k$.*

2. *The function $f$ is locally Lipschitz continuous with respect to $(x, \lambda)$ in the domain $\Omega \times \Lambda$.*

3. *The function $g\colon \mathbb{R}_+^2 \times \Omega \times \Lambda \longrightarrow \mathbb{R}^n$ is continuous in $(\tau_{k-1}, \tau_k] \times (\tau_{i-1}, \tau_i] \times \Omega \times \Lambda$, $k, i = 1, 2, \ldots$ and for $k, i = 1, 2, \ldots$ and $(x_0, \lambda_0) \in \Omega \times \Lambda$ there exists a finite limit of $g(t, s, x, \lambda)$ as $(t, s, x, \lambda) \longrightarrow (\tau_k, \tau_i, x_0, \lambda_0), t > \tau_k, s > \tau_i$.*

4. *The function g is locally Lipschitz continuous with respect to* $(x, \lambda)$ *in the domain* $\mathbb{R}^2_+ \times \Omega \times \Lambda$.

5. *For* $k = 1, 2, \ldots$ *the mapping* $\mathcal{I}_k : \Omega \times \Lambda \longrightarrow \Omega \times \Lambda$ *defined by means of the equality*

$$\mathcal{I}_k(x, \lambda) = x + I_k(x, \lambda), (x, \lambda) \in \Omega \times \Lambda$$

*is invertible in* $\Omega \times \Lambda$ *and* $\mathcal{I}_k$ *and* $\mathcal{I}_k^{-1}$ *are Lipschitz continuous with respect to* $(x, \lambda)$ *in* $\Omega \times \Lambda$.

6. *For* $\lambda = \lambda^* \in \Lambda$ *system (1) has a solution* $\varphi(t)$ *defined in the interval* $[\alpha, \beta], (\alpha, \beta \neq \tau_k, k = 1, 2, \ldots)$.

*Then there exists a number* $\delta > 0$ *and a set*

$$U = \{(t, x, \lambda) \in \mathbb{R}_+ \times \Omega \times \Lambda : \alpha \le t \le \beta, |x - \varphi(t^+)| < \delta, |\lambda - \lambda^*| < \delta\}$$

*such that:*

(A) *For any point* $(t_0, x_0, \lambda_0) \in U$ *there exists a unique solution* $x(t; t_0, x_0, \lambda_0)$ *of system (1) which is defined in the interval* $[\alpha, \beta]$.

(B) *The function* $x(t; t_0, x_0, \lambda_0)$ *is continuous for* $\alpha \le t \le \beta$, $(t_0, x_0, \lambda_0) \in U$, $t \neq \tau_k$, $t_0 \neq \tau_i$, $k, i = 1, 2, \ldots$

(C) *For* $k, i = 1, 2, \ldots$ $(x_0, \lambda_0) \in \Omega \times \Lambda$, $t \neq t_j$, $j = 1, 2, \ldots$ *the following relations are valid:*

$$\lim_{(s,\tau,y,\lambda) \to (t, \tau_k^+, x_0, \lambda_0)} x(s; \tau, y, \lambda) = x(t; \tau_k, x_0, \lambda_0), \tag{2}$$

$$\lim_{(s,\tau,y,\lambda) \to (t, \tau_k^-, x_0, \lambda_0)} x(s; \tau, y, \lambda) = x(t; \tau_k, x_0 + I_k(x_0, \lambda_0), \lambda_0), \tag{3}$$

$$\lim_{(s,\tau,y,\lambda) \to (\tau_k^+, t, x_0, \lambda_0)} x(s; \tau, y, \lambda) = x(\tau_k; t, x_0 \lambda_0) + I_k(x(\tau_k; t, x_0, \lambda_0), \lambda_0), \tag{4}$$

$$\lim_{(s,t,y,\lambda) \to (\tau_k^-, t, x_0, \lambda_0)} x(s; t, y, \lambda) = x(\tau_k; t, x_0, \lambda_0), \tag{5}$$

$$\lim_{(s,t,y,\lambda) \to (\tau_k^- \tau_k^+, x_0, \lambda_0)} x(s; t, y, \lambda) = x(\tau_i; \tau_k, x_0 \lambda_0), \tag{6}$$

$$\lim_{(s,t,y,\lambda) \to (\tau_i^-, \tau_k^-, x_0, \lambda_0)} x(s; t, y, \lambda) = x(\tau_i; \tau_k, x_0 + I_k(x_0, \lambda_0), \lambda_0), \tag{7}$$

(8)
$$\lim_{(s,t,y,\lambda)\to(\tau_i^+\tau_k^+,x_0,\lambda_0)} x(s,t,y,\lambda) = x(\tau_i;\tau_k,x_0,\lambda_0)+ \\ + I_i(x(\tau_i;\tau_k,x_0,\lambda_0),\lambda_0),$$

(9)
$$\lim_{(s,t,y,\lambda)\to(\tau_i^+\tau_k^-,x_0,\lambda_0)} x(s;t,y,\lambda) = x(\tau_i;\tau_k,x_0+I_k(x_0,\lambda_0),\lambda_0)+ \\ + I_i(x(\tau_i;\tau_k,x_0+I_k(x_0,\lambda_0),\lambda_0),\lambda_0).$$

The proof of Theorem 1 is analogous to the proof of Theorem 1 of [7] and we shall not give it (see also Theorem 3.10 of [5]).

