Rendiconti di Matematica, Serie VII
Volume 13 , Roma (1993), 13-26

Inequalities for generalized hypergeometric functions of three variables

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RIASSUNTO: *La tecnica, sviluppata da* Y.L. LUKE [5] *per ottenere disuguaglianze bilaterali, per le funzioni ipergeometriche generalizzate, a partire dalla loro rappresentazione mediante integrali euleriani, viene usata in questo lavoro per ottenere delle disuguaglianze per le funzioni di Lauricella* F_A, F_B, F_D. *Sono state sviluppate anche delle valutazioni numeriche che confermano la validità delle formule ottenute. Le stime ottenute sono migliori in quanto la determinazione delle disuguaglianze spesso implica un costo minore rispetto a quello delle serie triple.*

ABSTRACT: LUKE [5] *has developed a technique of obtaining two-sided inequalities for generalized hypergeometric functions through their Eulerian integral-representations. We exploit the thechnique suggested by him in obtaining inequalities for Lauricella functions* F_A, F_B *and* F_D. *Specific numerics have been given in support of the algebra involved and to lend credence to the validity of various formulas that are presented. The bounds obtained are worth while since evaluation of inequalities often takes much less effort than evaluation of a triple series. In the sequel, corrections in theorems 2 and 6 of* LUKE [5] *are also presented.*

KEY WORDS: *Generalized hypergeometric functions – Appell functions – Lauricella functions.*

A.M.S. CLASSIFICATION: 33C50 – 26D07

1 – Introduction

Using Padé approximation, LUKE [4] has developed inequalities for the generalized hypergeometric functions ${}_pF_q(\alpha_p;\beta_q - z), p = q, p = q+1$ and certain confluent forms under appropriate restrictions on parameters and variables. He [5] has exploited these notions in obtaining bounds for Appell function F_1, F_2 and F_3. As pointed out by him, the technique can be employed for deriving inequalities for hypergeometric functions of several variables such as those known as Lauricella's functions. We will, howerver, confine ourselves to the functions F_A, F_B in view of their applicability in the analysis of electron scattering from the nucleus [7] and F_D, which is useful in representing elliptic integrals [1].

In the investigation that follows we shall require the integral representation ([2], p.9 (59)):

$$(1) \quad B(x,y) = \frac{\Gamma x \Gamma y}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ R(x) > 0,\ R(y) > o,$$

and the result of LUKE ([4], 4.4):

$$(2) \quad \begin{aligned} &(1-az)^{-1} \leq (1+z)^{-a} \leq \frac{1-a}{1+a} + \frac{a}{\bar{a}}(1+\bar{a}z)^{-1}, \\ &0 \leq a \leq 1, z \geq 0, \bar{a} = \frac{1+a}{2}, \end{aligned}$$

which Luke used essentially to establish:

$$(3) \quad L \leq H \leq R,$$

where:

$$\begin{aligned} L &= \left(1 + cA + \left(\frac{a}{a+b}\right)cb\right)^{-1}, \\ R &= \frac{1-c}{1+c} + \frac{2cb}{(a+1)(a+b)(c+1)}(1+cA)^{-1} \\ &\quad + \frac{2ca(1+a+b)}{(a+1)(a+b)(c+1)}\left[1 + \bar{c}A + \frac{(1+a)}{(1+a+b)}\bar{c}B\right]^{-1}, \\ \bar{c} &= \frac{1+c}{2}, \end{aligned}$$

and:

$$\begin{aligned} &H = \frac{\Gamma(a+b)}{\Gamma a \Gamma b}\int_0^1 t^{a-1}(1-t)^{b-1}(1+A+tB)^{-c}dt, \\ &0 < a, b, 0 \leq c \leq 1, 0 \leq A, B, cB < 1 + cA, \end{aligned}$$

and also a, b, c, A, B are independent of t.

