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Equilateral sets and their central points

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RIASSUNTO: *In questa nota indichiamo alcune proprietà degli insiemi equilateri negli spazi normati, soprattutto riguardo ai loro punti "centrali" (centri, mediane, baricentri). In particolare, mostriamo che la situazione si presenta particolarmente semplice soltanto nel caso che la norma provenga da un prodotto scalare. Tramite esempi, in cui si considerano soprattutto triangoli equilateri, cerchiamo di descrivere in dettaglio le diverse situazioni che si possono presentare.*

ABSTRACT: *In this paper we indicate a few properties of equilateral sets in normed spaces, mainly with respect to some of their "central" points (centers, medians, barycenters). In particular, we show that we have an extremely simple situation only when the norm springs from an inner product; by means of examples, dealing mainly with equilateral triangles, we try to describe in details the different situations which may occur.*

KEY WORDS: *Equilateral sets – Minisum – Center.*

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1 – Introduction

We recall a few definitions. Let X be a normed space over the real field R. We say that X is *strictly convex* if $\|x+y\| = \|x\|+\|y\|$ $(x, y \in X)$ entails $x = ty$ for some $t \geq 0$. If A is a nonempty, bounded subset of X, then we set for $x \in X$:

$$r(A,x) = \sup\{\|a-x\|; a \in A\}, \tag{1.1}$$

and

$$r(A) = \inf\{r(A,x); x \in X\}. \tag{1.2}$$

A point $x \in X$ such that $r(A,x) = r(A)$ is called a *center* of A. Given $A \neq \emptyset$, we denote by $\mathrm{co}(A)$ its convex hull and by $\mathrm{aff}(A)$ the affine subspace spanned by the elements of A.

In the following we shall denote by F *a finite subset of* X, containing at least two points. We say that $F = \{x_1, \dots, x_n\}$ *is equilateral* if

$$\|x_i - x_j\| = \text{ constant for } i \neq j (1 \leq i,j \leq n). \tag{1.3.}$$

Given F, we set for $x \in X$

$$s(F,x) = \sum_{i=1}^{n} \|x_i - x\| \tag{1.4}$$

and

$$s(F) = \inf\{s(F,x); x \in X\}. \tag{1.5}$$

A point $x \in X$ such that $s(F,x) = s(F)$ is called a *median* of F.

The *barycenter* of F, $\sum_{i=1}^{n} \frac{x_i}{n}$, will be denoted by b.

If X is the euclidean plane and F is an equilateral set, then it is well known that the barycenter of F is at the same time the unique median of F. The same is true when X is any inner product space. In this paper we discuss the extent to which these and some other properties of equilateral sets can be generalized to more general spaces. Also, some natural, apparently simple questions are raised.

2 – Some general results

A simple application of the separation theorem implies the following result.

LEMMA 2.1.
Let X be an inner product space and let $F = \{x_1, \dots, x_n\} \subset X$.
Then x is a center of F if and only if x belongs to $\mathrm{co}(F_x)$, where $F_x = \{y \in F; \|x - y\| = r(F,x)\}$.

PROOF. If $x \notin \mathrm{co}(F_x)$ and M is hyperplane separating x from $\mathrm{co}(F_x)$, then $r(F, x)$ decreases when we move from x in a direction orthogonal to M, towards $\mathrm{co}(F_x)$.

REMARK. The above result is true in every two-dimensional strictly convex space, but not in general (see Lemma 4.6 for a similar result in a general normed space).

The following results are well known.

LEMMA 2.2. *Let X be a strictly convex space. Then for any finite set F there exist at most one center and - if F has at least three non collinear points - at most one median. If X is uniformly convex, then we have at most one center for any set A (the definition of uniform convexity is recalled in Section 4).*

LEMMA 2.3. *Let X be a reflexive space. Then for any finite set there exist at least one center and at least one median.*

Given a set A, we say (according to [2]) that a point $x \in X$ is a *minimal point* of A if no point $x' \neq x$ exists in X such that $\|x' - a\| \leq \|x - a\|$ for every $x \in A$. Denote by $\min(A)$ the set of minimal points of A. It is simple to see that, if X is strictly convex, then for every set A:

$$\overline{\mathrm{co}}(A) \subset \min(A)\,. \tag{2.1}$$

Concerning the reverse inclusion, we have the following result.

LEMMA 2.4. *(see e.g. [2]). If X is an inner product space, then equality holds in* (2.1) *for every subset A of X.*

Concerning medians and center, the following is true.

LEMMA 2.5. *(see [5] and [8]). If X is an inner product space, then the median of a set F always belongs to* $\mathrm{co}(F)$ *(thus also to* $\mathrm{aff}(A)$*); the same is true for the center of any bounded, nonempty set A.*

3 – Equilateral sets in inner product spaces

Now we shall prove some results in the special setting of inner product spaces.

THEOREM 3.1. *Let X be an inner product space with inner product $\langle .,. \rangle$ and let $A = \{a_1, \dots, a_n\}$ be an equilateral set. Then a point $c \in X$ is the center of A if and only if the following properties are satisfied:*

i) $c \in \mathrm{co}(A)$

ii) $\|c - a_i\|$ *is constant for* $i = 1, \dots, n$.

