Rendiconti di Matematica, Serie VII
Volume 13 , Roma (1993), 149-166

Heat propagation in a thin rod

A. SILI

RIASSUNTO: *Si studia il problema al contorno per l'equazione del calore in un dominio cilindrico sottile di raggio ε e lunghezza l. Si dimostra, per il tramite di un opportuno sviluppo asintotico, che per ε tendente a zero la soluzione del problema tende, in una opportuna topologia, alla soluzione di un problema al contorno per un'equazione del calore unidimensionale.*

ABSTRACT: *We consider the heat equation in a thin cylindric rod of radius ε and lenght l. We show that when ε tends to 0, the corresponding solution u^ε tends in a certain sense to the solution of some one-dimensional heat equation involving a zero - order term*

KEY WORDS: *Heat equation – Thin domains.*

A.M.S. CLASSIFICATION: 35K05

1 – Introduction: statement of the problem and of the results

In this work we consider a cylindrical rod of radius ε and lenght ℓ, the extremities of which are maintained at fixed temperatures a_0 and a_ℓ.

The rod is plunged in an exterior bath maintained at fixed temperature I. Its initial temperature is denoted by $d(x)$ where x denotes any point of $\mathbb{R}^3$.

For $x = (x_1, x_2, x_3)$, we set $y = (x_1, x_2)$ and $z = x_3$, in such a way that x is written as $x = (y, z)$.

In what follows, we denote by ω a regular bounded open set of $\mathbb{R}^2$

and by ω^ε its ε-homothetic defined by

$$\omega^\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 \colon (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \in \omega\}.$$

We also define the cylinder $\Omega^\varepsilon = \omega^\varepsilon X(0, \ell)$.

If $\partial\omega^\varepsilon$ denotes the boundary of ω^ε and $\overline{\omega}^\varepsilon$ the closure of ω^ε, we decompose the boundary Γ^ε of Ω^ε as follows:

$$\Gamma_0^\varepsilon = \Big\{(y, z) \in \mathbb{R}^3 \colon y \in \overline{\omega}^\varepsilon, z = 0\Big\},$$
$$\Gamma_\ell^\varepsilon = \Big\{(y, z) \in \mathbb{R}^3 \colon y \in \overline{\omega}^\varepsilon, z = \ell\Big\},$$
$$\Gamma_N^\varepsilon = \Big\{(y, z) \in \mathbb{R}^3 \colon y \in \partial\omega^\varepsilon, 0 < z < \ell\Big\}.$$

We then have: $\Gamma^\varepsilon = \Gamma_N^\varepsilon \cup \Gamma_\ell^\varepsilon \cup \Gamma_0^\varepsilon$

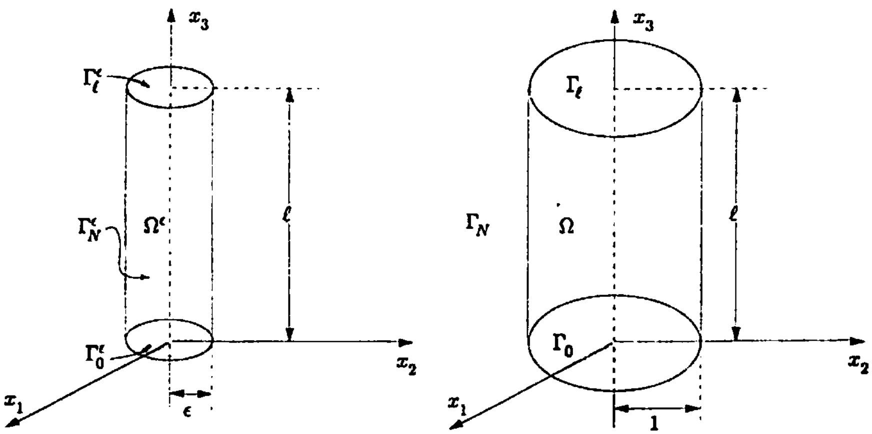

Fig. 1

We then set: $\Omega = \omega \times (0, \ell)$ (see fig. 1).
The boundary Γ of Ω is decomposed on:
$\Gamma = \Gamma_0 \cup \Gamma_\ell \cup \Gamma_N$ with:

$$\Gamma_0 = \Big\{(y, z) \in \mathbb{R}^3 \colon y \in \overline{\omega}, z = 0\Big\},$$
$$\Gamma_1 = \Big\{(y, z) \in \mathbb{R}^3 \colon y \in \overline{\omega}, z = \ell\Big\},$$
$$\Gamma_N = \Big\{(y, z) \in \mathbb{R}^3 \colon y \in \partial\omega, 0 < z < \ell\Big\}.$$

We also set: $\Gamma_D = \Gamma_0 \cup \Gamma_1$.