By specializing the various parameters and using the identification

$$(4)\qquad {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|x\right)=\frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt,$$
$$R(c)>R(b)>0,\ |\arg(1-x)|<\pi,$$

one obtains the one variable hypergeometric result:

$$(5)\qquad \begin{gathered}\left(1+\frac{abz}{c}\right)^{-1}\le {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|-z\right)\le 1-\frac{abd}{c}+\frac{abd}{c}\left(1+\frac{z}{d}\right)^{-1},\\ R(c)>R(b)>0,|\arg(1-z)|<\pi,0\le a\le 1\quad\text{and}\\ d=\frac{2(c+1)}{(1+a)(1-b)}.\end{gathered}$$

2 – Inequalities for F_D

Consider the function F_D, defined by

$$\begin{aligned}F_D=&F_D(a,b_1,b_2,b_3;c;-x,-y,-z)=\\ &=\sum_{m,np=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p(-x)^m(-y)^n(-z)^p}{(c)_{m+n+p}m!n!p!},\end{aligned}$$

$|x|<1$, $|y|<1$, $|z|<1$, c is not a negative integer or zero, which admits the integral representation:

$$\begin{aligned}F_D=\Gamma\left[\begin{matrix}c\\b_1,b_2,b_3,c-b_1-b_2-b_3\end{matrix}\right]\int_0^1\int_0^1\int_0^1 u^{b_1-1}(1-u)^{c-b_1-b_2-b_3-1}s^{b_2+b_3-1}\\ \cdot(1-s)^{c-b_2-b_3-1}t^{b_3-1}(1-t)^{b_2-1}\Big(1+xu(1-s)+ys(1-t)+zst\Big)^{-a}du\,ds\,dt,\end{aligned}$$

under the conditions:

$$(7)\qquad \begin{gathered}R(c)>R(b_1+b_2+b_3)>0,\ R(b_1)>0,\ R(b_2)>0,\ R(b_3)>0,\\ |\arg(1+x+y+z)|<\pi.\end{gathered}$$

If, therefore,

$$(8)\quad c>b_1+b_2+b_3>0,0\le a\le 1,b_1>0,b_2>0,b_3>0,z\ge y\ge x>0,$$

then applications of (1), (2), (4) and (5) in (7) yields:

$$\begin{aligned}
&\Gamma\left[\begin{matrix} c \\ b_2, b_3, c-b_2-b_3 \end{matrix}\right] \int_0^1 \int_0^1 s^{b_2+b_3-1}(1-s)^{c-b_2-b_3-1} t^{b_3-1}(1-t)^{b_2-1} \cdot \\
&\cdot \Big[1 + a(ys(1-t)+zst)\Big]^{-1} \cdot \\
&\cdot {}_2F_1\left[\begin{matrix} 1, b_1 ; \\ c-b_2-b_3 \end{matrix} - \frac{ax(1-s)}{1+a(ys(1-t)+zst)}\right] ds\, dt \leq \\
(9)\quad &\leq F_D \leq \frac{1-a}{1+a} + \frac{a}{\bar{a}} \Gamma\left[\begin{matrix} c \\ b_2, b_3, c-b_2-b_3 \end{matrix}\right] \int_0^1 \int_0^1 s^{b_2+b_3-1} \cdot \\
&\cdot (1-s)^{c-b_2-b_3-1} t^{b_3-1} (1-t)^{b_2-1} \Big[1 + \bar{a}(ys(1-t)+zst)\Big]^{-1} \cdot \\
&\cdot {}_2F_1\left[\begin{matrix} 1, b_1 ; \\ c-b_2-b_3 ; \end{matrix} - \frac{\bar{a}x(1-s)}{(1+\bar{a}(ys(1-t)+zst)}\right] ds\, dt\,.
\end{aligned}$$

Appropriate applications of 1,4 and 5 (with $b = 1$) in 9 and necessary simplication will lead to the following theorem:

THEOREM 1. *If* $c > b_1 + b_2 + b_3 > 0$, $0 \leq a \leq 1$, $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ *then for* $z \geq y \geq x > 0$,

$$\begin{aligned}
&\Big[1 + \frac{a}{c}(b_1 x + b_2 x + b_3 x)\Big]^{-1} < F_D < \frac{1-a}{1+a} + \\
(10)\quad &+ c_0\Big[1 + c_1\Big\{\frac{b_2}{c_2} + \frac{b_3}{c_3}(1+b_2+b_3)\Big\}\Big] + \\
&+ c_4\Big[\frac{1+b_3}{c_5} + \frac{(1+c)^2 b_2}{c_8} + \frac{b_3(1+c)^2}{c_9}\Big(1+b_2+b_3\Big)\Big],
\end{aligned}$$

where:

$$\begin{aligned}
c_0 &= \frac{a(c-b_1-b_2-b_3)}{\bar{a}c(1+b_1)(1+b_2+b_3)}, \\
c_1 &= \frac{(1+c)}{(1+b_3)(c-b_2-b_3)}, \\
c_2 &= 1 + \frac{\bar{a}(1+b_2+b_3)y}{(1+c)}, \\
c_3 &= 1 + \frac{\bar{a}}{(1+c)}(b_2 y + (1+b_3)z),
\end{aligned}$$