PROOF. "Only if" part. Let $A = \{a_1, \dots, a_n\}$ and let c be its center; it is not a restriction to assume that $c = 0$ (otherwise, we consider a translated of A). The necessity of i) is well known (see Lemma 2.5).

Set $r = \max_{1 \le i \le n} \|a_i\|$ and let $A_f = \{a \in A; \|a\| = r\}$; assume that $A_f \ne A$: for convenience, assume that $(2 \le p < n)$

$$\|a_1\| = \|a_2\| = \dots \|a_p\| = r > \max_{p+1 \le i \le n} \|a_i\| .$$

By Lemma 2.1 we have $c \in \mathrm{co}(A_f)$, thus $0 = \sum_{i=1}^{p} \lambda_i a_i$ with $0 \le \lambda_i \le 1$ for $1 \le i \le p$ and $\sum_{i=1}^{p} \lambda_i = 1$.

Therefore $0 = \| \sum_{i=1}^{p} \lambda_i a_i \|^2 = \sum_{i=1}^{p} \lambda_i^2 r^2 + \sum_{\substack{i=1 \\ i \ne j}}^{p} \sum_{j=1}^{p} \lambda_i \lambda_j \langle a_i, a_j \rangle$. This implies $\langle a_{i_0}, a_{j_0} \rangle < 0$ for at least one pair of different indices i_0, j_0 between 1 and p: assume again for simplicity of notations, $i_0 = 1$ and $j_0 = 2$, so

$$(*) \qquad \langle a_1, a_2 \rangle < 0 .$$

Let $p + 1 \le j \le n$ and $1 \le m \le p$; since A is equilateral, we have:

$$2r^2 - 2\langle a_1, a_2 \rangle = \|a_m - a_j\|^2 = r^2 - 2\langle a_m, a_j \rangle + \|a_j\|^2 <$$

$$2r^2 - 2\langle a_m, a_i \rangle, \text{ which implies (see (*)) } \langle a_m, a_j \rangle < \langle a_1, a_2 \rangle < 0 .$$

Since $\sum_{i=1}^{p} \lambda_i = 1$ we obtain, for any $j \ge p + 1$, $\sum_{i=1}^{p} \lambda_i \langle a_i, a_j \rangle < 0$: a contradiction, since $0 = \sum_{i=1}^{p} \lambda_i a_i$. Therefore $A_f = A$.

"If part". Let be $A = \{a_1, \dots, a_n\}$ an equilateral set and let $c \in \mathrm{co}(A)$; $\|c - a_i\| = \text{constant} = k$ for $1 \le i \le n$. Assume that $c' \ne c$ is the center of A; this implies $\|c' - a_i\| < \|c - a_i\| = k$ for $i = 1, \dots, n$. But this is impossible since $c \in \mathrm{co}(A)$ and $\mathrm{co}(A)$ is the set of minimal points of A (see Lemma 2.4). This concludes the proof.

We recall that if X is an inner product space, $A = \{a_1, \dots, a_n\} \subset X$ is equilateral and 0 is its center, then we have

$$\langle a_i, a_j\rangle = \langle a_1, a_2\rangle = \text{ constant for } 1 \le i \ne j \le n. \tag{3.1}$$

THEOREM 3.2. *Let be $A = \{a_1, \dots, a_n\}$ an equilateral set in an inner product space X; then the barycenter of A is, at the same time, its center and its median.*

PROOF. Let c be the center of A; for simplicity we assume $c = 0$, thus $0 = \sum_{i=1}^{n} \lambda_i a_i$ with $\sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} \lambda_i = 1$ and (according to Theorem 3.1) $\|a_i\| = r$ for $i = 1, \dots, n$. We obtain, for any i between 1 and n, $\lambda_i\, a_i = -\sum_{\substack{j=1\\ j\ne i}}^{n} \lambda_j a_j$; $\lambda_i r^2 = \lambda_i \|a_i\|^2 = -\langle \sum_{\substack{j=1\\ j\ne i}}^{n} \lambda_j a_j, a_i\rangle$. But then, by using (3.1), we have $\lambda_i r^2 = -\sum_{\substack{j=1\\ j\ne i}}^{n} \lambda_j \langle a_j, a_i\rangle = (\lambda_i - 1)\langle a_1, a_2\rangle$, thus $\lambda_i = \dfrac{\langle a_1, a_2\rangle}{\langle a_1, a_2\rangle - r^2} =$ constant ($i = 1, \dots, n$). Since $\sum_{i=1}^{n} \lambda_i = 1$ this implies $\dfrac{\langle a_1, a_2\rangle}{\langle a_1, a_2\rangle - r^2} = \frac{1}{n}$, so $c = b$. Now let s be the median of A, so $s = \sum_{i=1}^{n} \mu_i a_i$ with $\sum_{i=1}^{n} \mu_i = 1$. Now we consider the function $f(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \|\lambda_1 x_1 + \dots + \lambda_n x_n - x_i\|$: according to (3.1) and since A is equilateral, this is a symmetric function in its variables, so if it attains its minimum (for $\sum_{i+1}^{n} |\lambda_i| = \sum_{i=1}^{n} \lambda_i = 1$) when $\lambda_i = \mu_i$ $(1 \le i \le n)$, the same is true when we consider a permutation of $\mu_1, \dots, \mu_n$. The uniqueness of median then implies $\mu_1 = \dots = \mu_n = \frac{1}{n}$, so s is the barycenter of A. This completes the proof.