For the linear impulsive system of integro-differential equations

$$\text{(10)}\qquad \begin{cases} x'(t) = A(t,\lambda)x(t) + \int\limits_{t_0}^{t} B(t,s,\lambda)x(s)ds, t \neq \tau_k; \\ \Delta x|_{t=\tau_k} = B_k(\lambda)x(\tau_k), \\ x(t_0^+) = x_0 \end{cases}$$

where $A(t,\lambda)$, $B(t,s,\lambda)$ and $B_k(\lambda)$, $k = 1,2,\dots$ are $n \times n$-matrices, as a corollary of Theorem 1 the following theorem is obtained:

THEOREM 2. *Let the following conditions hold:*

1. *The matrix-valued function $A(t,\lambda)$ is continuous in $(\tau_{k-1},\tau_k] \times \Lambda$, $k = 1,2,\dots$ and for $k = 1,2,\dots$ and $\lambda_0 \in \Lambda$ there exists a finite limit of $A(t,\lambda)$ as $(t,\lambda) \longrightarrow (\tau_k,\lambda_0), t > \tau_k$.*

2. *The matrix-valued function $B(t,s,\lambda)$ is continuous in $(\tau_{k-1},\tau_k] \times (\tau_{i-1},\tau_i] \times \Lambda$, $k,i = 1,2\dots$ and for $x,i = 1,2,\dots$ and $\lambda_0 \in \Lambda$ there exists a finite limit of $B(t,s,\lambda)$ as $(t,s,\lambda) \longrightarrow (\tau_k,\tau_i,\lambda_0)$, $t > \tau_k, s > \tau_i$.*

3. *The matrix-valued functions $B_k(\lambda)$, $k = 1,2,\dots$ are continuous in $\Lambda$ and*

$$\text{(11)}\qquad \det(E + B_k(\lambda)) \neq 0, \ \lambda \in \Lambda; k = 1,2,\dots$$

*Then the solution $x(t;t_0,x_0,\lambda_0)$ of system (10) is a continuous function for $t,t_0 \in \mathbb{R}_+$, $t,t_0 \neq \tau_k$, $k = 1,2,\dots$, $x_0 \in \mathbb{R}^n$, $\lambda_0 \in \Lambda$ and relations (2) - (9) are valid for $I_k(x,\lambda) = B_k(\lambda)x$.*

REMARK. If any of the conditions (11) is not met, then the assertion of Theorem 2 is still valid for $t > t_0$.

Consider the linear impulsive integro-differential system

$$(12)\qquad \begin{cases} x'(t) = A(t,\lambda)x(t) + \int\limits_{t_0}^{t} B(t,s,\lambda)x(s)ds + F(t,\lambda), t \neq \tau_k\,; \\ \Delta x|_{t=\tau_k} = B_k(\lambda)x(\tau_k)\,, \\ x(t_0^+) = x_0\,, \end{cases}$$

where $F\colon \mathbb{R}_+ \times \Lambda \longrightarrow \mathbb{R}^n$.

For system (12) the following theorem is valid, which we shall use further on:

THEOREM 3. *Let the following conditions hold:*

1. *Conditions 1-3 of Theorem 2 are met.*

2. *The function $F\colon \mathbb{R}_+ \times \Lambda \longrightarrow \mathbb{R}^n$ is continuous in $(\tau_{k-1}, \tau_k] \times \Lambda$, $k = 1, 2, \dots$ and for $k = 1, 2, \dots$ and $\lambda_0 \in \Lambda$ there exists a finite limit of $F(t,\lambda)$ as $(t,\lambda) \longrightarrow (\tau_k, \lambda_0)$, $t > \tau_k$.*

3. *The function $F(t,\lambda)$ is locally Lipschitz continuous with respect to $\lambda$ in $\mathbb{R}_+ \times \Lambda$.*

*Then for the solution $x(t,\lambda)$ of system (12) the following formula is valid*

$$(13)\qquad \begin{aligned} x(t,\lambda) =& R(t,t_0^+;\lambda)x(t_0^+,\lambda) + \int\limits_{t_0}^{t} R(t,s;\lambda)F(s,\lambda)ds + \\ &+ \sum_{t_0<\tau_k<t} \Delta\mathcal{U}|_{s=\tau_k}, t > t_0, t \neq \tau_k, k = 1, 2, \dots \end{aligned}$$

*where $R(t,s;\lambda)$ satisfies the initial value problem*

$$(14)\qquad \begin{cases} \dfrac{\partial R(t,s;\lambda)}{\partial s} + R(t,s;\lambda)A(s,\lambda) + \int\limits_{s}^{t} R(t,\sigma;\lambda)B(\sigma,s,\lambda)d\sigma = 0\,, \\ R(t,t;\lambda) = E \end{cases}$$

*for $0 \le s \le t < \infty$, $t, s \neq \tau_k$, $k = 1, 2, \dots$ and*

$$\Delta\mathcal{U}\big|_{s=\tau_k} = R(t,\tau_k^+;\lambda)x(\tau_k^+,\lambda) - R(t,\tau_k^-;\lambda)x(\tau_k,\lambda)\,.$$

PROOF.

Let $R(t,s;\lambda)$ be a solution of (14). Set $\mathcal{U}(s)=R(t,s;\lambda)x(s,\lambda)$, where $x(s,\lambda)=x(s;t_0,x_0,\lambda)$ is a solution of (12). Then for $s\neq\tau_k$, $k=1,2,\dots$ we have

$$\mathcal{U}'(s)=\frac{\partial R(t,s;\lambda)}{\partial s}+R(t,s;\lambda)[A(s,\lambda)x(s,\lambda)+ \\ +\int_{t_0}^{s}B(s,\sigma,\lambda)x(\sigma,\lambda)d\sigma+F(s,\lambda)]\,.$$

Integrating from $t_0$ to $t$ and using the fact that

$$\int_0^t \mathcal{U}'(s)ds=\mathcal{U}(t^-)-\mathcal{U}(t_0^+)-\sum_{t_0<\tau_k<t}\Delta\mathcal{U}|_{s=\tau_k}\,,$$

we obtain

$$R(t,t^-;\lambda)x(t,\lambda)-R(t,t_0^+;\lambda)x(t_0^+)=$$

$$=\int_{t_0}^{t}\left[\frac{\partial R(t,s;\lambda)}{\partial s}+R(t,s;\lambda)A(s,\lambda)\right]x(s,\lambda)ds+ \\ +\int_{t_0}^{t}\left[R(t,s;\lambda)\int_{t_0}^{s}B(s,\sigma,\lambda)x(\sigma,\lambda)d\sigma\right]ds+ \\ +\int_{t_0}^{t}R(t,s;\lambda)F(s,\lambda)ds+\sum_{t_0<\tau_k<t}\Delta\mathcal{U}|_{s=\tau_k}\,.$$