$$\begin{aligned}
c_4 =&\frac{ab_1(1+c-b_2-b_3)^2}{\bar{a}c(1+b_2)(1+b_3)(1+b_2+b_3)(1-b_2-b_3)}\,,\\
c_5 =&(1+c-b_2-b_3)+\bar{a}(1+b_1)x\,,\\
c_6 =&(1+c)(1+c-b_2-b_3)\,,\\
c_7 =&(c-b_2-b_3)\bar{a}(1+b_1)x\,,\\
c_8 =&c_6+c_7+(1+b_2+b_3)(1+c-b_2-b_3)\bar{a}y\,,\\
c_9 =&c_6+c_7+(1+c-b_2-b_3)\bar{a}(b_2y+(1+b_3)z)\,.
\end{aligned}$$

The function F_D is symmetric with respect to the parameters b_1, b_2, b_3 and variables x, y, z i.e. a simultaneous interchange of b_1, b_2, b_3 and x, y, z, leaves the function unaffected, whereas this symmetry is noticeable in the left side of the inequlity, the right side lacks this feature which is virtually obvious. This observation enables us to construct inequalities for different choices of ordering between x, y, z. For example, to obtain inequality for $z \le y \le x$ one should replace b_1 and x by b_3 and z and vice-versa in the right side of (10).

In the next place, starting with the integral representation

(11)

$$F_D=\Gamma\begin{bmatrix} c \\ a, c-a\end{bmatrix}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1+ux)^{-b_1}(1+uy)^{-b_2}(1+uz)^{-b_3}du\,,$$

$$R(c) > R(a) > 0\,, |\arg(1+x)| < \pi, |\arg(1+y)| < \pi, |\arg(1+z)| < \pi\,,$$

and noting that for $\alpha < \beta < \gamma$:

$$\frac{1}{(1+\alpha u)(1+\beta u)(1+\gamma u)} = \frac{A_1}{(1+\alpha u)} + \frac{A_2}{(1+\beta u)} + \frac{A_3}{(1+\gamma u)}\,,$$

where:

$$(12)\qquad \begin{aligned}
A_1 =&\frac{\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}\,, \quad A_2 = \frac{\beta^2}{(\gamma-\beta)(\beta-\alpha)}\\
A_3 =&\frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\,,
\end{aligned}$$

we have, in the manner of theorem (1) the following theorem:

THEOREM 2. *If* $c > a > 0$, $0 \le b_1 \le 1$, $0 \le b_2 \le 1$, $0 \le b_3 \le 1$, $0 < (1+b_1)x < (1+b_2)y < (1+b_3)z$, $0 < b_1x < b_2y < b_3z$, *then:*

$$\begin{aligned}&\sum_{i=1}^{3} d_{i+4}e_i < F_D < \frac{d_0}{d_1} + \frac{1}{d_1}\Big[d_{i+1}(f_8 + f_9e_{i+3}) - f_1y_0e_8 + \\ (13)\qquad &+z_0(f_8+f_9e_6)(f_1+f_2) - (f_2+f_3)x_0e_7 + f_3y_0(f_8+f_9e_5) - \\ &-f_7e_8 + \sum_{i=1}^{2} f_{4+i}(f_8+f_9e_{2i+2})\Big],\end{aligned}$$

where:

$$a_1 = b_2y - b_1x, \quad a_2 = b_3z - b_1x, \quad a_3 = b_3z - b_2y,$$

$$d_0 = \prod_{i=1}^{3}(1-b_i), \quad d_1 = \prod_{i=1}^{3}(a+b_i), \quad d_2 = 2b_1\prod_{i=1}^{2}(1-b_{i+1}),$$

$$d_3 = 2b_2(1-b_1)(1-b_3), \quad d_4 = 2b_3\prod_{i=1}^{2}(1-b_i), \quad d_5 = \frac{(b_1x)^2}{a_1a_2},$$

$$d_6 = \frac{(b_2y)^2}{a_1a_3}, \quad d_7 = \frac{(b_3z)^2}{a_2a_3},$$

$$e_0 = \left(1 + \frac{(1+a)}{(1+c)}b_2y\right)^{-1}, \quad e_1 = \left(1 + \frac{ab_1x}{c}\right)^{-1},$$