If $v^\varepsilon = v^\varepsilon(y,z,t)$ denotes the temperature in the rod, the propagation of the heat is described by

$$
(1.1)\qquad
\begin{cases}
\dfrac{\partial v^\varepsilon}{\partial t} - \Delta v^\varepsilon = f(x,t) \quad \text{in } \Omega^\varepsilon \times (0,T), T>0. \\
\dfrac{\partial v^\varepsilon}{\partial n} + k^\varepsilon(v^\varepsilon - I) = 0 \quad \text{on } \Gamma_N^\varepsilon \times (0,T) \\
v^\varepsilon = a_0 \quad \text{on } \Gamma_0^\varepsilon \times (0,T) \\
v^\varepsilon = a_\ell \quad \text{on } \Gamma_\ell^\varepsilon \times (0,T) \\
v^\varepsilon(x,0) = d(x) \quad \text{on } \Omega^\varepsilon \times \{0\}
\end{cases}
$$

where the physical meaning of the data is as follows:

k^ε is the thermic conductivity of the rod: we will always assume $k^\varepsilon > 0$. $f(x,t)$ describes the production of heat by sources distributed in the rod, physically $f \equiv 0$ a_0 and a_1 are two given constants.

In order to deal with a problem in a fixed domain with homogeneous boundary conditions we define:

$$
(1.2)\qquad u^\varepsilon(y,z,t) = v^\varepsilon(\varepsilon y, z, t) - r(z)
$$

where

$$
(1.3)\qquad r(z) = (a_1 - a_0)\frac{z}{\ell} + a_0 \qquad z \in (0,\ell)
$$

If we denote by Δ' and ∇' respectively Laplace's operator and the gradient with respect to the variables $y = (x_1, x_2)$ and since Γ depends only on z, the function u^ε is the solution of:

$$
(1.4)\qquad
\begin{cases}
\dfrac{\partial u^\varepsilon}{\partial t} - \dfrac{1}{\varepsilon^2}\Delta' u^\varepsilon - \dfrac{\partial^2 u^\varepsilon}{\partial z^2} = f(\varepsilon y, z, t) \text{ in } \Omega \times (0,T) \\
\dfrac{1}{\varepsilon}\dfrac{\partial u^\varepsilon}{\partial n} + k^\varepsilon(u^\varepsilon + r(z) - I) = 0 \text{ on } \Gamma_N \times (0,T) \\
u^\varepsilon(y,0,t) = 0 \text{ on } \Gamma_0 \times (0,T) \\
u^\varepsilon(y,\ell,t) = 0 \text{ on } \Gamma^\ell \times (0,T) \\
u^\varepsilon(x,0) = g^\varepsilon(x) \text{ on } \Omega \times \{0\}
\end{cases}
$$

where $g^\varepsilon(x) = d(\varepsilon y, z) - r(z)$.

We will study the behaviour of u^ε when ε goes to 0.

If the conductivity coefficient is small [in the sense there exists some constant C^* such that $0 \le C^* < +\infty$ and $\frac{k^\varepsilon}{\varepsilon} \longrightarrow C^*$], then u^ε tends in a convenient topology to the solution u of the problem:

(1.5)
$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial z^2} + \dfrac{|\partial\omega|}{|\omega|} C^* u = -\dfrac{|\partial\omega|}{|\omega|} C^*(r(z) - I) + \tilde{f}(z,t) \text{ in } (0,\ell)\times(0,T) \\ u(0,t) = 0 \text{ on } \{0\} \times (0,T) \\ u(\ell,t) = 0 \text{ on } \ell \times (0,T) \\ u(z,0) = \tilde{g}(z) \text{ on } \Omega \times \{0\} \end{cases}$$

where the functions $\tilde{f}$ and $\tilde{g}$ are defined by:

$$\tilde{f}(z,t) = f(0,z,t), \tilde{g}(z) = d(0,z) - r(z).$$

This result is obtained under the hypotheses:

$$(1.6) \quad \begin{cases} \text{i)} & (x_1,x_2) \longrightarrow f(x_1,x_2,z,t) \text{ is continuous} \\ & \text{for almost } (z,t) \in (0,\ell) \times (0,T) \\ \text{ii)} & (z,t) \longrightarrow f(x_1,x_2,z,t) \text{ is measurable for any} (x_1,x_2) \in \omega \\ \text{iii)} & |f(x_1,x_2,z,t)| \le F(z,t) \text{ for almost } x_1,x_2 \text{ and any } z,t, \\ & \text{with } F \in L^2((0,\ell) \times (0,T)). \end{cases}$$

