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The paraquaternionic projective spaces

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RIASSUNTO: *Si studiano gli spazi proiettivi paraquaternionali che sono gli spazi simmetrici para-Hermitiani corrispondenti alla coppia simmetrica* $(sl(n+1, \mathbb{C}), sl(n, \mathbb{C})+\mathbb{C})$ *nella classificazione di Kaneyuki-Kozai.*

ABSTRACT: *The paraquaternionic projective spaces, which are the para-Hermitian symmetric spaces corresponding to the symmetric pair* $(sl(n+1, \mathbb{C}), sl(n, \mathbb{C}) + \mathbb{C})$ *in the Kaneyuki-Kozai classification, are studied.*

KEY WORDS: *Para-Kählerian manifold – Para-Hermitian symmetric space.*

A.M.S. CLASSIFICATION: 53C15 – 53C35 – 53C50

1 – Introduction

Para-Kählerian manifolds were introduced by Rasevskii [11] and Libermann [10], and studied by several authors (see [1] and the large list of references therein). Para-Hermitian symmetric spaces, which are para-Kählerian manifolds, were defined by Kaneyuki and Kozai [8], who obtained the infinitesimal classification of such symmetric manifolds with semisimple group, up to paraholomorphic equivalence.

Moreover, the spaces $P_n(B)$, which are para-Kählerian manifolds such that for $n > 1$ are models of spaces of constant paraholomorphic sectional curvature, were introduced in [2] and studied in [1], [3], [4], [5], [6].

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The spaces $P_n(B)$ and $P_n(B)/\mathbb{Z}_2$ (see [3]) are the symmetric spaces corresponding to the Kaneyuki-Kozai symmetric pair $(sl(n+1,\mathbb{R}), sl(n,\mathbb{R})+\mathbb{R})$. They are diffeomorphic to the cotangent bundles T^*S^n and $T^*P_n(\mathbb{R})$, respectively ([8], p. 97).

The purpose of the present paper is to study the spaces $P_{n,n}(\mathbb{C})$, which are similar to the spaces $P_n(B)/\mathbb{Z}_2$ for the ring of *complex dual numbers* $B_{\mathbb{C}} = \mathbb{C}[j] = \mathbb{R}[i][j]$, with $i^2 = -1$, $j^2 = 1$, and mainly their underlying real manifolds, $P_{n,n}(\mathbb{C})_{\mathbf{R}}$, which we call *paraquaternionic projective spaces.*

There are three basic facts justifying this terminology.

First,the group $\{\pm 1, \pm i, \pm j, \pm(ij)\}$ plays the role of quaternionic group in the paracomplex setting. Secondly, as it is proved below, we have $P_{n,n}(\mathbb{C})_{\mathbf{R}} = Gl(n+1,\mathbb{C})/Gl(n,\mathbb{C}) \times Gl(1,\mathbb{C})$, which is the similar presentation of $P_{n,n}(\mathbb{C})_{\mathbf{R}}$ as a symmetric space to that of the quaternionic projective space, $P_n(\mathbb{H}) = Sp(n+1)/Sp(n) \times Sp(1)$ ([7], p. 454). Finally, the local expression of the canonical Riemannian metric of constant quaternionic sectional curvature on $P_n(\mathbb{H})$ is ([12], p. 147):

$$\mathrm{Re}\left[\frac{1}{1+<q,\bar{q}>}\Big(dq_\alpha \cdot d\bar{q}_\alpha - \frac{1}{1+<q,\bar{q}>}\bar{q}_\alpha q_\beta \, dq_\alpha \cdot d\bar{q}_\beta\Big)\right],$$

and the metric induced on the symmetric space $P_{n,n}(\mathbb{C})_{\mathbf{R}}$ by the Killing form on $sl(n+1,\mathbb{C})$ is

$$\mathrm{Re}\left[\frac{1}{1+<z,w>}\Big(dz_\alpha \cdot dw_\alpha - \frac{1}{1+<z,w>} z_\beta w_\alpha dz_\alpha \cdot dw_\beta\Big)\right],$$

as it is proved below.

The spaces $P_{n,n}(\mathbb{C})_{\mathbf{R}}$ are the para-Hermitian symmetric spaces corresponding to the Kaneyuki-Kozai symmetric pair $(sl(n+1,\mathbb{C}), sl(n,\mathbb{C})+\mathbb{C})$. Furthermore, the spaces $P_{n,n}(\mathbb{C})_{\mathbf{R}}$ are diffeomorphic to the cotangent bundle $T^*P_n(\mathbb{C})$ of the complex projective space $P_n(\mathbb{C})$. Actually, they are examples of manifolds with non-trivial para-Hodge structure ([9]). From the earlier considerations, it becomes clear that the spaces under study have a rich structure.