$$e_2 = -(f_8 + f_9e_0), \quad e_3 = \left(1 + \frac{ab_3z}{c}\right)^{-1}, \quad e_4 = \left(1 + \frac{\bar{a}x_0}{(1+c)}\right)^{-1},$$

$$e_5 = \left(1 + \frac{\bar{a}y_0}{(1+c)}\right)^{-1}, \quad e_6 = \left(1 + \frac{\bar{a}z_0}{(1+c)}\right)^{-1}, \quad e_7 = \left(1 + \frac{ax_0}{2c}\right)^{-1},$$

$$e_8 = \left(1 + \frac{ay_0}{2c}\right)^{-1},$$

$$f_1 = \frac{4b_2b_3(1-b_1)}{g_2}, \quad f_2 = \frac{4b_1b_3(1-b_2)}{g_3}, \quad f_3 = \frac{4b_1b_2(1-b_3)}{g_1},$$

$$f_4 = 8b_1b_2b_3, \quad f_5 = \frac{f_4(x_0)^2}{g_1g_3}, \quad f_6 = \frac{f_4(z_0)^2}{g_2g_3}, \quad f_7 = \frac{f_4(y_0)^2}{g_1g_2},$$

$$f_8 = \frac{(c-a)}{c(1+a)}, \quad f_9 = \frac{a(1+c)}{c(1+a)},$$

$$g_1 = y_0 - x_0, \quad g_2 = z_0 - y_0, \quad g_3 = z_0 - x_0,$$

$$x_0 = (1+b_1)x, \quad y_0 = (1+b_2)y, \quad z_0 = (1+b_3)z.$$

A closer examination of conditions of existence of theorems 1 and 2 reveals that both the theorems hold provided that $c > b_1 + b_2 + b_3 > 0$,

$$\begin{aligned}&c > a > 0,\ 0 \le a \le 1,\ 0 \le b_1 \le 1,\ 0 \le b_2 \le 1,\ 0 \le b_3 \le 1,\\ &b_3 z > b_2 y > b_1 x > 0,\ (1+b_3)z > (1+b_2)y > (1+b_1)x > 0,\\ &z \ge y \ge x \ge 0\,.\end{aligned} \tag{14}$$

Under the set of conditions (14), if we choose $a = .5$, $b_1 = .1$, $b_2 = .2$, $b_3 = .3$, $c = .7$, $x = .2$, $y = .6$, $z = .8$, then from (10):

$$.7865168 < F_D < .8067879\,, \tag{15}$$

and from (13):

$$.773462 < F_D < .8101716\,. \tag{16}$$

In support of the discussion about symmetry aspect pertainig to (10) and (13), we may consider the set of values $a = .3$, $b_1 = .4$, $b_2 = .5$, $b_3 = .7$, $c = 2$, $x = .8$ $y = .4$, $z = .1$, which gives from (10):

$$.9186955 < F_D < .9226166\,, \tag{17}$$

and from (13):

$$.9128956 < F_D < .9313927\,. \tag{18}$$

It may be pointed out that the conditions for theorem 2 of [5] should be read as:

$$0 \le \beta \le 1\,,\ 0 \le \beta' \le 1\,, \gamma \ge \alpha > 0\,,\ \beta' > \beta > 0\,,\ y \ge x > 0\,. \tag{19}$$

Furthermore, it may also be pointed out that the inequalities of theorem 2 of [5] will still hold for the set of conditions:

$$\begin{aligned}&0 \le \beta \le 1\,,\ 0 \le \beta' \le 1\,, \gamma \ge \alpha > 0\,,\\ &(1+\beta')y > (1+\beta)x > 0\,,\ \beta' y > \beta x > 0\,.\end{aligned} \tag{20}$$

These conditions are obviously weaker than (19) since there exists set of values $\beta = .1$, $\beta' = .9$, $x = .8$, $y = .7$, which satisfies (20) byt not (19).