$$(1.7) \quad \begin{cases} \text{i)} & (x_1,x_2) \longrightarrow d(x_1,x_2,z,t) \text{ is continuous} \\ & \text{for almost } z \in (0,\ell). \\ \text{ii)} & z \longrightarrow d(x_1,x_2,z) \text{ is measurable for any} (x_1,x_2) \in \omega \\ \text{iii)} & |d(x_1,x_2,z)| \le D(z) \text{ for almost } x_1,x_2 \text{ and any } z, \\ & \text{with } D \in L^2((0,\ell)). \end{cases}$$

Turning back to the temperature v^ε in the thin rod, this result means that in some sense (see section 4), v^ε tends to the solution v of the problem:

$$(1.8) \quad \begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial^2 v}{\partial z^2} + \dfrac{|\partial\omega|}{|\omega|} C^* v = \tilde{\tilde{f}}\,(z,t) \text{ in } (0,\ell) \times (0,T) \\ v(0,t) = a_0 \text{ on } \{0\} \times (0,T) \\ v(\ell,t) = a_\ell \text{ on } \ell \times (0,T) \\ v(z,0) = \tilde{\tilde{g}}(z) \text{ on } (0,\ell) \times \{0\} \end{cases}$$

where:

$$\tilde{\tilde{f}}\,(z,t) = \tilde{f}(z,t) + \frac{|\partial\omega|}{|\omega|}C^*I = f(0,z,t) + \frac{|\partial\omega|}{|\omega|}C^*I$$
$$\tilde{\tilde{g}}(z) = \tilde{g}(z) + r(z) = d(0,z)\,.$$

In the case where k^ε is large [that is if $\dfrac{k^\varepsilon}{\varepsilon} \longrightarrow +\infty$] we show that u^ε tends in a convenient topology to $u(z) = I - r(z)$.

Turning back to the solution v^ε of initial problem (1.1) we show that in some sense, (section 4) v^ε tends to the constant function I.

We obtain this last result assuming hypotheses (1.6) and (1.7) where f is moreover assumed to satisfy:

$$f \in L^\infty(\Omega \times (0,T))\,. \tag{1.9}$$

This hypothesis will be used in conjunction with the maximum principle in order to obtain an L^∞ a priori-estimate on u^ε.

Note that the limit equation (which corresponds to the situation where the rod is infinitely thin) is posed on the segment $(0,\ell)$, and that the limit u does not depend on x_1, x_2.

In addition note that in (1.5), the term $\dfrac{|\partial\omega|}{|\omega|}C^*u$ appears in the left hand side of the equation and the source term $-\dfrac{|\partial\omega|}{|\omega|}C^*(r(z)-I)$ appears in the right hand side.

This last term takes into consideration the temperature I of the bath. If k^ε is too small ($k^\varepsilon << \varepsilon$, i.e. $C^* = 0$) these effects are not seen at the limit.

When k^ε is large, ($k^\varepsilon/\varepsilon \longrightarrow +\infty$) the initial condition on u^ε is ignored and only the exterior bath is determinant since $u \equiv I - r(z)$ in this case. This result corresponds formally, to take $C^* = +\infty$ in equation (1.5).

The method employed to study this problem (passing to a fixed domain by an homothety in certain directions) has been widely used in recent years for studying various problems in elasticity; see eg. [1,3,4,5,6].

Note that the weak formulation of problem (1.4) is:

$$
(1.10)\qquad \begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u^\varepsilon(x,t)v(x)dx + \frac{1}{\varepsilon^2}\int_\Omega \nabla' u^\varepsilon.\nabla' v dx + \\ \qquad + \displaystyle\int_\Omega \frac{\partial u^\varepsilon}{\partial z}\cdot\frac{\partial v}{\partial z}dx + \frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}\gamma u^\varepsilon.\gamma v d\sigma = \\ = -\dfrac{k^\varepsilon}{\varepsilon}\displaystyle\int_{\Gamma_N}(r(z)-I)\gamma v d\sigma + \int_\Omega f(\varepsilon y, z, t)v dx . \quad \forall v \in V \\ u^\varepsilon(x,0) = g^\varepsilon(x) \end{cases}
$$

where $V = \{u \in H^1(\Omega), u/\Gamma_D = 0\}$ and γ denotes the trace application from $H^1(\Omega)$ to $L^2(\Gamma)$.