2 – The paraquaternionic projective spaces

Let us consider the bilinear form $<,>$ on $\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$ given by

$$< (z,w)\,,\,(z',w') >= \sum_{j=0}^{n}(z_j w'_j + z'_j w_j)\,.$$

We also consider the complex almost product tensor $J_0(z,w) = (z,-w)$. We define:

$$P_{n,n}(\mathbb{C}) = \{([z],[w]) \in P_n(\mathbb{C}) \times P_n(\mathbb{C})\ ;\ < (z,0)\,,\,(0,w) > \neq 0\}$$

and

$$S = \{(z,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}\ ;\ < (z,0)\,,\,(0,w) >= 1\}\,.$$

$Gl(n+1,\mathbb{C})$ acts on $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ by setting:

$$A \cdot (z,w) = (A(z)\,,\,{}^t\!A^{-1}(w))\,,$$

and it is easily checked that $Gl(n+1,\mathbb{C})$ acts transitively on S. Hence, $(S, Gl(n+1,\mathbb{C}))$ is a homogeneous manifold, and we have a principal bundle

$$\pi : S = Gl(n+1,\mathbb{C})/Gl(1,\mathbb{C}) \longrightarrow P_{n,n}(\mathbb{C})$$

whose structure group is $Gl(1,\mathbb{C}) = \mathbb{C}^*$. Similarly, one can prove that

$$P_{n,n}(\mathbb{C}) = Gl(n+1,\mathbb{C})/Gl(n,\mathbb{C}) \times Gl(1,\mathbb{C})\,.$$

We want to endow the real manifold underlying $P_{n,n}(\mathbb{C})$ with a para-Hermitian structure. Let U_j be the open subset of $P_{n,n}(\mathbb{C})$ defined by

$$U_j = \{([z]\,,\,[w])\ ;\ z_j w_j \neq 0\}, \qquad 0 \le j \le n\,.$$

We define a system of coordinates on U_j by the formulas:

$$z_j^a = z_a/z_j \quad , \quad w_j^a = w_a/w_j \qquad (a \neq j\,, 0 \le a \le n)\,.$$

Hence, $P_{n,n}(\mathbb{C})$ is an open submanifold of $P_n(\mathbb{C}) \times P_n(\mathbb{C})$ with the atlas $(U_j\,;\,z_j^a\,,\,w_j^a)$. Let $P_{n,n}(\mathbb{C})_{\mathbb{R}}$ be the C^ω real manifold underlying $P_{n,n}(\mathbb{C})$, which will be called *the paraquaternionic projective space.*

First of all, we shall construct a $Gl(n+1,\mathbb{C})$-invariant pseudo-Riemannian metric on $P_{n,n}(\mathbb{C})_{\mathbf{R}}$.

$Gl(n+1,\mathbb{C})$ is an isometry group of $<,>$ on $\mathbb{C}^{n+1}\oplus\mathbb{C}^{n+1}$, and preserves S. Consequently, the above group induces an isometry group of S.

LEMMA. *The restriction of the metric* $<,>$ *to* $T_{(z,w)}(\pi^{-1}(\pi(z,w)))$ *is non-degenerate in* $T_{(z,w)}(S)$. *Accordingly, given* $Y\in T_{\pi(z,w)}(P_{n,n}(\mathbb{C}))$, *there exists a unique* $Y^H\in T_{(z,w)}S$ *such that:*

(i) $\pi_*(Y^H)=Y$,

(ii) $Y^H\in T^{\perp}_{(z,w)}(\pi^{-1}(\pi(z,w)))$.

Moreover, for every $\lambda\in Gl(1,\mathbb{C})$ *one has:*

$$(R_\lambda)_*(Y^H_{(z,w)})=Y^H_{(\lambda z,\lambda^{-1}w)}\,.$$

PROOF. Let $\mu: Gl(1,\mathbb{C})\longrightarrow S$ be $\mu(\lambda)=(\lambda z^0,\lambda^{-1}w^0)$. Then,

$$\mu_*(T_I(Gl(1,\mathbb{C})))=T_{(z^0,w^0)}(\pi^{-1}(\pi(z,w)))\,,$$

and as a simple computation shows,

$$\mu_*\Big(\frac{\partial}{\partial\lambda}\Big)_I=z_j^0\Big(\frac{\partial}{\partial z_j}\Big)_{(z^0,w^0)}-w_j^0\Big(\frac{\partial}{\partial w_j}\Big)_{(z^0,w^0)}\,,$$

thus proving that the tangent space to the fibre through (z^0,w^0) is non-singular.

The last part in the statement follows taking into account that R_λ is an isometry of $<,>$.