From the Fubini theorem it follows that

$$\int_{t_0}^{t}\left[R(t,s;\lambda)\int_{t_0}^{s}B(s,\sigma,\lambda)x(\sigma,\lambda)d\sigma\right]ds=\int_{t_0}^{t}\left[\int_{s}^{t}R(t,\sigma;\lambda)B(\sigma,s,\lambda)d\sigma\right]ds\,.$$

Hence for $s, t \neq \tau_k$, $k = 1, 2, \dots$ we have

$$\begin{aligned} x(t,\lambda) =& R(t,t_0^+;\lambda)x(t_0^+,\lambda) + \int_{t_0}^{t} \Big[\frac{\partial R(t,s;\lambda)}{\partial s} + \\ &+ R(t,s;\lambda)A(s,\lambda) + \int_s^t R(t,\sigma;\lambda)B(\sigma,s,\lambda)d\sigma\Big] x(s,\lambda)ds + \\ &+ \int_{t_0}^{t} R(t,s;\lambda)F(s,\lambda)ds + \sum_{t_0<\tau_k<t} \Delta\mathcal{U}|_{s=\tau_k}, \end{aligned}$$

whence using (14) we obtain (13).

Theorem 3 is proved. □

THEOREM 4. *Let the following conditions hold:*

1. *The function* $f\colon \mathbb{R}_+ \times \Omega \times \Lambda \longrightarrow \mathbb{R}^n$ *is continuous in* $(\tau_{k-1}, \tau_k] \times \Omega \times \Lambda$, $k = 1, 2, \dots$ *and for* $k = 1, 2, \dots$ *and* $(x_0, \lambda_0) \in \Omega \times \Lambda$ *there exists a finite limit of* $f(t, x, \lambda)$ *as* $(t, x, \lambda) \longrightarrow (\tau_k, x_0, \lambda_0), t > \tau_k$.

2. *The derivatives* $f_x$ *and* $f_\lambda$ *exist and are continuous in* $(\tau_{k-1}, \tau_k] \times \Omega \times \Lambda$, $k = 1, 2, \dots$ *and for* $k = 1, 2, \dots$ *and* $(x_0, \lambda_0) \in \Omega \times \Lambda$ *there exist finite limits of* $f_x(t, x, \lambda)$ *and* $f_\lambda(t, x, \lambda)$ *as* $(t, x, \lambda) \longrightarrow (\tau_k, x_0, \lambda_0)$, $t < \tau_k$ *and as* $(t, x, \lambda) \longrightarrow (\tau_k, x_0, \lambda_0)$, $t > t_k$.

3. *The function* $g\colon \mathbb{R}_+^2 \times \Omega \times \Lambda \longrightarrow \mathbb{R}^n$ *is continuous in* $(\tau_{k-1}, \tau_k] \times (\tau_{i-1}, \tau_i] \times \Omega \times \Lambda$, $k, i = 1, 2, \dots$ *and for* $k, i = 1, 2, \dots$ *and* $(x_0, \lambda_0) \in \Omega \times \Lambda$ *there exists a finite limit of* $g(t, s, x, \lambda)$ *as* $(t, s, x, \lambda) \longrightarrow (\tau_k, \tau_i, x_0, \lambda_0)$, $t > \tau_k$, $s > \tau_i$.

4. *The derivatives* $g_x$ *and* $g_\lambda$ *exist and are continuous in* $(\tau_{k-1}, \tau_k] \times (\tau_{i-1}, \tau_i] \times \Omega \times \Lambda$, $k, i = 1, 2, \dots$ *and for* $k, i = 1, 2, \dots$ *and* $(x_0, \lambda_0) \in \Omega \times \Lambda$ *there exist finite limits of* $g_x$ *and* $g_\lambda$ *as* $(t, s, x, \lambda) \longrightarrow (\tau_k, \tau_i, x_0, \lambda_0)$, $t < \tau_k$, $s < \tau_i$ *and as* $(t, s, x, \lambda) \longrightarrow (\tau_k, \tau_i, x_0, \lambda_0)$, $t > \tau_k$, $s > \tau_i$.

5. *For* $k = 1, 2, \dots$ *the mapping* $\mathcal{I}_k\colon \Omega \times \Lambda \longrightarrow \Omega \times \Lambda$ *defined by* $\mathcal{I}_k(x, \lambda) = x + I_k(x, \lambda)$ *is a diffeomorphism in* $\Omega \times \Lambda$ *and the following relation are valid*

$$\det\Big(E + \frac{\partial I_k}{\partial x}(x, \lambda)\Big) \neq 0, \; k = 1, 2, \dots$$

6. *For* $\lambda = \lambda^* \in \Omega$ *system (1) has a solution* $\varphi(t)$ *defined in the interval* $[\alpha, \beta]$, $(\alpha, \beta \neq \tau_k, k = 1, 2, \dots)$.

*Then there exists a number* $\delta > 0$ *and a set*

$$U = \Big\{(t, x, \lambda) \in \mathbb{R}_+ \times \Omega \times \Lambda \colon \alpha < t < \beta, |x - \varphi(t^+)| < \delta, |\lambda - \lambda^*| < \delta\Big\}$$

*such that:*

(A) *For* $t \in (\alpha, \beta)$ *and* $(t_0, x_0, \lambda_0) \in U$, $t, t_0 \neq \tau_k$, $k = 1, 2, \dots$ *the function* $x = x(t) = x(t; t_0, x_0, \lambda_0)$ *has continuous derivatives* $\partial x/\partial t$, $\partial x/\partial t_0$, $\partial x/\partial x_0$, $\partial x/\partial \lambda_0$.