REMARK.3.3. In inner product spaces, for any equilateral set $A = \{a_1, \dots, a_n\}$ the set of all elements of X which have the same distance from every point of A is a linear variety L of codimension $n-1$. The intersection of L with aff(A) is the barycenter of A (as well as its center and median).

The following consequence of Theorem 3.2 could be proved also directly, for example by induction; note that this result does not hold in a general normed space, as we shall see later (see Examples 4.3, 4.9 and 4.10).

COROLLARY 3.4. *If F is an equilateral set in an inner product space, then its barycenter (so its median and center) is equidistant from all points of A.*

4 – Equilateral sets in normed spaces

Let $A = \{x_1, \dots, x_n\}$ be an equilateral set in A; if $\|x_i - x_j\| = d$ for $i \neq j$, then fixed $i_0 \in \{1, \dots, n\}$ we have: $\sum_{\substack{j=1 \\ j\neq i_0}}^{n} \|x_{i_0} - x_j\| = d(n-1)$; also $\|x_{i_0} - b\| = \left\| \sum_{i=1}^{n} \frac{x_{i_0} - x_i}{n} \right\| \leq \sum_{i=1}^{n} \left\| \frac{x_{i_0} - x_i}{n} \right\| = \frac{(n-1)d}{n}$; in both ways we obtain the estimate $\min_{x \in X} \sum_{i=1}^{n} \|x_i - x\| \leq d(n-1)$ (the last number is always the maximum of the convex function $\sum_{i=1}^{n} \|x_i - x\|$ over the convex hull of A). Next Lemma gives a lower estimate.

LEMMA 4.1. *If $A = \{x_1, \dots, x_n\}$ is an equilateral set, and $\|x_i - x_j\| = d$ for $i \neq j$, then we have $\sum_{i=1}^{n} \|x_i - m\| \geq \frac{nd}{2}$ for any $m \in X$.*

PROOF. The proof will be given by induction. For $n = 2$ the statement is true. Now assume that it is true for some $n \geq 2$ and let $A = \{x_1, \dots, x_n, x_{n+1}\}$ be equilateral with $\|x_i - x_j\| = d$ for $i \neq j$.

For $m \in X$ we have: $n \sum_{i=1}^{n+1} \|x_i - m\| = (n-1) \sum_{i=1}^{n} \|x_i - m\| + (n-1)\|x_{n+1} - m\| + \sum_{i=1}^{n+1} \|x_i - m\| \geq (n-1)\frac{nd}{2} + n\|x_{n+1} - m\| + \sum_{i=1}^{n} \|x_i - m\| =$

$(n^2-n)\frac{d}{2}+\sum_{i=1}^{n}\left(\|x_i-m\|+\|x_{n+1}-m\|\right)\geq(n^2-n)\frac{d}{2}+\sum_{i=1}^{n}\|x_i-x_{n+1}\|=$ $d(\frac{n^2-n}{2}+n)=d\frac{n^2+n}{2}$; thus $\sum_{i=1}^{n+1}\|x_i-m\|\geq\frac{(n+1)d}{2}$, and the inequality is proved.

Concerning the upper bound for $s(F)$ a slightly better estimate holds if X is a *uniformly convex* space. This means that for the function of $\epsilon\in[0,2]$:

$$\delta(\epsilon)=\inf\{1-\frac{\|x+y\|}{2};\ \|x\|=\|y\|=1,\ \|x-y\|\geq\epsilon\}$$

we have $\delta(\epsilon)>0$ for $\epsilon>0$. Under this assumption, if $\|x_1-y\|\leq d$, $\|x_2-y\|\leq d$ and $\|x_1-x_2\|\geq\epsilon$, then we have:

$$\left\|\frac{x_1+x_2}{2}-y\right\|\leq d-d\delta(\frac{\epsilon}{d}).$$

PROPOSITION 4.2. *Let $F=\{x_1,\dots,x_n\}$ be an equilateral set contained in a uniformly convex space X. Then*

$$s(F)\leq d[(n-1)-(n-2)\delta(1)]. \tag{4.1}$$

Better estimates are possible for $n\geq 4$ (see (4.2)).

PROOF. We have $s(F,x_i)=(n-1)d$ for $i=1,\dots,n$. Let $y_1=\frac{x_1+x_2}{2}$; since $\|x_1-x_2\|=d=a_1$; $\|x_1-x_i\|=\|x_2-x_i\|=d$ for $i=3,\dots,n$ then we have $\|y_1-x_i\|\leq d-d\delta(1)$ for $i=3,\dots,n$. This implies $s(F)\leq s(F,y_1)\leq d+(n-2)d(1-\delta(1))=(n-1)d-(n-2)d\delta(1)$, which is (4.1). Note that the same estimate holds for $y_2=\frac{x_2+x_3}{2},\dots,y_n=\frac{x_n+x_1}{2}$.