This lemma allows one to define a metric g on $P_{n,n}(\mathbb{C})$ by the formula:

$$g(Y,Z)=<Y^H_{(z,w)}\,,\,Z^H_{(z,w)}>\,,\qquad Y,Z\in T_{\pi(z,w)}(P_{n,n}(\mathbb{C}))\,.$$

PROPOSITION.
Let $f:\mathbb{C}^{n+1}\oplus\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ *be* $f(z,w)=<(z,0),(0,w)>$. *We have:*

$$T^{\perp}_{(z^0,w^0)}(S)=\mathbb{C}\cdot(\operatorname{grad} f)_{(z^0,w^0)}\,,$$

and

$$(\operatorname{grad} f)_{(z^0,w^0)}=z_j^0\Big(\frac{\partial}{\partial z_j}\Big)_{(z^0,w^0)}+w_j^0\Big(\frac{\partial}{\partial w_j}\Big)_{(z^0,w^0)}\,.$$

PROOF. The first formula follows directly from the equation $S = f^{-1}(1)$, and the second from a calculation.

COROLLARY. *The metric g on $P_{n,n}(\mathbb{C})$ defined above is non-degenerate.*

The complex manifold $P_{n,n}(\mathbb{C})$ has a complex almost product structure J defined below.
$Gl(1,\mathbb{C})\times Gl(1,\mathbb{C})$ acts on $\mathbb{C}^{n+1}\times\mathbb{C}^{n+1}-Q$, $Q=\{(z,w)\in\mathbb{C}^{n+1}\times\mathbb{C}^{n+1};\ <(z,0),(0,w)>=0\}$, and $P_{n,n}(\mathbb{C})=(\mathbb{C}^{n+1}\times\mathbb{C}^{n+1}-Q)/Gl(1,\mathbb{C})\times Gl(1,\mathbb{C})$.
Set:

$$J_0(u,v) = (u,-v)\,,\ (u,v)\in T_{(z,w)}(\mathbb{C}^{n+1}\times\mathbb{C}^{n+1}-Q) = \\ = T_z(\mathbb{C}^{n+1}\times T_w(\mathbb{C}^{n+1})\,.$$

Hence, $(J_0\circ R_{(\lambda,\mu)*} - R_{(\lambda,\mu)*}\circ J_0)(u,v) = 0$.

Consequently, J_0 defines a $Gl(n+1,\mathbb{C})$-invariant structure J on $P_{n,n}(\mathbb{C})$, which satisfies the further condition of being an anti-isometry; i.e.,

$$g(X,JY)+g(JX,Y)=0,\ \ X,Y\in T(P_{n,n}(\mathbb{C}))\,.$$

The pair (g,J) induces a real para-Hermitian structure $(g_{\mathbf{R}}, J_{\mathbf{R}})$ on $P_{n,n}(\mathbb{C})_{\mathbf{R}}$ by setting,

$$g_{\mathbf{R}} = \mathrm{Re}\ g\ ,\ \ J_{\mathbf{R}}(X) = \mathrm{Re}\ J(X - iI(X))\,,$$

$X\in T(P_{n,n}(\mathbb{C})_{\mathbf{R}})$ and I being the almost complex structure induced by the complex manifold structure of $P_{n,n}(\mathbb{C})$.

THEOREM. *The local expressions of the metric $g_{\mathbf{R}}$ and the almost product structure $J_{\mathbf{R}}$ on the open subset U_0 of $P_{n,n}(\mathbb{C})_{\mathbf{R}}$ with coordinates (z_0^a, w_0^a), introduced above, are as follows:*

$$g_{\mathbf{R}} = \mathrm{Re}\left[\sum_{u,v=1}^{n}\frac{1}{1+\sum_{a=1}^{n} z_0^a w_0^a}\left(\delta_{uv} - \frac{z_0^v w_0^u}{1+\sum_{a=1}^{n} z_0^a w_0^a}\right)dz_0^u\cdot dw_0^v\right]$$

$$J_{\mathbf{R}}(X) = \mathrm{Re}\left[\left(\frac{\partial}{\partial z_0^u}\otimes dz_0^u - \frac{\partial}{\partial w_0^u}\otimes dw_0^u\right)(X - iI(X))\right].$$

Consequently, $(P_{n,n}(\mathbb{C})_{\mathbf{R}}, g_{\mathbf{R}}, J_{\mathbf{R}})$ is a para-Kählerian manifold.

PROOF. With the above notations, we have:

$$\left(\frac{\partial}{\partial z_0^a}\right)^H = z_0\left(\frac{\partial}{\partial z_a} - z_j w_a \frac{\partial}{\partial z_j}\right) , \left(\frac{\partial}{\partial w_0^a}\right)^H = w_0\left(\frac{\partial}{\partial w_a} - w_j z_a \frac{\partial}{\partial w_j}\right).$$

Hence: $g(\frac{\partial}{\partial z_0^a}, \frac{\partial}{\partial w_0^b}) = z_0 w_0(\delta_{ab} - z_b w_a)$. Taking into account that $1 = \sum_{a=0}^n z_a w_a$ on S , we obtain $z_0 w_0 = 1/(1 + \sum_{a=1}^n z_0^a w_0^a)$, and the local expression for the metric follows.