(B) *The derivative* $\Phi(t, t_0, x_0, \lambda_0) = \dfrac{\partial x(t; t_0, x_0, \lambda_0)}{\partial x_0}$ *is a solution of the initial value problem*

$$\text{(15)} \quad \begin{cases} y'(t) = f_x(t, x(t), \lambda_0)y(t) + \int\limits_{t_0}^{t} g_x(t, s, x(s), \lambda_0)y(s)ds, \ t \neq \tau_k\,; \\ \Delta y|_{t=\tau_k} = \dfrac{\partial I_k(x(\tau_k), \lambda_0)}{\partial x} y(\tau_k)\,, \\ y(t_0^+) = E \end{cases}$$

(C) *The derivative* $\Psi(t, t_0, x_0, \lambda_0) = \dfrac{\partial x(t; t_0, x_0, \lambda_0)}{\partial t_0}$ *is a solution of the initial value problem*

$$\text{(16)} \quad \begin{cases} y'(t) = f_x(t, x(t), \lambda_0)y(t) + \int\limits_{t_0}^{t} g_x(t, s, x(s), \lambda_0)y(s)ds - \\ \quad -g(t, t_0, x_0, \lambda_0), \ t \neq \tau_k\,; \\ \Delta y|_{t=\tau_k} = \dfrac{\partial I_k(x(\tau_k), \lambda_0)}{\partial x} y(\tau_k)\,, \\ y(t_0) = -f(t_0, x_0, \lambda_0)\,. \end{cases}$$

(D) *The derivative* $\dfrac{\partial x(t; t_0, x_0, \lambda_0)}{\partial \lambda_0}$ *is a solution of the initial value problem*

$$\text{(17)} \quad \begin{cases} y'(t) = f_x(t, x(t), \lambda_0)y(t) + f_{\lambda_0}(t, x(t), \lambda_0) + \\ \quad + \int\limits_{t_0}^{t} \Big[g_x(t, s, x(s), \lambda_0)y(s)ds + g_{\lambda_0}(t, s, x(s), \lambda_0)\Big]ds, t \neq \tau_k\,; \\ \Delta y|_{t=\tau_k} = \dfrac{\partial I_k(x(\tau_k), \lambda_0)}{\partial x} y(\tau_k) + \dfrac{\partial I_k(x(\tau_k), \lambda_0)}{\partial \lambda_0} \\ y(t_0^+) = 0_{nm}\,. \end{cases}$$

(E) *The following relation is valid*

$$
(18)\qquad \begin{aligned} &\Psi(t,t_0,x_0,\lambda_0)+\Phi(t,t_0,x_0,\lambda_0)f(t_0,x_0,\lambda_0)+\\ &+\int_{t_0}^{t} R(t,\sigma;t_0,x_0,\lambda_0)g(\sigma,t_0,x_0,\lambda_0)d\sigma=\\ &=\sum_{t_0<\tau_k<t}\Big[\Delta V(t,\tau_k;t_0,x_0,\lambda_0)f(t_0,x_0,\lambda_0)+\Delta W(t,\tau_k;t_0,x_0,\lambda_0)\Big] \end{aligned}
$$

*for* $t,t_0\neq\tau_k$ $k=1,2,\dots$, *where* $R(t,s;t_0,x_0,\lambda_0)$ *satisfies the initial value problem*

$$
\begin{cases} \dfrac{\partial R(t,s;t_0,x_0,\lambda_0)}{\partial s}+R(t,s;t_0,x_0,\lambda_0)f_x(s,x(s),\lambda_0)+\\ \quad+\int_s^t R(t,\sigma;t_0,x_0,\lambda_0)g_x(\sigma,s,x(s),\lambda_0)d\sigma=0\,,\\ R(t,t;t_0,x_0,\lambda_0)=E \end{cases}
$$

*for* $t,s,t_0\neq\tau_k$ $k=1,2,\dots$ *and*

$$
\begin{aligned} \Delta V(t,\tau_k;t_0,x_0,\lambda_0)&=R(t,\tau_k^+;t_0,x_0,\lambda_0)\Phi(\tau_k^+,t_0,x_0,\lambda_0)-\\ &\quad-R(t,\tau_k^-;t_0,x_0,\lambda_0)\Phi(\tau_k^-,t_0,x_0,\lambda_0)\,,\\ \Delta W(t,\tau_k;t_0,x_0,\lambda_0)&=R(t,\tau_k^+;t_0,x_0,\lambda_0)\Psi(\tau_k^+,t_0,x_0,\lambda_0)-\\ &\quad-R(t,\tau_k^-;t_0,x_0,\lambda_0)\Psi(\tau_k^-,t_0,x_0,\lambda_0)\,. \end{aligned}
$$

PROOF. Applying Theorem 1, we conclude that there exists a number $\delta>0$ and a set $U=\{(t,x,\lambda)\in\mathbb{R}_+\times\Omega\times\Lambda\colon \alpha<t<\beta,\ |x-\varphi(t^+)|<\delta,\ |\lambda-\lambda^*|<\delta\}$ such that the function $x(t;t_0,x_0,\lambda_0)$ is defined and continuous for $\alpha<t<\beta$, $(t_0,x_0,\lambda_0)\in U$, $t,t_0\neq\tau_k$, $k=1,2,\dots$

From the continuity of $f_x$, $f_\lambda$, $g_x$ and $g_\lambda$ it follows that the solutions of the initial value problem (15)-(17) exist and are unique in the interval $(\alpha,\beta)$.

First we shall prove the existence of the derivative of $x(t;t_0,x_0,\lambda_0)$ with respect to the $i$-th coordinate of $x_0$.