Going a step further in the previous reasoning, if $n\geq 4$ we may consider $z_1=\frac{y_1+y_2}{2}$. Set $d(1-\delta(1))=a_2$ and $F_3=\{x_1,x_2,x_3\}$; we have $s(F_3,z_1)\leq\max(s(F_3,y_1),s(F_3,y_2))\leq d+a_2$. For $i\geq 4$ we obtain: $\|y_1-x_i\|\leq a_2$, $\|y_2-x_i\|\leq a_2$, $\|y_1-y_2\|=\frac{1}{2}\|x_1-x_3\|=\frac{d}{2}$, so: $\|z_1-x_i\|\leq a_2(1-\delta(\frac{d}{2a_2}))=a_3$; proceeding in the same way, if $n\geq 5$, by using $u_1=\frac{z_1+z_2}{2}$ where $z_2=\frac{y_2+y_3}{2}$ and $F_4=\{x_1,x_2,x_3,x_4\}$ we obtain: $s(F_4,u_1)\leq\max(s(F_4,z_1),s(F_4,z_2))\leq d+a_2+a_3$; $\|u_1-x_i\|\leq a_3(1-\delta(\frac{d}{4a_3}))=a_4$

for $i \geq 5$. Finally, we obtain in this way: $s(F_4) \leq \sum_{i=1}^{n-1} a_i$ where $a_1 = d$ and, for $i \geq 2$,

$$a_{i+1} = a_i\Big(1 - \delta(\frac{d}{2^{i-1}a_i})\Big). \tag{4.2}$$

If we majorize $a_3, a_4, \ldots$ with the simpler terms: $a_3' = a_2(1 - \delta(\frac{1}{2}))$, $a_4' = a_3'(1 - \delta(\frac{1}{2^2})), \ldots$, we obtain:

$$\begin{aligned} s(F_n) \leq &d + d(1-\delta(1)) + d(1-\delta(1))(1-\delta(\frac{1}{2})) + \ldots + \\ &+ d(1-\delta(1))\ldots\Big(1-\delta\Big(\frac{d}{2^{n-3}}\Big)\Big). \end{aligned} \tag{4.2'}$$

REMARK. The above estimates are not sharp. In fact, for example if X is an inner product space then we have $\delta(\epsilon) = 1 - \sqrt{1-\epsilon^2/4}$; thus, if $d = 1$, then for $n = 3$ the right value of $s(F_3) = 1 + \frac{\sqrt{3}}{2}$; for $n = 4$ the right value of $s(F_4)$ is $\sqrt{6}$; our estimate (4.2) gives $s(F_4) \leq 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}(1 - \delta(\frac{1}{\sqrt{3}})) = 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{11}}{4} > \sqrt{6}$.

In general centers and medians do not coincide, also for equilateral sets, as the following examples show.

EXAMPLE 4.3. Let be X the space $\mathbb{R}^2$ endowed with the norm $\|(x_1, x_2)\| = \frac{1}{2}(|x_1| + |x_2|) + \frac{1}{2}\sqrt{x_1^2 + x_2^2}$. Consider an equilateral set $A = \{x, y, z\}$ where $x = (0, 1)$, $y = (a, -b)$, $z = (-a - b)$; $\|x\| = 1$; $a > 0$; $b \geq 0$; we must have $\|x - y\| = \|x - z\| = \|z - y\|$ and $\|x\| = \|y\| = \|z\| = 1$, so $\frac{1}{2}(1 + a + b + \sqrt{(1+b)^2 + a^2}) = 2a$; $\frac{1}{2}(a + b + \sqrt{a^2 + b^2}) = 1$. We obtain successively, from the first equation: $(3a - 1 - b)^2 = (1 + b)^2 + a^2; 8a^2 - 6a - 6ab = 0$; $b = \frac{4}{3}a - 1$. By the second equation we thus obtain: $2 - a - \frac{4}{3}a + 1 = \sqrt{a^2 + (\frac{4}{3}a - 1)^2}$; $(3 - \frac{7}{3}a)^2 = a^2 + \frac{16}{9}a^2 + 1 - \frac{8}{3}a$; $\frac{24}{9}a^2 - (14 - \frac{8}{3})a + 8 = 0$ and so: $a = \dfrac{51 \pm \sqrt{873}}{24} \cong 0,894$ and $b \cong 0,192$. It is not difficult to see that the unique center of A is $(0, 0)$, while the unique median of A is $(0, -b)$ and $s(A) = 1 + b + 2a \cong 2,98 < 3$. The barycenter is $(0, \frac{1-2b}{3})$. Both the median and the barycenter have different distances from the elements of A.

Now consider again $X = \mathbb{R}^2$, but with the norm: $\|(x_1, x_2)\| = \frac{1}{2}(|x_1| + |x_2|) + \frac{1}{2}\max(|x_1|, |x_2|)$. Take $x = (0, 1)$; $y = (\frac{6}{7}, \frac{-2}{7})$; $z =$

$(\frac{-6}{7}, \frac{-2}{7})$. This triplet forms an equilateral set, whose unique center is again $(0,0)$; all points $(0,m)$ with $\frac{-2}{7} \le m \le \frac{4}{7}$ are medians.

The example which follows shows how barycenters of equilateral sets — in general — are not centers nor medians.

EXAMPLE 4.4. Let $X = \mathbb{R}^3$ with the max norm. Let $A = \{x,y,z\}$ where $x = (1,1,1)$, $y = (0,0,0)$, $z = (0,1,0)$. The set A is equilateral and its barycenter is $b = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$. We have: $\|b-x\| = \|b-y\| = \frac{2}{3}$; $\|b-z\| = \frac{1}{3}$. The unique center and median of A is the point $c = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (we have $r(A,c) = \frac{1}{2} < r(A,b) = \frac{2}{3}$; $s(A,c) = \frac{3}{2} < s(A,b) = 2$).