The local expression for $J_{\mathbf{R}}$ directly follows from its very definition.

Moreover, since $g_{\mathbf{R}}$ and $J_{\mathbf{R}}$ are $Gl(n+1, \mathbb{C})$-invariant, the 2-form $F_{\mathbf{R}}(X, Y) = g_{\mathbf{R}}(J_{\mathbf{R}}X, Y)$ and its exterior differential are also $Gl(n+1, \mathbb{C})$-invariant. As $d(F_{\mathbf{R}})$ vanishes at the point $z_i = w_i = 0$, it follows that $d(F_{\mathbf{R}})$ vanishes identically, thus finishing the proof of the theorem.

THEOREM. *$(P_{n,n}(\mathbb{C})_{\mathbf{R}}, g_{\mathbf{R}}, J_{\mathbf{R}})$ is the para-Hermitian symmetric space corresponding to the symmetric pair $(sl(n+1, \mathbb{C}), sl(n, \mathbb{C}) + \mathbb{C})$ in the Kaneyuki-Kozai classification ([8], p. 97).*

PROOF. Let σ be the involutive automorphism of $G = Gl(n+1, \mathbb{C})$ given by

$$\sigma(A) = \begin{pmatrix} -1 & 0 \\ 0 & -I_n \end{pmatrix} A \begin{pmatrix} -1 & 0 \\ 0 & -I_n \end{pmatrix} , A \in Gl(n+1, \mathbb{C}).$$

Then, $G^\sigma = Gl(1, \mathbb{C}) \times Gl(n, \mathbb{C})$. Moreover, the inner product $<,>$ in $g = gl(n+1, \mathbb{C})$ given by $< X, Y >= Re\,Tr\,(XY)$ is $Ad(G)$-invariant, and we have $< d\sigma(X), d\sigma(Y) >=< X, Y >$ for every $X, Y \in g$. Also,

$$m = \{X \in g; d\sigma(X) = -X\} = \left\{\begin{pmatrix} 0 & u \\ {}^t v & 0 \end{pmatrix} ; u, v \in \mathbb{C}^n\right\}.$$

Set

$$J_0 \begin{pmatrix} 0 & u \\ {}^t v & 0 \end{pmatrix} = \begin{pmatrix} 0 & -u \\ {}^t v & 0 \end{pmatrix} ,$$

and $g_0 = <,>|_m$, thus obtaining $g_0(J_0 X, Y) + g_0(X, J_0 Y) = 0$ for $X, Y \in m$. Taking into account the natural identification $m \approx T_0(G/G^\sigma)$, we conclude that the extensions via left translations of g_0 and J_0 are $g_{\mathbf{R}}$ and $J_{\mathbf{R}}$, respectively.

REFERENCES

[1] C. BEJAN,C.: *Structuri hiperbolice pe diverse spatii fibrate*, Ph.D. Thesis, Iasi, 1990.

[2] P.M. GADEA - A. MONTESINOS AMILIBIA: *Spaces of constant paraholomorphic sectional curvature*, Pacific J. Math. **136** (1989), 85-101.

[3] P.M. GADEA - A. MONTESINOS AMILIBIA: *Some geometric properties of para-kählerian space forms*, Rend. Sem. Mat. Univ. Cagliari **59** (1989), 131-145.

[4] P.M. GADEA - J. MUÑOZ MASQUÉ: *Classification of almost para-hermitian manifolds*, Rend. Mat. VII, **11**, (1991), 377-396.

[5] P.M. GADEA - J. MUÑOZ MASQUÉ: *Classification of homogeneous para-Kählerian space forms*, Nova J. Alg. Geom. Iowa, 1992 (in press).

[6] P.M. GADEA - J. MUÑOZ MASQUÉ: *Classification of non-flat para-Kählerian space forms*, (preprint).

[7] S. HELGASON: *Differential Geometry, Lie groups and symmetric spaces*, Academic Press, 1978.

[8] S. KANEYUKI - M. KOZAI: *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985), 81-98.

[9] S. KANEYUKI - F.L. WILLIAMS: *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985), 173-187.

[10] P. LIBERMANN: *Sur le problème d'équivalence de certaines structures infinitésimales*, Ann. di Mat. **36** (1954), 27-120.

[11] P.K. RASEVSKII: *The scalar field in a stratified space*, Trudy Sem. Vekt. Tenz. Anal. **6** (1948), 225-248.

[12] Y. WATANABE: *On the characteristic function of quaternion Kählerian spaces of constant Q-sectional curvature*, Rev. Roum. Mat. Pur. Appl. **12** (1977), 131-148.

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