We deduce easily from the theorem of J.L. LIONS (see [2]; [7]) the following result:

LEMMA 1.1. *There exists a unique solution u^ε of (1.10) such that:*

$$
u^\varepsilon \in L^2((0,T);V) \cap C((0,T];L^2(\Omega)), \qquad \frac{\partial u^\varepsilon}{\partial t} \in L^2([0,T];V') .
$$

The paper is organized as follows:

In section 2 we study the case where $\dfrac{k^\varepsilon}{\varepsilon} \longrightarrow C^*$, $0 \le C^* < +\infty$.

The section 3 is devoted to study the case $\dfrac{k^\varepsilon}{\varepsilon} \longrightarrow +\infty$.

We show finally in section 4 that the mean value with respect to (x_1, x_2) of the solution v^ε of (1.1) converges to the solution v of problem (1.8), in the case $\frac{k^\varepsilon}{\varepsilon} \longrightarrow C^* < +\infty$ and converges to I in the case $\frac{k^\varepsilon}{\varepsilon} \longrightarrow +\infty$.

2 – The case $k^\varepsilon/\varepsilon \longrightarrow C^*; 0 \le C^* < +\infty$

Suppose that $\frac{k^\varepsilon}{\varepsilon}$ is bounded. Then there exists a constant C which depends on T and C^*, such that:

i) $\displaystyle\sup_{(0,T)}\int_\Omega |u^\varepsilon(x,t)|^2 dx \le C$

ii) $\displaystyle\int_0^T\int_\Omega |\frac{\partial u^\varepsilon}{\partial z}|^2 dxdt \leq C$

iii) $\displaystyle\frac{1}{\varepsilon^2}\int_0^T\int_\Omega |\nabla' u^\varepsilon|^2 dxdt \leq C$

REMARK 2.2. From these estimates one deduce:

$$\|u^\varepsilon\|_{L^2((0,T);H^1(\Omega))} \leq C\,.$$

PROOF. we take $v = u^\varepsilon(t)$ in (1.10). Integrating with respect to t, we obtain:

$$\frac{1}{2}\int_\Omega |u^\varepsilon(x,t)|dx + \frac{1}{\varepsilon^2}\int_0^t\int_\Omega |\nabla' u^\varepsilon(x,s)|^2 dxds+$$
$$+\int_0^t\int_\Omega \left|\frac{\partial u^\varepsilon}{\partial z}(x,s)\right|^2 dxds + \frac{k^\varepsilon}{\varepsilon}\int_0^t\int_{\Gamma_N} |\gamma u^\varepsilon(x,s)|^2 d\sigma ds =$$
$$= -\frac{k^\varepsilon}{\varepsilon}\int_0^t\int_{\Gamma_N} (r(z)-I)\gamma u^\varepsilon(x,s)d\sigma ds+$$
$$+\int_0^t\int_\Omega f(\varepsilon y,z,s)u^\varepsilon(x,s)dxds + 1/2\int_\Omega |g^\varepsilon(x)|^2 dx\,.$$

The second member is bounded by:

$$\frac{k^\varepsilon}{2\varepsilon}\int_0^T\int_{\Gamma_N} (r(z)-I)^2 d\sigma ds + \frac{k^\varepsilon}{2\varepsilon}\int_0^t\int_{\Gamma_N} |\gamma u^\varepsilon(x,s)|^2 d\sigma ds+$$
$$+1/2\int_0^T\int_\Omega f(\varepsilon y,z,s)|^2 dxds + 1/2\int_0^t\int_\Omega |u^\varepsilon(x,s)|^2 dxds+$$
$$+1/2\int_\Omega |g^\varepsilon(x)|^2 dx\,.$$

As f and d satisfy hypotheses (1.6) and (1.7), we have:

$$\int_\Omega |g^\varepsilon(x)|^2 dx \le 2\int_\Omega |d(\varepsilon y, z)|^2 dx + 2\int_\Omega |r(z)|^2 dx \le$$
$$\le 2\int_\Omega |D(z)|^2 dx + 2\int_\Omega |r(z)|^2 dx \le C$$

and:

$$\int_0^T\int_\Omega |f(\varepsilon y, z, s)|^2 dxds \le \int_0^T\int_\Omega |F(z,s)|^2 dxds \le C$$

for some constant C.