If we slightly modify the norm in the above examples we can see that we have the same pathology also when the norm of X is uniformly convex. A more specific example of this situation, for barycenters, is given by next example.

EXAMPLE 4.5. Let $X = \mathbb{R}^3$ with the l^p norm. Consider the set $A = \{e_1, e_2, e_3\}$ where e_i is the i-th element of the natural basis. We want to minimize $s(A,x)$ with $x \in X$; for symmetry reasons, the unique median must be a point x with $x_1 = x_2 = x_3 = \alpha$, $0 \le \alpha \le 1$. Thus we must look for $\alpha \in [0,1]$ minimizing $s(A,x) = 3[(1-\alpha)^p + 2\alpha^p]^{\frac{1}{p}} = f(\alpha)$. Since $f'(\alpha) = 3[(1-\alpha)^p + 2\alpha^p]^{\frac{1}{p}-1}\,[2\alpha^{p-1} - (1-\alpha)^{p-1}]$ is 0 when $\alpha = \bar{\alpha} = \frac{1}{1+2^{1/(p-1)}}$, then $x = (\bar{\alpha}, \bar{\alpha}, \bar{\alpha})$ is the unique median of A. Note that x is also the barycenter $b = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of A only for $p = 2$ (note that b is equidistant from the points of A). Moreover x is also the unique center of A, according to what we shall prove later (Corollary 4.8).

We shall base the proof of next result on a general lemma we are proving: this lemma is an extension of Theorem 1 in [4]; in some sense it will replace, in general spaces, Lemma 2.1.

LEMMA 4.6. *Let $\emptyset \neq A$ be a closed and bounded set and let c be a center of A; set, for $\sigma > 0$, $A_\sigma = \{a \in A; \|c-a\| \ge r(A) - \sigma\}$. Then we have $r(A_\sigma) = r(A)$ for every $\sigma > 0$. So in particular, c is also a center for A_σ, $\sigma > 0$.*

PROOF. Assume the contrary; i.e., let there exist $\sigma > 0$ such that $r(A_\sigma) = r(A) - \epsilon$ ($\epsilon > 0$) (in fact, the inequality $r(A_\sigma) \le r(A)$ is always true). Take $b \in X$ such that: $r(A_\sigma, b) < r(A) - \frac{\epsilon}{2}$. Now we consider $z_\lambda = c + \lambda(b-c)$ $(0 < \lambda < 1)$. If $a \in A_\sigma$, then we have: $\|a - z_\lambda\| =$

$\|a-c-\lambda(b-c)\| \leq \lambda\|a-b\|+(1-\lambda)\|a-c\| \leq \lambda(r(A)-\frac{\epsilon}{2})+(1-\lambda)r(A) = r(A)-\frac{\lambda\epsilon}{2}$. If $a \in A \setminus A_\sigma$, then we have $\|a-z_\lambda\| \leq \|a-c\| + \lambda\|b-c\| < r(A)-\sigma+\lambda\|b-c\|$. This shows that if we take λ small enough ($\lambda > 0$), then we obtain $r(A, z_\lambda) < r(A)$. This contradicion proves the lemma.

REMARK. The result given by the above lemma is true if we consider $r_A(A) = \inf_{x \in A} r(A, x)$ instead of $r(A)$ with A also convex; in fact, if we have $c \in A$ and $b \in A_\sigma$, then $z_\lambda \in A$ (which is convex).

THEOREM 4.7. *Let $A = \{x_1, x_2, x_3\}$ be an equilateral set. If c is a center for A, then $\|c-x_1\| = \|c-x_2\| = \|c-x_3\|$.*

PROOF. Let A' be the convex hull of A; set $r(A') = r(A) = r(A, c) = r$ and $B_r(c) = \{x \in X; \|c-x\| = r\}$. According to Lemma 4.6, $A \cap B_r(c)$ contains at least two points, say x_1, x_2. If $\|c-x_3\| < r$, then we must have: $r(\{x_1, x_2\}, c) = r$; but $r(\{x_1, x_2\}) = r$, so the segment joining them is a diameter of $B_r(c)$; i.e., $\|x_1-x_2\| = 2r$. But then $\|c-x_3\| < r$ would imply: $\|x_1-x_3\| \leq \|c-x_1\| + \|c-x_3\| < 2r$, which is contradiction. Therefore all three points have the same distance r from c.

As we shall see later (Example 4.10) the last result cannot be extended to equilateral sets containing more than three elements.

COROLLARY 4.8. *If a median m of a three - point equilateral set $A = \{x_1, x_2, x_3\}$ is equidistant from x_1, x_2, x_3, then it is also a center for A.*

PROOF. Under our assumptions, assume that $r(A, c) < \frac{s(A,m)}{3} = \|m-x_i\|$ for $i = 1, 2, 3$ and some $c \neq m$: then $s(A, c) < 3r(A, c) \leq s(A, m) = s(A)$. This contradiction proves the corollary.

REMARK. In general medians and barycenter are not equidistant from the points of an equilateral set (see Example 4.3); the center is equidistant from the three points of an equilateral set (Theorem 4.7), but it is not in general a median (see again Example 4.3). To have a more complete picture of all situations which may occur we give another example.