Let $e_i=(e_i^1,\dots,e_i^n)$, $i=1,2,\dots,n$ be the vectors for which $e_i^j=0$ for $i\neq j$ and $e_i^i=1$. Then for sufficiently small $h$ the function $x(t,h)=$

$x(t; t_0, x_0 + he_i, \lambda_0)$ is defined in $(\alpha, \beta)$ and $\lim_{h\to 0} x(t,h) = x(t; t_0, x_0, \lambda_0)$, uniformly in $(\alpha, \beta)$. Set

$$x(t) = x(t; t_0, x_0, \lambda_0);\ \Delta x(t,h) = x(t,h) - x(t)\,;$$
$$z(t,h) = \frac{1}{h}\Delta x(t,h),\ h \neq 0\,.$$

Applying the Newton-Leibniz formula, we obtain

$$\frac{dz(t,h)}{dt} = \int_0^1 f_x(t, x(t) + s\Delta x(t,h), \lambda_0)ds.z(t,h) +$$
$$+ \int_{t_0}^t \Big[\int_0^1 g_x(t,s,x(s) + \sigma\Delta x(s,h), \lambda_0)d\sigma.z(s,h)\Big]ds,\ t \neq \tau_k,\ k = 1, 2, \ldots$$
$$z(\tau_k^+, h) = \frac{1}{h}\Big[x(\tau_k, h) - x(\tau_k)\Big] + \frac{1}{h}\Big[I_k(x(\tau_k,h),\lambda_0) - I_k(x(\tau_k),\lambda_0)\Big] =$$
$$= z(\tau_k, h) + \int_0^1 \frac{\partial I_k(x(\tau_k) + s\Delta x(\tau_k, h), \lambda_0)}{\partial x} ds.z(\tau_k, h),$$
$$k = 1, 2, \ldots$$
$$z(t_0^+, h) = e_i$$

i.e. $z(t,h)$ is a solution of the linear impulsive initial value problem

$$\text{(19)} \qquad \begin{cases} z'(t) = A(t,h)z(t) + \int_{t_0}^t B(t,s,h)z(s)ds\,,\ t \neq \tau_k\,; \\ \Delta z|_{t=\tau_k} = B_k(h)z(\tau_k)\,, \\ z(t_0^+) = e_i\,, \end{cases}$$

where

$$A(t,h) = \int_0^1 f_x(t, x(t) + s\Delta x(t,h), \lambda_0)ds\,,$$
$$B(t,s,h) = \int_0^1 g_x(t,s,x(s) + \sigma\Delta x(s,h),\lambda_0)d\sigma\,,$$
$$B_k(h) = \int_0^1 \frac{\partial I_k(x(\tau_k) + s\Delta x(\tau_k,h),\lambda_0)}{\partial x}ds\,.$$

The matrix-valued functions $A(t,h)$, $B(t,s,h)$ and $B_k(h)$, $k=1,2,\dots$ satisfy the conditions of Theorem 2. Hence $\lim_{h\to 0} z(t,h)=y(t)$, where $y(t)$ is a solution of the initial value problem

$$(20)\qquad \begin{cases} y'(t)=f_x(t,x(t),\lambda_0)y(t)+\int\limits_{t_0}^{t} g_x(t,s,x(s),\lambda_0)y(s)ds, \ t\neq\tau_k\,; \\ \Delta y|_{t=\tau_k}=\dfrac{\partial I_k(x(\tau_k),\lambda_0)}{\partial x}y(\tau_k)\,, \\ y(t_0^+)=e_i\,, \end{cases}$$

which is obtained from (19) for $h=0$. Thus we proved the existence of the derivative of $x(t;t_0,x_0,\lambda_0)$ with respect to the $i$-th coordinate of $x_0$ and therefore the existence of $\frac{\partial x(t;t_0,x_0,\lambda_0)}{\partial x_0}$. Taking into account (20), we conclude that the matrix $\Phi(t,t_0,x_0,\lambda_0)=\frac{\partial x(t;t_0,x_0,\lambda_0)}{\partial x_0}$ satisfies the initial value problem (15) and from Theorem 2 it follows that the solution $\Phi(t,t_0,x_0,\lambda_0)$ of (15) is continuous for $\alpha<t<\beta$, $(t_0,x_0,\lambda_0)\in U$, $t,t_0\neq\tau_k$, $k=1,2,\dots$

Now we will prove the existence and continuity of the derivative. Set

$$x(t)=x(t;t_0,x_0,\lambda_0);\ x(t,h)=x(t;t_0,h,x_0,\lambda_0)\,;$$
$$\Delta x(t,h)=x(t,h)-x(t);\ z(t,h)=\frac{1}{h}\Delta x(t,h),\ h\neq 0\,.$$

Applying the Newton-Leibniz formula, we obtain

$$\frac{dz(t,h)}{dt}=\int\limits_0^1 f_x(t,x(t)+s\Delta x(t,h),\lambda_0)ds.z(t,h)+$$
$$+\int\limits_{t_0+h}^{t}\Big[\int\limits_0^1 g_x(t,s,x(s)+\sigma\Delta x(s,h),\lambda_0)d\sigma.z(s,h)\Big]ds-$$
$$-\frac{1}{h}\int\limits_{t_0}^{t_0+h} g(t,s,x(s),\lambda_0)ds,\ t\neq\tau_k,\ k=1,2,\dots\,,$$

$$z(\tau_k^+,h)=z(\tau_k,h)+\int\limits_0^1\frac{\partial I_k(x(\tau_k)+s\Delta x(\tau_k,h),\lambda_0)}{\partial x}ds.z(\tau_k,h)\,,$$
$$k=1,2,\dots\,,$$

$$z(t_0+h,h) = -\frac{1}{h}\int\limits_{t_0}^{t_0+h} f(t,x(t),\lambda_0)dt - \frac{1}{h}\int\limits_{t_0}^{t_0+h}\left[\int\limits_{t_0}^{\sigma} g(\sigma,s,x(s),\lambda_0)ds\right]d\sigma\,.$$