Since we assumed that $\frac{k^\varepsilon}{\varepsilon} \longrightarrow C^*$, that is $\frac{k^\varepsilon}{\varepsilon}$ is bounded, we conclude using Gronwall's lemma that:

$$\text{i)} \quad \sup_{(0,T)} \int_\Omega |u^\varepsilon(x,t)|^2 dx \le C$$

we then deduce:

$$\text{ii)} \quad \int_0^T\int_\Omega \left|\frac{\partial u^\varepsilon}{\partial z}(x,s)\right|^2 dxds \le C \cdot$$

and

$$\text{iii)} \quad 1/\varepsilon^2 \int_0^T\int_\Omega |\nabla' u^\varepsilon(x,s)|^2 dxds \le C$$

Here C denotes various positive constant which depend on T and C^*. □

Now, we establish the following result:

THEOREM 2.3: PASSING TO THE LIMIT. *Suppose that* $\dfrac{k^\varepsilon}{\varepsilon} \longrightarrow C^*$, $0 \le C^* < +\infty$

Then the solution u^ε *of* (1.10) *converges weakly in* $L^2((0,T);H^1(\Omega))$ *to the unique weak solution* u *of* (1.5); *furthermore,*

$$u \in C((0,T);L^2(0,\ell) \cap L^2((0,T);H_0^1(0,\ell)).$$

PROOF. Since u^ε is bounded in $L^2((0,T);H^1(\Omega))$ (Remark 2.2), there exists a subsequence u^{ε_k} wich weakly converges to some $\hat{u}$ in $L^2((0,T);H^1(\Omega))$.

Since we will see that $\hat{u}$ is unique, we will actually obtain that the whole sequence u^ε converges to $\hat{u}$ and we thus drop the subscript k.

We see by proposition 2.1, iii) that $\nabla' u^\varepsilon$ tends to 0 strongly in $L^2((0,T);(L^2(\Omega))^2)$, and then $\nabla'\hat{u} = 0$. Therefore $\hat{u}$ can be identified with a function u of $L^2((0,T);H^1(0,\ell))$ by:

$$\hat{u}(y,z,t) = u(z,t)\,.$$

On the other hand, $u^\varepsilon_{|\Gamma_D} = 0$ implies that $\hat{u}_{|\Gamma_D} = 0$.

Then $u \in L^2((0,T);H^1_0(0,\ell))$.

Consider now $v \in \mathcal{D}(0,\ell)$ and $\varphi \in \mathcal{D}(0,T)$. Take $v\varphi$ as a test function in equation (1.10) and integrate by parts with respect to t. We obtain:

$$-\int_0^T\int_\Omega u^\varepsilon(t).v\varphi' dxdt + \int_0^T\int_\Omega \frac{\partial u^\varepsilon}{\partial z}\cdot\frac{\partial v}{\partial z}\varphi dxdt +$$
$$+\frac{k^\varepsilon}{\varepsilon}\int_0^T\int_{\Gamma_N} \gamma u^\varepsilon.\gamma v\varphi d\sigma dt = -\frac{k^\varepsilon}{\varepsilon}\int_0^T\int_{\Gamma_N}(r(z)-I)\gamma v.\varphi dxdt +$$
$$+\int_0^T\int_\Omega f(\varepsilon y,z,t)v\varphi dxdt\,.$$

Using hypothesis (1.6), the weak convergence of u^ε to $\hat{u}$ in $L^2((0,T);H^1(\Omega))$, and $\frac{k^\varepsilon}{\varepsilon}\longrightarrow C^*$, we obtain after passing to the limit in the last equation:

$$-\int_0^T\int_\Omega \hat{u}v\varphi'(t)dxdt + \int_0^T\int_\Omega \frac{\partial \hat{u}}{\partial z}\cdot\frac{\partial v}{\partial z}\varphi(t)dxdt +$$
$$+C^*\int_0^T\int_{\Gamma_N}\gamma\hat{u}\gamma v\varphi d\sigma dt = -C^*\int_0^T\int_{\Gamma_N}(r(z)-I)\gamma v\varphi dzdt +$$
$$+\int_0^T\int_\Omega f(0,z,t)v\varphi dxdt\,.$$

Since $\hat{u}$ depends only on z and t, the last equation becomes:

$$-|\omega|\int_0^T\int_0^\ell uv\varphi'(t)dzdt+|\omega|\int_0^T\int_0^\ell \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}\varphi(t)dzdt+$$
$$+C^*|\partial\omega|\int_0^T\int_0^\ell uv\varphi dzdt=-C^*|\partial\omega|\int_0^T\int_0^\ell (r(z)-I)v\varphi dzdt+$$
$$+\int_0^T\int_0^\ell f(0,z,t)v\varphi dxdt\,.$$