EXAMPLE 4.9. Let $X = \mathbb{R}^3$ with the max norm. Let $A = \{e_1, e_2, e_3\}$ where e_i is the i-th element of the natural basis. The barycenter b is equidistant from the three points of A and of course, belongs to co(A). The unique center and median of A is the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \neq b$. If we consider in the same space $A = \{e_1, e_2, 0\}$, then the set of centers and medians has diameter $1 =$ diameter (A).

The result given by Theorem 4.7 is not true for sets containing more than three points: see next example.

EXAMPLE 4.10 Consider the space $X = \{(x_1, x_2, x_3, x_4); \sum_{i=1}^{4} x_i = 0; x_i \in \mathbb{R}\}$, with the norm: $\|x\| = \sum_{i=1}^{4} |x_i|$. Let A be the three points set containing $P_1 = (1, 0, 0, -1)$; $P_2 = (0, 1, 0, -1)$; $P_3 = (0, 0, 1, -1)$. We have: $r(A) \geq \frac{4}{3}$. In fact, it is not difficult to see that in order to minimize $r(A, x)$, $x = (x_1, x_2, x_3, x_4)$ it is necessary to take $x_1 = x_2 = x_3 = \alpha \in [0, 1]$ (see Theorem 4.7), so $x_4 = -3\alpha$. The minimization of the following expression: $|\alpha - 1| + |2\alpha| + |3\alpha + 1|$ gives $\alpha = \frac{1}{3}$, so $r(A) = \frac{4}{3}$ and the unique center of A is $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Now let $A' = A \cup \{P_4\}$ where $P_4 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2})$; of course $r(A') \geq \frac{4}{3}$: since $r(A', x) = \frac{4}{3}$, then x is also the unique center of A', which is equilateral, but we have $\|x - P_i\| = \frac{4}{3}$ for $i = 1, 2, 3$ and $\|x - P_4\| = 1$. Also, x is the unique median of A': in fact, to have a median $y = (y_1, y_2, y_3, y_4)$ we must have (for symmetry reasons) $y_1 = y_2 = y_3 = a \in [0, 1]$; $y_4 = b = -3a \in [-\frac{3}{2}, -1]$. So $\sum_{i=1}^{4} \|y - P_i\| = 3(1 - a + 2a + |1 - b|) + 3|a - \frac{1}{2}| + b + \frac{3}{2} = 9a + 3|a - \frac{1}{2}| + \frac{3}{2}$, $a = -\frac{b}{3} \geq \frac{1}{3}$. This shows that the median is again $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$ and $S(A') = 5$. The barycenter is $b = (\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, -\frac{9}{8})$ and $\|b - P_i\| = \frac{3}{2}$ for $i = 1, 2, 3$; $\|b - P_4\| = \frac{3}{4}$.

5 – Centerable spaces and nonexistence of centers and medians

We say that a bounded, nonempty subset A of X is *centered* if $r(A) = \frac{1}{2}$ diameter (A) and there exists at least a center of A. A space X is said to be *centerable* if every bounded, nonempty subset A of X is centered (see e.g. [6]; [8]). For example, the spaces c_0, $C[0, 1]$, $L^\infty(X)$ are centerable. In particular, the space of Example 4.4 is centerable.

We indicate a simple result.

PROPOSITION 5.1. *For an equilateral set* $F = \{x_1, \dots, n\}$, $\|x_i - x_j\| = d$ *for* $i \neq j$, *the conditions* $r(F) = \frac{1}{2}d$ *and* $s(F) = \frac{nd}{2}$ *are equivalent. In these cases, if* $n \geq 2$ *then centers and medians coincide (but in general they are not unique).*