3 – Inequalities for F_A

We have the integral representation

(21) $$F_A \equiv F_A(a, b_1, b_2, b_3; c_1, c_2, c_3; -x, -y, -z) =$$
$$= \Gamma\begin{bmatrix} c_1, & c_2, & c_3 \\ b_1, & b_2, & b_3, & c_1-b_1, & c_2-b_2, c_3-b_3 \end{bmatrix} \int_0^1\int_0^1\int_0^1 u^{b_1-1}(1-u)^{c_1-b_1-1}\cdot$$
$$\cdot v^{b_2-1}\cdot(1-v)^{c_2-b_2-1}w^{b_3-1}(1-w)^{c_3-b_3-1}(1+ux+vy+wz)^{-a}dudvdw\,,$$
$$R(c_1) > R(b_1) > 0\,, R(c_2) > R(b_2) > 0\,, R(c_3) > R(b_3) > 0\,,$$
$$|\arg(1+x+y+z)| < \pi\,,$$

where

$$F_A = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_{p(-x)^m(-y)^n(-z)^p}}{(c_1)_m(c_2)_n(c_3)_p\; m!n!p!}\,,$$

$|x|+|y|+|z|<1$, c_1, c_2, c_3 are neither negative integers nor zeros.

Considering $c_1 > b_1 > 0$, $c_2 > b_2 > 0$, $c_3 > b_3 > 0$, $0 \le a \le 1$, $x > 0$, $y > 0$, $z > 0$, application of (2) to $(1+ux+vy+wz)^{-a}$ in (21) with the help of (1) and (3) yields

(23)
$$=\Gamma\begin{bmatrix} c_2, & c_3 \\ b_2, & b_3, & c_2-b_2, & c_3-b_3 \end{bmatrix} \int_0^1\int_0^1 v^{b_2-1}(1-v)^{c_2-b_2-1}w^{b_3-1}(1-w)^{c_3-b_3-1}\cdot$$
$$\cdot(1+avy+awz)^{-1}{}_2F_1\left(\begin{matrix}1, b_1\,; \\ c_1;\end{matrix}\middle| -\frac{ax}{(1+auy+awz)}\right)dvdw <$$
$$< F_A < \frac{1-a}{1+a}\Gamma\begin{bmatrix} c_2, & c_3 \\ b_2, & b_3, & c_2-b_2, & c_3-b_3 \end{bmatrix} \int_0^1\int_0^1 v^{b_2-1}(1-v)^{c_2-b_2-1}\cdot$$
$$\cdot w^{b_3-1}(1-w)^{c_3-b_3-1}(1+\bar{a}(vy+wz))^{-1}$$
$${}_2F_1\left(\begin{matrix}1, b_1; \\ c_1;\end{matrix}\middle| -\frac{\bar{a}}{(1+\bar{a}(vy+wz))}\right)dvdw\,.$$

Now, appropriate application of (1), (4) and (5) in (23) leads to the following theorem:

THEOREM 3. *Let* $c_1 > b_1 > 0$, $c_2 > b_2 > 0$, $c_3 > b_3 > 0$, $0 \le a \le 1$, $x, y, z > 0$, *then:*

$$\begin{aligned}(24)\qquad &\Big(1 + a\sum_{j=1}^{3}\frac{b_jx_j}{h_j}\Big)^{-1} < F_A < \frac{1-a}{1+a} + \frac{a}{\bar{a}}\prod_{i=1}^{3}\Big(\frac{k_i}{h_i l_i}\Big)\cdot \\ &\cdot\Big[1 + \sum_{j=1}^{3}\frac{b_jh_j}{k_j}\Big(1 + \frac{\bar{a}l_jx_j}{h_j}\Big)^{-1}\Big] + \sum_{j=1}^{3}\frac{b_jb_{j+1}h_jh_{j+1}}{k_jk_{j+1}}\cdot \\ &\cdot\Big[1 + \bar{a}\Big(\frac{l_jx_j}{h_j} + \frac{l_{j+1}x_{j+1}}{h_{j+1}}\Big)\Big]^{-1} + \prod_{i=1}^{3}\frac{b_ih_i}{k_i}\Big[1 + \bar{a}\Big(\sum_{j=1}^{3}\frac{l_jx_j}{h_j}\Big)^{-1}\Big],\end{aligned}$$

where:

$$\begin{aligned}&x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = x_1,\ b_4 = b_1,\ h_4 = h_1,\ k_4 = k_1,\ l_4 = l_1,\\ &h_1 = 1 + c_1,\ h_2 = 1 + c_2,\ h_3 = 1 + c_3,\ k_1 = c_1 - b_1,\ k_2 = c_2 - b_2,\\ &k_3 = c_3 - b_3,\ l_1 = 1 + b_1,\ l_2 = 1 + b_2,\ l_3 = 1 + b_3\,.\end{aligned}$$