Hence $z(t,h)$ is a solution of the linear impulsive initial value problem

$$\begin{cases} y'(t) = H(t,t_0,x_0,\lambda_0,h)y(t)+ \\ \quad + \int\limits_{t_0+h}^{t} G(t,s,t_0,x_0,\lambda_0,h)y(s)ds - L(t,t_0,x_0,\Lambda_0,h), \;\; t\neq\tau_k \\ \Delta y|_{t=\tau_k} = B_k(t_0,x_0,\lambda_0,h)y(\tau_k)\,, \\ y(t_0+h) = a(t_0,x_0,\lambda_0,h)\,, \end{cases}$$

where

$$H(t,t_0,x_0,\lambda_0,h) = \int\limits_0^1 f_x(t,x(t)+s\Delta x(t,h),\lambda_0)ds\,,$$

$$G(t,s,t_0,x_0,\lambda_0,h) = \int\limits_0^1 g_x(t,s,x(s)+\sigma\Delta x(s,h),\lambda_0)d\sigma\,,$$

$$L(t,t_0,x_0,\lambda_0,h) = \frac{1}{h}\int\limits_{t_0}^{t_0+h} g(t,s,x(s),\lambda_0)ds\,,$$

$$B_k(t_0,x_0,\lambda_0,h) = \int\limits_0^1 \frac{\partial I_k(x(\tau_k)+s\Delta x(\tau_k,h),\lambda_0)}{\partial x}ds,\;\; k=1,2,\dots\,,$$

$$a(t_0,x_0,\lambda_0,h) = -\frac{1}{h}\int\limits_{t_0}^{t_0+h} f(t,x(t),\lambda_0)dt - \frac{1}{h}\int\limits_{t_0}^{t_0+h}\left[\int\limits_{t_0}^{\sigma} g(\sigma,s,x(s),\lambda_0)d\sigma\right]ds\,.$$

From the conditions of Theorem 4 it follows that

$$\lim_{h\to 0} H(t,t_0,x_0,\lambda_0,h) = f_x(t,x(t),\lambda_0)\,,$$
$$\lim_{h\to 0} G(t,s,t_0,x_0,\lambda_0,h) = g(t,s,x(s),\lambda_0)\,,$$
$$\lim_{h\to 0} L(t,t_0,x_0,\lambda_0,h) = g(t,t_0,x_0,\lambda_0)\,,$$

$$\lim_{h\to 0} B_k(t_0,x_0,\lambda_0,h) = \frac{\partial I_k(x(\tau_k),\lambda_0)}{\partial x}, \quad k=1,2,\dots,$$
$$\lim_{h\to 0} a(t_0,x_0,\lambda_0,h) = -f(t_0,x_0,\lambda_0).$$

Using the same arguments as in the proof of assertion (B) of Theorem 4, we obtain that the derivative $\Psi(t,t_0,x_0,\lambda_0) = \frac{\partial x(t;t_0,x_0,\lambda_0)}{\partial t_0}$ exists and is continuous for $\alpha < t < \beta$, $(t_0,x_0,\lambda_0) \in U$, $t, t_0 \neq \tau_k$, $k=1,2,\dots$ and is a solution of the initial value problem (16).

Now we will prove the existence and continuity of the derivative $\partial x/\partial \lambda_0$.

Let $e_i = (e_i^1,\dots,e_i^m)$, $i=1,2,\dots,m$ be the vector for which $e_i^j = 0$ for $i \neq j$ and $e_i^i = 1$. Set

$$x(t) = x(t;t_0,x_0,\lambda_0);\ x(t,h) = x(t;t_0,x_0,\lambda_0+he_i);$$
$$\Delta x(t,h) = x(t,h)-x(t);\ z(t,h) = \frac{1}{h}\Delta x(t,h), h \neq 0,$$

where $h$ is a sufficiently small number. Applying the Newton-Leibniz formula, we obtain

$$\frac{dz(t,h)}{dt} = \int_0^1 f_x(t,x(t)+s\Delta x(t,h),\lambda_0)ds.z(t,h)+$$
$$+\int_0^1 f_{\lambda_0}(t,x(t,h),\lambda_0+she_i)ds + \int_{t_0}^{t}\Big[\int_0^1 (g_x(t,s,x(s)+\sigma\Delta x(s,h),\lambda_0)+$$
$$+g_{\lambda_0}(t,s,x(s,h),\lambda_0+\sigma he_i))d\sigma\Big]ds,\ t \neq \tau_k;$$

$$z(\tau_k^+,h) = z(\tau_k,h) + \int_0^1 \Big[\frac{\partial I_k(x(\tau_k)+s\Delta x(\tau_k,h),\lambda_0)}{\partial x} z(\tau_k,h)+$$
$$+\frac{\partial I_k(x(\tau_k,h),\lambda_0+she_i)}{\partial \lambda_0}\Big]ds,$$
$$z(t_0^+,h) = (0,\dots,0) \in \mathbb{R}^n.$$

Hence $z(t,h)$ is a solution of the linear impulsive initial value problem

$$\begin{cases} y'(t) = H_1(t,t_0,x_0,\lambda_0)y(t) + H_2(t,t_0,x_0,\lambda_0,h) + \\ \quad + \int\limits_{t_0}^{t} \Big[G_1(t,s,t_0,x_0,\lambda_0,h)y(s) + G_2(t,s,t_0,x_0,\lambda_0,h)\Big]ds, \ t \neq \tau_k\,; \\ \Delta y|_{t=\tau_k} = B_k(t_0,x_0,\lambda_0,h)y(\tau_k) + C_k(t_0,x_0,\lambda_0,h)\,, \\ y(t_0^+) = 0 \in \mathbb{R}^n \end{cases}$$

where

$$H_1(t,t_0,x_0,\lambda_0,h) = \int\limits_0^1 f_x(t,x(t)+s\Delta x(t,h),\lambda_0)ds\,,$$

$$H_2(t,t_0,x_0,\lambda_0,h) = \int\limits_0^1 f_{\lambda_0}(t,x(t,h),\lambda_0+she_i)ds\,,$$

$$G_1(t,s,t_0,x_0,\lambda_0,h) = \int\limits_0^1 g_x(t,s,x(s)+\sigma\Delta x(s,h),\lambda_0 d\sigma\,,$$

$$G_2(t,s,t_0,x_0,\lambda_0,h) = \int\limits_0^1 g_{\lambda_0}(t,s,x(s,h),\lambda_0+\sigma he_i)d\sigma\,,$$

$$B_k(t_0,x_0,\lambda_0,h) = \int\limits_0^1 \frac{\partial I_k(x(\tau_k)+s\Delta x(\tau_k,h),\lambda_0)}{\partial x}ds\,,$$

$$C_k(t_0,x_0,\lambda_0,h) = \int\limits_0^1 \frac{\partial I_k(x(\tau_k,h),\lambda_0+she_i)}{\partial \lambda_0}ds\,.$$

The proof of assertion (D) of Theorem 4 is completed as the proof of assertion (B).