PROOF. For F as above, let $r(F) = \frac{1}{2}d$; then, for any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that $r(F, x_\epsilon) < \frac{1}{2}d + \epsilon$, thus $s(F, x_\epsilon) \leq n(\frac{1}{2}d + \epsilon)$; since ϵ is arbitrary, then we have $s(F) \leq \frac{nd}{2}$. According to Lemma 4.1, this implies $s(F) = \frac{nd}{2}$. Now we assume $s(F) = \frac{nd}{2}$; given $\epsilon > 0$ there exists $m_\epsilon \in X$ such that: $s(F, m_\epsilon) < \frac{n}{2}d + \epsilon$. We want to show, by contradiction, that $r(F, m_\epsilon) \leq \frac{1}{2}d + \epsilon$. Let $\|x_i - m_\epsilon\| > \frac{1}{2}d + \epsilon$ for some i between 1 and n: assume, for simplicity, that $i = 1$. If n is odd, then we have: $s(F, m_\epsilon) = \|x_1 - m_\epsilon\| + \sum_{j=2}^{n} \|x_j - m_\epsilon\| > \frac{1}{2}d + \epsilon + \|x_2 - x_3\| + \dots + \|x_{n-1} - x_n\| \geq \frac{1}{2}d + \epsilon + \frac{n-1}{2}d = \frac{n}{d} + \epsilon$, which is a contradiction, proving that $r(F) \leq r(F, m_\epsilon) \leq \frac{1}{2}d + \epsilon$. Since ϵ is arbitrary, this proves the second part of Proposition 5.1, for n odd. Now let n be even. For $n = 2$ there is nothing to prove since $r(F) = \frac{1}{2}d$, so we assume $n \geq 4$. From $s(F, m_\epsilon) < \frac{n}{2}d + \epsilon$ we obtain: $\frac{n}{2}d + \epsilon \geq \|x_1 - m_\epsilon\| + \|x_2 - m_\epsilon\| + \sum_{j=3}^{n} \|x_j - m_\epsilon\| \geq \|x_1 - x_2\| + \frac{n-2}{2}d = \frac{n}{2}d$. This implies $\|x_1 - m_\epsilon\| + \|x_2 - m_\epsilon\| \leq d + \epsilon$. Now assume that $\|x_1 - m_\epsilon\| > \frac{1}{2}d + \epsilon$, thus $\|x_2 - m_\epsilon\| = c < \frac{1}{2}d$; we obtain $\|x_j - m_\epsilon\| \geq d - c$ for $j = 3, \dots, n$. Thus $\frac{n}{2}d + \epsilon \geq \frac{1}{2}d + \epsilon + c + (n-2)(d-c)$, so $0 \geq \frac{1}{2}d(1 - n + 2(n-2)) + c(1 - n + 2)$, then $\frac{1}{2}d(n-3) \leq c(n-3)$, against $c < \frac{1}{2}d$: this contradiction proves that $\|x_1 - m_\epsilon\| \leq \frac{1}{2}d + \epsilon$. A similar reasoning concerning x_j, $j \geq 2$, shows that $r(F, m_\epsilon) \leq \frac{d}{2} + \epsilon$, thus the conclusion from the arbitrariness of ϵ. In case we set $\epsilon = 0$ in the above proofs, we obtain the last part of the thesis. Note that for $n = 2$, all points of the segment joining two points x and y are medians, while only the middle point is a center.

The above proposition shows that for equilateral sets such that $r(F) = \frac{d}{2}$, existence of centers is equivalent to existence of medians. In particular, in this case (according to Theorem 4.7), a median of a three-point set F is equidistant from the points of F.

Now we want to discuss the existence of centers and medians for equilateral sets. For an example of a three-point, equilateral set without a center, see [7]. A slightly different example is given below.

EXAMPLE 5.2 Consider c_0 (the usual space of sequences of real numbers $(x_1, x_2, \dots, x_n, \dots)$ with the sup norm). Set $X = \{x \in c_0; \sum_{n=1}^{\infty} f_n x_n = 0\}$ where $f_n = 1$ for $n = 1,2,3,4$ and $f_n = \frac{1}{2^{n-4}}$ for $n \geq 5$. Denote by e_i the elements of the natural basis of c_0. Let $A = \{a, b, c\}$ where $a = e_1 - e_4$; $b = e_2 - \frac{1}{2}e_4 - e_5$; $c = e_3 - \frac{1}{2}e_4 - e_5$. We have $\|a-b\| = \|a-c\| = \|c-b\| = 1$. Now set, for $\epsilon > 0$: $x^\epsilon = (\frac{1}{2}-\epsilon, \frac{1}{2}-\epsilon, \frac{1}{2}-\epsilon, -1-\epsilon, -\frac{1}{2}-\epsilon, \dots, -\frac{1}{2}-\epsilon, x_{n+5}, \dots)$ where $x_k = 0$ for $k \geq n+5$. If $\epsilon < \frac{1}{2}$, we have $\|x^\epsilon - a\| = \|x^\epsilon - b\| = \|x^\epsilon - c\| = \frac{1}{2}+\epsilon$; also, we shall have $f(x^\epsilon) = 0$ if we take $\epsilon = \frac{1}{2(5 \cdot 2^n - 1)}$. This show that, for n large, ϵ becomes as small as we wish: thus $s(A) \leq \frac{3}{2}$, and so we have $s(A) = \frac{3}{2}$ (see Lemma 4.1). Now, a median $m = (m_1, \dots, m_n, \dots)$ could exist only if m satisfies $\|m-a\| = \|m-b\| = \|m-c\| = \frac{1}{2}$. But this is impossible since we should have: $m_1 = m_2 = m_3 = \frac{1}{2}$, $-1 \leq m_4 \leq -\frac{1}{2}$; $m_5 = -\frac{1}{2}$; $-\frac{1}{2} \leq m_n \leq \frac{1}{2}$ for $n \geq 6$: but this would imply, since $\sum_{n=1}^{\infty} f_n m_n = 0$, $m_n = \frac{1}{2}$ for $n \geq 6$, so m cannot belong to c_0. Thus A has no median. According to Proposition 5.1, the set A has no center either.

6 – Points lying on a sphere

According to [3], we say that the points $x_1, \dots, x_n$ of X lie on a sphere if there exists $x \in X$ such that $\|x - x_i\| =$ constant. In [3] it is shown that there exist 'bad' normed spaces, as well as some 'nice', but incomplete normed spaces, such that n arbitrary given points ($n \in N$) always lie on a shere (of course, three collinear points in a strictly convex normed space cannnot lie on a sphere). In fact, the general "positive" result given in [3], Proposition 2, relies upon a characterization of spaces whose dual is strictly convex, given in [11]. In [10] it is shown that given an equilateral set $\{x_1, x_2, x_3\}$ in any space of dimension at least three, there always exists a point x_4 such that $\{x_1, x_2, x_3, x_4\}$ is equilateral; in particular x_1, x_2, x_3 lie on a sphere. However, the construction given to prove this fact cannot be carried over, in general, to create an equilateral set with 5 points, whichever the dimension of the space is. In fact, an example of a four point set which do not lie on a sphere is given below.