We now remark that the tensorial product $\mathcal{D}(0,\ell)\otimes\mathcal{D}(0,T)$ is dense in $\mathcal{D}[(0,\ell)\times(0,T)]$. We thus obtain:

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial z^2}+C^*\frac{|\partial\omega|}{|\omega|}u=-C^*\frac{|\partial\omega|}{|\omega|}(r(z)-I)+\tilde{f}(z,t)\text{ in }\mathcal{D}'[(0,\ell)\times(0,T)]\,.$$

As u is an element of $L^2((0,T);H_0^1(0,\ell))$ this equation shows that $\frac{\partial u}{\partial t}\in L^2((0,T);H^{-1}(0,\ell))$, and therefore $u\in C([0,T];L^2(\Omega))$. In order to look of the initial condition on u, we introduce for a fixed v in $H_0^1(0,\ell)$, the following function:

$$Z^\varepsilon(t)=\int_\Omega u^\varepsilon(x,t)v(z)dx$$

Since $u^\varepsilon\in C([0,T];L^2(\Omega))$, we have $Z^\varepsilon\in C([0,T];\mathbb{R})$. On the other hand, the equation:

$$\frac{d}{dt}Z^\varepsilon(t)=-\int_\Omega\frac{\partial u^\varepsilon}{\partial z}\cdot\frac{\partial v}{\partial z}dx-\frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}\gamma u^\varepsilon\gamma vdx-$$
$$-\frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}(r(z)-I)\gamma vdx+\int_\Omega f(\varepsilon y,z,t)vdx$$

shows that $\frac{dZ^\varepsilon}{dt}$ is bounded in $L^2(0,\ell)$.

We then deduce that Z^ε tends uniformly to some Z in $C\ ([0,T])$. But if $\varphi\in\mathcal{D}(0,T)$, we have:

$$\langle Z^\varepsilon,\varphi\rangle_{\mathcal{D}',\mathcal{D}(0,T)}\longrightarrow\int_0^T\int_\Omega\hat{u}(z,t)v(z)\varphi(t)dxdt=|\omega|\int_0^T\int_0^\ell u(z,t)v(z)\varphi(t)dzdt$$

which implies that: $Z(t) = |\omega| \int_0^\ell u(z,t)v(z)dz$, $\forall t \in (0,T)$.

In particular, we have

$$Z^\varepsilon(0) = \int_\Omega g^\varepsilon(x)v(z)dx \longrightarrow Z(0) = |\omega| \int_0^\ell u(z,0)vdz$$

On the other hand, we have:

$$\int_\Omega g^\varepsilon(x)v(z)dx \longrightarrow \int_\Omega \tilde{g}(x)v(z)dx = |\omega| \int_0^\ell \tilde{g}(z)v(z)dz$$

So:

$$Z(0) = |\omega| \int_0^\ell u(z,0)v(z)dz = |\omega| \int_0^\ell \tilde{g}(z)v(z)dz, \quad \forall v \in H_0^1(0,\ell).$$

We deduce that: $u(z,0) = \tilde{g}(z)$ for almost all $z \in (0,\ell)$.

This completes the proof of theorem 2.3.

3 – The case $k^\varepsilon/\varepsilon \longrightarrow +\infty$

In this case the proof used in section 2 does not work. In order to establish analogous estimates to those given in proposition 2.1, we need the following lemma which is based on maximum principle.

LEMMA 3.1. *Assume that f and d satisfy hupotheses* (1.6), (1.7) *and* (1.9). *Then there exists some constant C which depends on T, such that the solution u^ε of* (1.10) *satisfies.*

$$\|u^\varepsilon\|_{L^\infty(\Omega\times(0,T))} \le C$$

PROOF. Set

$$\lambda = \text{Max}\left\{\|d+r\|_{L^\infty(\Omega)};\ \|I-r(z)\|_{L^\infty(0,\ell)};\ \sqrt{\|f\|_{L^\infty(\Omega\times(0,T))}}\right\}$$

and define: $w^\varepsilon = u^\varepsilon e^{-\lambda t}$.