Finally we shall show that equality (18) is valid.

From assertion (B) of Theorem 4 and from Theorem 3 it follows that

$$\begin{aligned}\Phi(t,t_0,x_0,\lambda_0) =& R(t,t_0^+;t_0,x_0,\lambda_0)+ \\ &+ \sum_{t_0<\tau_k<t} \Delta V(t,\tau_k;t_0,x_0,\lambda_0),\ t,t_0 \neq \tau_k\,, k=1,2,\ldots\end{aligned} \tag{21}$$

From assertion (C) of Theorem 4 and from Theorem 3 it follows that

$$\Psi(t,t_0,x_0,\lambda_0)=-R(t,t_0^+;t_0,x_0,\lambda_0)f(t_0,x_0,\lambda_0)- \tag{22}$$
$$-\int_{t_0}^{t} R(t,s;t_0,x_0,\lambda_0)g(s,t_0,x_0,\lambda_0)ds+$$
$$+\sum_{t_0<\tau_k<t} \Delta W(t,\tau_k;t_0,x_0,\lambda_0),\ t,t_0\neq\tau_k\,,k=1,2,\ldots$$

From (21) and (22) we obtain (18).

Theorem 4 is proved. □

## 3.2 – Variation of parameters formula and analogue of Alekseev's formula for impulsive systems of integro-differential equations

Consider the impulsive integro-differential systems

$$\begin{cases} x'(t) = f(t,x(t)) + \int_{t_0}^{t} g(t,s,x(s))ds,\ t\neq\tau_k\,; \\ \Delta x|_{t=\tau_k} = I_k(x(\tau_k))\,, \\ x(t_0^+) = x_0\,, \end{cases} \tag{23}$$

$$\begin{cases} y'(t) = f(t,y(t)) + \int_{t_0}^{t} g(t,s,y(s))ds+ \\ \qquad +F(t,y(t),(Sy)(t)),t\neq\tau_k\,; \\ \Delta y|_{t=\tau_k} = I_k(y(\tau_k)) + h_k(y(\tau_k))\,, \\ y(t_0^+) = x_0 \end{cases} \tag{24}$$

where $f\colon \mathbb{R}_+\times\Omega\longrightarrow\mathbb{R}^n, g\colon \mathbb{R}_+^2\times\Omega\mathbb{R}^n, I_k\colon\Omega\longrightarrow\mathbb{R}^n,$

$$F\colon R_+\times\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^n, (Sy)(t)=\int_{t_0}^{t} K(t,s,y(s))ds, K\colon\mathbb{R}_+^2\times\Omega\longrightarrow\mathbb{R}^n\,.$$

THEOREM 5. *Let systems (23) and (24) satisfy the conditions of Theorem 4 and let $x(t;t_0,x_0)$ be a solution of system (23) which is defined in the interval $\mathcal{I}$.*

*Then for any solution $y(t) = y(t;t_0,x_0)$ of system (24) which is defined in $\mathcal{I}$ the following formula is valid*

$$
(25)\quad
\begin{aligned}
y(t;t_0,x_0) &= x(t;t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s))F(s,y(s),(Sy)(s))ds + \\
&+ \int_{t_0}^{t}\int_{s}^{t} \Big[\Big(\Phi(t,\sigma,y(\sigma)) - R(t,\sigma;s,y(s))\Big)g(\sigma,s,y(s))\Big]d\sigma ds + \\
&+ \int_{t_0}^{t}\Big[\sum_{s<\tau_k<t}\Big(\Delta V(t,\tau_k;s,y(s))f(s,y(s)) + \Delta W(t,\tau_k;s,y(s))\Big)\Big]ds + \\
&+ \sum_{t_0<\tau_k<t}\int_0^1 \Phi\Big(t,\tau_k^-,y(\tau_k)+I_k(y(\tau_k))+sh_k(y(\tau_k))\Big)ds.h_k(y(\tau_k))
\end{aligned}
$$

*for $t > t_0$, $t, t_0 \neq \tau_k$, $k = 1,2,\dots$ and*

$$
(26)\quad
\begin{aligned}
y(t;t_0,x_0) &= x(t;t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s))F(s,y(s),(Sy)(s))ds + \\
&+ \int_{t_0}^{t}\int_{s}^{t} \Big[\Big(\Phi(t,\sigma,y(\sigma)) - R(t,\sigma;s,y(s))\Big)g(\sigma,s,y(s))\Big]d\sigma ds + \\
&+ \int_{t_0}^{t}\Big[\sum_{t<\tau_k<s}\Big(\Delta V(t,\tau_k;s,y(s))f(s,y(s)) + \Delta W(t,\tau_k;s,y(s))\Big)\Big]ds + \\
&+ \sum_{t<\tau_k<t_0}\int_0^1 \Phi\Big(t,\tau_k^-,y(\tau_k)+I_k(y(\tau_k))+sh_k(y(\tau_k))\Big)ds.h_k(y(\tau_k))
\end{aligned}
$$

*for $t \le t_0$, $t, t_0 \neq \tau_k$, $k = 1,2,\dots$ where $R(t,s;t_0,x_0)$ is a solution of the initial value problem*

$$
\begin{cases}
\dfrac{\partial R(t,s;t_0,x_0)}{\partial s} + R(t,s;t_0,x_0)f_x(s,x(s;t_0,x_0)) + \\
\quad + \int_s^t R(t,\sigma;t_0,x_0)g_x(\sigma,s,x(s;t_0,x_0))d\sigma = 0 \\
R(t,t;t_0,x_0) = E
\end{cases}
$$