EXAMPLE 6.1 Consider l^1, the space of all summable real sequence, with the sum norm; let $X = \{x = (x_1, \dots, x_n, \dots) \in l^1; x_1 + x_2 +$

$x_3 = 0\}$. Consider the equilateral set $A = \{P_1, P_2, P_3, P_4\}$ where $P_1 = (0,0,0,0,\dots)$, $P_2 = (1,-1,0,0,\dots)$, $P_3 = (1,0,-1,0,\dots)$, $P_4 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, 0, \dots)$; we have $\|P_i - P_j\| = 2$ for $i \neq j$. We look for $P \in X$ such that $\|P - P_1\| = \|P - P_2\| = \|P - P_3\| = \|P - P_4\|$. If $P = (x_1, \dots, x_n, \dots)$, set $\sum_{n=4}^{\infty} |x_n| = \alpha$. We must have: $|x_1| + |x_2| + |x_3| + \alpha = |x_1 - 1| + |x_2 + 1| + |x_3| + \alpha = |x_1 - 1| + |x_2| + |x_3 + 1| + \alpha$ so $|x_1| - |x_1 - 1| = |x_2 + 1| - |x_2| = |x_3 + 1| - |x_3|$. Now, by examining the graph of the function $f(x) = |x| - |x - 1|$, we see that the following cases are possible: $x_1 \leq 0$, $x_2 + 1 \leq 0$, $x_3 + 1 \leq 0$: this is impossible since we must have: $x_1 + x_2 + x_3 = 0$; or: $x_1 \geq 1$, $x_2 + 1 \geq 0$, $x_3 + 1 \geq 0$, which is again impossible for the same reason. So we must have $0 \leq x_1 \leq 1$; $x_1 = x_2 + 1 = x_3 + 1$; therefore the condition: $x_1 + x_2 + x_3 = 0$ implies $x_1 = \frac{2}{3}$, $x_2 = x_3 = -\frac{1}{3}$, and then $\|P - P_1\| = \|P - P_2\| = \|P - P_3\| = \frac{4}{3} + \alpha$. But then the condition: $\|P - P_4\| = \frac{4}{3} + \alpha$ would imply: $|x_4 - \frac{2}{3}| + \sum_{n=5}^{\infty} |x_n| = \frac{4}{3} + \alpha = \frac{4}{3} + |x_4| + \sum_{n=5}^{\infty} |x_n|$, thus $|x_4 - \frac{2}{3}| = \frac{4}{3} + |x_4|$, which is impossible. So the four points P_1, P_2, P_3, P_4 do not lie on a sphere of X.

7 – Some open questions

Many characterizations of inner product spaces (i.p.s.), whose dimension is at least three, deal with properties concerning "central points" of sets. For example, the following characterizations are known.

PROPOSITION 7.1. *Let* $\dim(X) \geq 3$. *Then each of the following conditions is (necessary and) sufficient for* X *to be an inner product space.*

a) *for every set* F *with three elements* $\{x_1, x_2, x_3\}$ *and every triplet of real numbers* $\{\lambda_1, \lambda_2, \lambda_3\}$ *there exists a solution of* $\min_{x \in X} \sum_{n=1}^{3} \lambda_n \|x - x_n\|$ *in* $\mathrm{aff}(A)$.

b) *for every set* F, *there exists a median in* $\mathrm{co}(F)$.

c) *for every set* F, *there exists a median in* $\mathrm{aff}(F)$.

d) *for every set* F *with three elements* $\{x_1, x_2, x_3\}$ *and every triplet of real numbers* $\{\lambda_1, \lambda_2, \lambda_3\}$, $\bigcap_{i=1}^{3} B(x_i, \lambda_i) \neq \emptyset$ *implies* $\bigcap_{i=1}^{3} B(x_i, \lambda_i) \bigcap \{x_1, x_2, x_3\} \neq \emptyset$.

e) *for every set F containing three elements $\{x_1, x_2, x_3\}$ we have:* $r(F) = \inf\{r(F, y); y \in \text{aff}(F)\}$.

f) *for every finite set $F = \{x_1, \dots, x_n\}$, the expression* $\sum_{i=1}^{n} \|x - x_i\|^2$ *attains its minimum for x the barycenter of F.*

For the first three conditions, see [5]; for the remaining three, see [1]. Note that concerning f) the same proof given in [1] works by using the condition only for $n = 3$.

REMARK. Note that, according to b) and e) of Proposition 7.1, barycenters are also medians and centers of any finite set only in i.p.s. The following natural questions can arise:
Is condition f) sufficient for X to be an i.p.s. if we assume it to hold only for equilateral sets, or equilateral triplets?
Can we use as characterizations weakenings of conditions b) and c) which use only triplets, or equilateral sets, or equilateral triplets?
Can we weaken conditions a), c), d) by using only equilateral triplets? If not, what about using equilateral (finite) sets?
Similar questions can be considered concerning characterizations of centerable spaces (considered in section 5), or reflexive spaces: a characterization of reflexivity in terms of centers for triplets was given in [9]; probably some similar results concerning medians, or also equilateral triplets could be proved.

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