It is clear that w^ε satisfies the equation:

$$\frac{d}{dt}\int_\Omega w^\varepsilon v dx + \lambda \int_\Omega w^\varepsilon v dx + \frac{1}{\varepsilon^2}\int_\Omega \nabla' w^\varepsilon . \nabla' v dx +$$

$$+\int_\Omega \frac{\partial w^\varepsilon}{\partial z}\cdot\frac{\partial v}{\partial z}dx + \frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}\gamma w^\varepsilon . \gamma v d\sigma =$$

$$= e^{-\lambda t}\Big[-\frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}(r(z)-I)\gamma v d\sigma + \int_\Omega f(\varepsilon y, z, t) v dx\Big], \qquad \forall v \in V,$$

wich is conveniently rewritten as:

$$\frac{d}{dt}\int_\Omega (w^\varepsilon-\lambda) v dx + \lambda \int_\Omega (w^\varepsilon-\lambda) v dx + \lambda^2 \int_\Omega v dx +$$

$$+\frac{1}{\varepsilon^2}\int_\Omega \nabla'(w^\varepsilon-\lambda).\nabla' v dx +$$

$$+\int_\Omega \frac{\partial}{\partial z}(w^\varepsilon-\lambda)\cdot\frac{\partial v}{\partial z}dx + \frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}\gamma(w^\varepsilon-\lambda).\gamma v d\sigma +$$

$$+\frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}\lambda\gamma v d\sigma = e^{-\lambda t}\Big[-\frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N}(r(z)-I)\gamma v d\sigma +$$

$$+\int_\Omega f(\varepsilon y, z, t) v dx\Big], \qquad \forall v \in V,$$

Remark now that $\lambda > 0$ and $w^\varepsilon_{|\Gamma_D} = 0$, so the positive part $(w^\varepsilon - \lambda)^+$ is an element of V. Then we can take $v = (w^\varepsilon - \lambda)^+$ in the last equation, to obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(w^{\varepsilon}-\lambda)^{+2}dx \le e^{-\lambda t}\Big[-\frac{k^{\varepsilon}}{\varepsilon}\int_{\Gamma_N}(r(z)-I)\gamma(w^{\varepsilon}-\lambda)^{+}d\sigma +$$

$$+\int_{\Omega} f(\varepsilon y,z,t)(w^{\varepsilon}-\lambda)^{+}dx\Big]-\lambda^2\int_{\Omega}(w^{\varepsilon}-\lambda)^{+}dx-\frac{k^{\varepsilon}}{\varepsilon}\int_{\Gamma_N}\lambda\gamma(w^{\varepsilon}-\lambda)^{+}d\sigma \le$$

$$\le \frac{k^{\varepsilon}}{\varepsilon}\int_{\Gamma_N}\Big(\|r(z)-I\|_{L^{\infty}(0,\ell)}-\lambda\Big)\gamma(w^{\varepsilon}-\lambda)^{+}d\sigma +$$

$$+\int_{\Omega}\Big(\|f\|_{L^{\infty}(\Omega)}-\lambda^2\Big)(w^{\varepsilon}-\lambda)^{+}dx \le 0$$

according to the choice of λ.

So, we have:

$$\int_{\Omega}(w^{\varepsilon}-\lambda)^{+2}(x,t)dx \le \int_{\Omega}(w^{\varepsilon}-\lambda)^{+2}(x,0)dx\,.$$

Since $(w^{\varepsilon}-\lambda)^{+}(0)=(u^{\varepsilon}-\lambda)^{+}(0)=(d(\varepsilon y,z)+r(z)-\lambda)^{+}=0$ we obtain:

$$w^{\varepsilon}\le\lambda \quad \text{a.e. in} \quad \Omega\times(0,T)$$

This implies that: $u^{\varepsilon}\le\lambda e^{\lambda T}$ a.e. in $\Omega\times(0,T)$

A similar calculation shows that:

$$u^{\varepsilon}\ge-\lambda e^{\lambda T} \qquad \text{a.e. in } \Omega\times(0,T)$$

This completes the proof of lemma 3.1.

In the following, we set for fixed $\eta>0$:

$$\Omega^{\eta}=\omega\times]\eta,\ell-\eta[$$

$$\text{and} \qquad \Gamma_N^{\eta}=\partial\omega\times]\eta,\ell-\eta[\,.$$

We also define: $\bar{u}^{\varepsilon}=u^{\varepsilon}+r(z)-I$ where u^{ε} denotes the solution of (1.10).