The above inequality is symmetric in the sense of F_A. In particular for

$$\begin{aligned}&a = .2,\ b_1 = .1,\ b_2 = .3,\ b_3 = .6,\ c_1 = .4,\ c_2 = .5,\ c_3 = .8,\ x = .1,\\ &y = .2,\ z = .4\,.\end{aligned}$$

Theorem 3 gives

$$(25)\qquad .9182736 < F_A < .9338971\,.$$

4 – Inequalities for F_B

The function F_B defined by

$$\begin{aligned}(26)\qquad F_B &\equiv F_B(a_1, a_2, a_3, b_1, b_2, b_3; c; -x, -y, -z)\\ &= \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_m(b_2)_n(b_3)_p(-x)^m(-y)^n(z)^p}{(c)_{m+n+p}\, m!n!p!}\end{aligned}$$

$|x| < 1$, $|y| < 1$, $|z| < 1$, is not negative integer or zero, has the integral representation

(27)

$$F_B = \Gamma\left[\begin{matrix} c \\ b_1, b_2, b_3, c - b_1 - b_2 - b_3 \end{matrix}\right] \int_0^1 \int_0^1 \int_0^1 u^{c-b_1-b_3-1}(1-u)^{b_1-1} v^{b_2-1} \cdot$$

$$\cdot (1-v)^{c-b_1-b_2-b_3-1} . w^{c-b_3-1}(1-w)^{b_3-1}(1+(1-u)wx)^{-a_1}(1+uvwy)^{-a_2} \cdot$$

$$\cdot (1+(1-w)z)^{-a_3} du dv dw ,$$

$$R(c) > R(b_1+b_2+b_3) > 0,\ R(b_1) > 0,\ R(b_2) > 0,\ R(b_3) > 0 ,$$

$$|\arg(1+x)| < \pi,\ |\arg(1+y)| < \pi,\ |\arg(1+z)| < \pi .$$

Although the presence of three binomial factors in the triple integral representation makes the result quite cumbersome, yet it is worthwhile to investigate the inequalities in veiw of its applicability.

Formula (27) can be written as

$$F_B = \Gamma\left[\begin{matrix} c \\ b_3, c - b_3 \end{matrix}\right] \int_0^1 w^{c-b_3-1}(1-w)^{b_3-1}(1+(1-w)z)^{-a_3} . \tag{28}$$

$$\cdot F_3(a_1, a_2, b_1, b_2; c - b_3; -wx, -wy) dw ,$$

where F_3 is defined by ([2]; p. 224 (8)):

$$F_3 \equiv F_3(a_1, a_2; b_1, b_2; c, -x, -y) \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n (-x)^m (-y)^n}{(c)_{m+n}\, m! n!} ,$$

$|x| < 1$, $|y| < 1$, c is not a negative integer or zero. The two-sided inequality for F_B would evidently follows using the two-sided inequality for F_3. Here we shall use theorem 6[5] of Luke. But we observe that the left side of theorem 6[5] is incorrect whereas there is a typographical error in right side of theorem 6[5].

Rectifying the errors, the corrected version of theorem 6[5], reads as follows:

$$\left(1+\frac{\alpha\beta x}{\gamma}\right)^{-1}\left(1+\frac{\alpha'\beta' y}{\gamma}\right)^{-1} < F_3 < \frac{(1-\alpha)(1-w_4)}{(1+\alpha)}+$$
$$+\left(1-\frac{\alpha}{w_1}\right)w_4(w_5+w_6)+\frac{\alpha}{w_1}(1-w_4)(w_2+w_3)+$$
$$+\frac{\alpha w_4[w_1x(w_2+w_3)+w_7y(w_5+w_6)]}{w_1[w_7y+w_1x(1+w_7y)]},$$

where:

$$w_1=\frac{1+\alpha}{2},\qquad w_2=\frac{\gamma-\beta}{\gamma(1+\beta)},\qquad w_3=\frac{\beta(1+\gamma)}{(1+\beta(1+\mu))},$$
$$w_4=\frac{(\gamma-\beta+1)2\alpha'\beta'}{(\gamma-\beta)(1+\beta')(1+\alpha')},\quad w_5=\frac{\beta}{\gamma(\gamma-\beta+1)},$$
$$w_6=\frac{(\gamma-\beta)(1+\gamma)}{\gamma(\gamma-\beta+1)(1+\mu)},\qquad w_7=\frac{(\alpha'+1)(\beta'+1)}{2(\gamma-\beta+1)},$$
$$\mu=\frac{(\alpha+1)(\beta+1)x}{2(\gamma+1)},\qquad \mu'=\frac{(\alpha'+1)(\beta'+1)y}{2(\gamma+1)},$$