*for $t_0 \le s \le t$, $s,t \neq \tau_k$, $k = 1,2,\dots$ and*

$$\Delta V(t,\tau_k;s,y(s)) = R(t,\tau_k^+;s,y(s))\Phi(\tau_k^+,s,y(s)) - \\ - R(t,\tau_k^-;s,y(s))\Phi(\tau_k^-,s,y(s)),$$

$$\Delta W(t,\tau_k;s,y(s)) = R(t,\tau_k^+;s,y(s))\Psi(\tau_k^+,s,y(s)) - \\ - R(t,\tau_k^-;s,y(s))\Psi(\tau_k^-,s,y(s)).$$

PROOF. Set $p(s) = x(t;s,y(s))$, where $y(s) = y(s;t_0,x_0)$. Then for $s \neq \tau_k$, $k = 1,2,\dots$ we have

$$\begin{aligned} p'(s) =& \Psi(t,s,y(s)) + \Phi(t,s,y(s))y'(s) = \\ =& \Psi(t,s,y(s)) + \Phi(t,s,y(s))[f(s,y(s)) + \\ &+ \int_{t_0}^{s} g(s,\sigma,y(\sigma))d\sigma + F(s,y(s),(Sy)(s))\Big]; \end{aligned} \tag{27}$$

$$\begin{aligned} \Delta p|_{s=\tau_k} =& x(t;\tau_k^+,y(\tau_k^+)) - x(t;\tau_k^-,y(\tau_k)) = \\ =& x(t;\tau_k,y(\tau_k) + I_k(y(\tau_k)) + h_k(y(\tau_k))) - \\ &- x(t;\tau_k,y(\tau_k) + I_k(y(\tau_k))) = \\ =& \int_0^1 \Phi(t,\tau_k^-,y(\tau_k) + I_k(y(\tau_k)) + sh_k(y(\tau_k)))ds.h_k(y\tau_k)). \end{aligned} \tag{28}$$

Integrating (27) from $t_0$ to $t$ and using (28) and (18) and that $p(t) = x(t;t,y(t)) = y(t)$ for $t \neq \tau_k$, $k = 1,2,\dots$ we obtain formula (25). In an analogous way formula (26) is proved.

Theorem 5 is proved. □

In the special case when $g(t,s,x) \equiv 0$ and $F(t,x,y) = h(t,x)$ then $R(t,s;t_0,x_0) \equiv 0$ and formulae (25) and (26) respectively take the form

$$y(t;t_0,x_0) = x(t;t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s))h(s,y(s))ds +$$

$$+ \sum_{t_0<\tau_k<t} \int_0^1 \Phi(t,\tau_k^-,y(\tau_k) + I_k(y(\tau_k)) + sh_k(y(\tau_k)))ds.h_k(y(\tau_k))$$

for $t > t_0$, $t \neq \tau_k$, $k = 1,2,\dots$ and

$$y(t;t_0,x_0) = x(t;t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s))h(s,y(s))ds -$$
$$- \sum_{t<\tau_k<t_0} \int_0^1 \Phi(t,\tau_k^-,y(\tau_k) + I_k(y(\tau_k)) + sh_k(y(\tau_k)))ds.h_k(y(\tau_k))$$

for $t < t_0$, $t \neq \tau_k$, $k = 1,2,\dots$ which represent an anlogue of Alekseev's formula for impulsive systems of differential equations (see formula (39) of [7] or formula (2.5.4) of [5]).

If, moreover, $f(t,x) = A(t)x$, where $A(t)$ is a piecewise continuous $n \times n$-matrix with points of discontinuity $t = \tau_k$, $k = 1,2,\dots$ at which it is continuous from the left, and $I_k(x) = B_k x$, where $B_k$, $k = 1,2,\dots$ are constant $n \times n$-matrices, for which $\det(E + B_k) \neq 0$, then formulae (25) and (26) respectively take the form:

$$\begin{aligned} y(t;t_0,x_0) =& W(t,t_0^+)x_0 + \int_{t_0}^{t} W(t,s)h(s,y(s))ds + \\ &+ \sum_{t_0<\tau_k<t} W(t,\tau_k^+)h_k(y(\tau_k)) \quad \text{for } t > t_0 \end{aligned} \tag{29}$$

and

$$\begin{aligned} y(t;t_0,x_0) =& W(t,t_0^+)x_0 + \int_{t_0}^{t} W(t,s)h(s,y(s))ds - \\ &- \sum_{t<\tau_k<t_0} W(t,\tau_k^+)h_k(y(\tau_k)) \quad \text{for } t < t_0, \end{aligned} \tag{30}$$

where

$$W(t,s) =$$
$$= \begin{cases} U_k(t,s) \quad \text{for } t,s \in (\tau_{k-1},\tau_k], \\ U_{k+1}(t,\tau_k^+)(E+B_k)U_k(\tau_k,s) \quad \text{for } \tau_{k-1} < s \le \tau_k < t \le \tau_{k+1}, \\ U_k(t,\tau_k(E+B_k)^{-1}U_{k+1}(\tau_k^+,s) \quad \text{for } \tau_{k-1} < t \le \tau_k < s \le \tau_{k+1}, \\ U_{k+1}(t,\tau_k^+)\left[\prod_{j=k}^{i+1}(E+B_j)U_j(\tau_j,\tau_{j-1}^+)\right] \circ (E+B_i)U_i(\tau_i,s) \\ \qquad \text{for } \tau_{i-1} < s \le \tau_i < \tau_k < t \le \tau_{k+1}, \\ U_i(t,\tau_i)\left[\prod_{j=i}^{k-1}(E+B_j)^{-1}U_{j+1}(\tau_j^+,\tau_{j+1})\right] \circ (E+B_k)^{-1}U_{k+1}(\tau_k^+,s) \\ \qquad \text{for } \tau_{i-1} < t \le \tau_i < \tau_k < s \le \tau_{k+1} \end{cases}$$

and $U_k(t,s)$ $(t,s \in (\tau_{k-1},\tau_k])$ is the Cauchy matrix of the linear system

$$\dot{x} = A(t)x, (\tau_{k-1} < t \leq \tau_k).$$

Formulae (29) and (30) represent variation of parameters formulae for impulsive systems of differential equations (see formulae (35) and (36) of [7] or formula (2.5.2) of [5]).

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