$$0\le\alpha\le 1,\ 0\le\alpha'\le 1,\ \gamma>\beta>0,\ \beta'>0,\ x>0,\ y>0.$$

It may also be noted here that theorem 6[5] is not an independent result itself and infact can be deduced from theorem 4[5]. Indeed if one substitutes $\rho=\beta$ and $\rho'=\gamma-\beta$ in equation (17) [5], one would be led to equation (29) above. As a further defence to our argument if we can consider the set of value as in ([5], p. 48)

$$\alpha=1/4,\ \alpha'=1/2,\ \beta=1/3,\ \beta'=3/4,\ \gamma=2,\ x=1/2,\ y=2/3,$$

we have from (29) above

$$.8707483 < F_3 < .8861676, \tag{30}$$

whereas from theorem 6[5], we get

$$.8944794 < F_3 < .7700386. \tag{31}$$

Now appropriate application of (1), (2), (4), (5), and (29) in (28) yields the theorem:

THEOREM 4. *If* $0 \le a_1 \le 1$, $0 \le a_2 \le 1$, $0 \le a_3 \le 1$, $c > b_1 + b_2 + b_3 > 0$, $b_1 > 0$, $b_2 > 0$, $b_3 > 0$, $a_1 b_1 x > a_2 b_2 y > 0$, $(1+a_1)b_1 x > (1+a_2)(1+b_2)y > 0$. $x > 0, y > 0, z > 0$, *then:*

$$\begin{aligned}
&t_1\left[r_1^2 s_1\left(1+\frac{r_5}{c}\right)^{-1} + (r_8)^2 t_2\left(1+\frac{r_7}{c}\right)^{-1}\right]- \\
&-(r_6)^2 t_2 s_1\left(\beta_3 + \beta_4\left(1+\frac{r_6}{\beta_4 c}\right)^{-1}\right) < \\
&< F_B < \gamma_2(\gamma_3\beta_8 + \beta_7) + \gamma_1\gamma_2\gamma_3\alpha_2 s_2 + \alpha_1(1-\beta_1)\gamma_1\gamma_2 s_4 + \\
&+\gamma_4\left[\frac{\gamma_3\beta_8}{c(1+b_3)} + s_3\beta_7\right] + \gamma_1\gamma_3\gamma_4\alpha_2 t_3(r_9 s_3 + \alpha_8 y s_2) + \\
&+\gamma_4\alpha_1(1-\beta_1)\gamma_1 t_4(r_9 s_3 + \alpha_7 x s_4) + \gamma_2\gamma_5\beta_1 t_7 s_5 s_6 + \\
&+\frac{\gamma_4\beta_1 t_7 s_6 s_7 t_5}{(c-b_3)}(r_9 s_3 + r_4 s_7) + t_6\gamma_2 r_3(\alpha_7 x s_4 - r_4 t_1) + \\
&+t_8(1-a_3)r_1(r_4 s_4 - \alpha_8 y s_8) + t_6\gamma_4\left[r_9^2 t_3 t_4 s_3 + (\alpha_7 x)^2 t_4 s_4 r_2 - \right. \\
&\left. -(r_4)^2 r_2 t_5 s_7\right] + 2a_3 t_8\left(r_9^2 t_3 t_4 s_3 + (r_4)^2 t_5 r_1 s_5 - (\alpha_8 y)^2 t_3 r_1 s_8\right),
\end{aligned}$$

where:

$$\begin{aligned}
&\alpha_1 = \frac{2a_1 b_1}{(1+a_1)(1+b_1)},\ \alpha_2 = \frac{2a_2 b_2}{(1+a_2)(1+b_2)},\ \alpha_3 = \frac{2a_3 b_3}{(1+a_3)(1+b_3)}, \\
&\alpha_4 = \frac{(1+a_1)(1+b_1)}{2(1+c)},\ \alpha_5 = \frac{(1+a_2)(1+b_2)}{2(1+c)},\ \alpha_6 = \frac{(1+a_3)(1+b_3)}{2(1+c)}, \\
&\alpha_7 = \frac{(1+a_1)(1+b_1)}{2(1+c-b_3)},\ \alpha_8 = \frac{(1+a_2)(1+b_2)}{2(1+c-b_3)},\ \alpha_9 = \frac{(1+a_2)(1+b_2)}{2(1+c-b_1-b_3)}, \\
&\beta_1 = \frac{\alpha_2(1+c-b_1-b_3)}{(c-b_1-b_3)},\ \beta_2 = \frac{1+a_1}{2},\ \beta_3 = \frac{b_3}{c(1+c-b_3)}, \\
&\beta_4 = \frac{(1+c)(c-b_3)}{c(1+c-b_3)},\ \beta_5 = \frac{(c-b_3)}{c(1+b_3)},\ \beta_6 = \frac{b_3(1+c)}{c(1+b_3)}, \\
&\beta_7 = \frac{a_1(1-\beta_1)(c-b_1-b_3)}{\beta_2(c-b_3)(1+b_1)},\ \beta_8 = 1-\beta_1+\frac{\beta_1 b_1}{(c-b_3)(1+c-b_1-b_3)},
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= \frac{1+c-b_3}{c-b_3}, \quad \gamma_2 = \frac{1-a_3}{1+a_3}, \quad \gamma_3 = \frac{1-a_1}{1+a_1}, \\
\gamma_4 &= \frac{2a_3}{1+a_3}, \quad \gamma_5 = \frac{a_1}{\beta_1}, \\
r_1 &= (r_4 - \alpha_8 y)^{-1}, \quad r_2 = (\alpha_7 x - r_4)^{-1}, \quad r_3 = (\alpha_7 x - \alpha_8 y)^{-1}, \\
r_4 &= \beta_2 \alpha_9 x y (\beta_2 x + \alpha_9 y)^{-1}, \quad r_5 = a_1 b_1 x, \quad r_6 = a_2 b_2 y, \\
r_7 &= a_3 b_3 z, \quad r_8 = a_3 (c - b_3) z, \quad r_9 = \frac{(1+a_3)z}{2}, \\
s_1 &= (r_5 - r_6)^{-1}, \quad s_2 = \beta_3 + \beta_4 (1 + \alpha_5 y)^{-1}, \\
s_3 &= \beta_5 + \beta_6 (1 + \alpha_6 z)^{-1}, \quad s_4 = \beta_3 + \beta_4 (1 + \alpha_4 x)^{-1}, \\
s_5 &= \beta_3 + \beta_4 \Big(1 + \frac{(1+c-b_3)}{(1+c)} r_4\Big)^{-1}, \\
s_6 &= (\beta_2 x + \alpha_9 y)^{-1}, \quad s_7 = \Big(1 + \frac{(c-b_3)}{c} r_4\Big)^{-1}, \\
s_8 &= \Big(1 + \frac{(c-b_3)}{c} \alpha_8 y\Big)^{-1}, \\
t_1 &= (r_8 + r_5(1 + a_3 z))^{-1}, \quad t_2 = (r_8 + r_6(1 + a_3 z))^{-1}, \\
t_3 &= \Big[(r_9 + \alpha_8 y (1 + r_9))\Big]^{-1}, \quad t_4 = \Big[(r_9 + \alpha_7 x (1 + r_9))\Big]^{-1}, \\
t_5 &= \Big[r_9 (1 + r_4) + r_4\Big]^{-1}, \quad t_6 = \beta_2 \gamma_1 \beta_1 \alpha_1 x s_6, \\
t_7 &= \frac{\beta_2 (c - b_1 - b_3) x}{(1+b_1)} + \frac{b_1 \alpha_9 y}{(1+c-b_1-b_3)}, \quad t_8 = \frac{a_1 (1+c-b_3) \alpha_2 \alpha_9 s_6 y}{\beta_2 (1+a_3) c - b_3)}.
\end{aligned}$$

A verification of the formula presented can be done by considering $a_1 = .8$, $a_2 = .3$, $a_3 = .9$, $b_1 = .5$, $b_2 = .1$, $b_3 = .4$, $c = 2$, $x = .7$, $y = .3$, $z = .6$, we have:

$$.7881504 < F_B < .809902.$$

The symmetry aspect can also be discussed in a manner to that of F_D. We, however omit the details for reason of brevity.

In conclusion it may be added that, if in place of (2), we use the inequalities (4.13, 4.16 of [4]) inequalities for F_A, F_B and F_D may be obtained with different ranges of parameters. In general, however, the domain of validity of the aforementioned inequalities can be extended by recourse of transformation theory ([15] p. 83).

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Lavoro pervenuto alla redazione il 18 settembre 1991
ed accettato per la pubblicazione il 3 dicembre 1992

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