We verify easily that u^{ε} satisfies the equation:

$$(3.1)\quad \begin{cases} \dfrac{d}{dt}\displaystyle\int_{\Omega}\bar{u}^{\varepsilon}v dx+1/\varepsilon^2\int_{\Omega}\nabla'\bar{u}^{\varepsilon}.\nabla' v dx+\int_{\Omega}\Big(\frac{\partial\bar{u}^{\varepsilon}}{\partial z}-\frac{dr}{dz}\Big)\frac{\partial v}{\partial z}dx+ \\ +\dfrac{k^{\varepsilon}}{\varepsilon}\displaystyle\int_{\Gamma_N}\gamma\bar{u}^{\varepsilon}.\gamma v d\sigma=\int_{\Omega}f(\varepsilon y,z,t)v dx, \quad \forall v\in V \end{cases}$$

We have then the following estimates:

PROPOSITION 3.2. *If $\frac{k^\varepsilon}{\varepsilon} \longrightarrow +\infty$ and f and d satisfy the hypotheses* (1.6), (1.7) *and* (1.9), *there is a constant $C(\eta)$ depending on η and T such that:*

$$\text{i)} \qquad \int_0^T \int_{\Omega^\eta} |\nabla' \bar{u}^\varepsilon|^2 dx dt \leq \varepsilon^2 C(\eta)$$

$$\text{ii)} \qquad \int_0^T \int_{\Omega^\eta} |\frac{\partial \bar{u}^\varepsilon}{\partial z}|^2 dx dt \leq C(\eta)$$

$$\text{iii)} \qquad \frac{k^\varepsilon}{\varepsilon} \int_0^T \int_{\Gamma_N^\eta} |\gamma \bar{u}^\varepsilon|^2 d\sigma dt \leq C(\eta)$$

PROOF. Let α some function of $\mathcal{D}(0,\ell)$ such that $\alpha \equiv 1$ on $]\eta, \ell-\eta[$.
The function $\bar{u}^\varepsilon \alpha^2(z)$ belongs to V since $\bar{u}^\varepsilon \alpha^2(z) = 0$ on Γ_D. We thus can use $v = \bar{u}^\varepsilon \alpha^2(z)$ as test function in equation (3.1). We obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\bar{u}^\varepsilon|^2 \alpha^2(z) dx + 1/\varepsilon^2 \int_\Omega |\nabla' \bar{u}^\varepsilon|^2 \alpha^2(z) dx +$$
$$+ \int_\Omega \Big(\frac{\partial \bar{u}^\varepsilon}{\partial z} - \frac{dr}{dz}\Big)\Big(\alpha^2 \frac{\partial \bar{u}^\varepsilon}{\partial z} + 2\alpha\alpha' \bar{u}^\varepsilon\Big) dx +$$
$$+ \frac{k^\varepsilon}{\varepsilon} \int_{\Gamma_N} |\gamma \bar{u}^\varepsilon|^2 . \alpha^2(z) d\sigma = \int_\Omega f(\varepsilon y, z, t) \bar{u}^\varepsilon \alpha^2(z) dx \,.$$

This equation can be rewritten as:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\bar{u}^\varepsilon|^2 \alpha^2(z) dx + \frac{1}{\varepsilon^2} \int_\Omega |\nabla' \bar{u}^\varepsilon|^2 \alpha^2(z) dx +$$
$$+ \int_\Omega \alpha^2(z) \Big|\frac{\partial \bar{u}^\varepsilon}{\partial z}\Big|^2 dx + \frac{k^\varepsilon}{\varepsilon} \int_{\Gamma_N} |\gamma \bar{u}^\varepsilon|^2 \alpha^2(z) d\sigma =$$
$$= \int_\Omega f(\varepsilon y, z, t) \bar{u}^\varepsilon \alpha^2(z) dx - 2 \int_\Omega \alpha\alpha' \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial z} dx +$$
$$+ \int_\Omega \alpha^2 \frac{\partial \bar{u}^\varepsilon}{\partial z} \frac{dr}{dz} dx + \int_\Omega 2\alpha\alpha' \bar{u}^\varepsilon \frac{dr}{dz} dx$$

We now use Young's inequality and the L^∞ estimate obtained in lemma 3.1 to obtain:

$$\Big| -2\int_\Omega \alpha\alpha'\bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial z} dx\Big| \le \int_\Omega \frac{\alpha^2}{4}\Big|\frac{\partial \bar{u}^\varepsilon}{\partial z}\Big|^2 dx + \int_\Omega 4\alpha'|\bar{u}^\varepsilon|^2 dx \le$$
$$\le 1/4 \int_\Omega \alpha^2(z)\Big|\frac{\partial \bar{u}^\varepsilon}{\partial z}\Big|^2 dx + C(\alpha)$$

where $C(\alpha)$ denotes some constant depending on α. On the other hand:

$$\int_\Omega \alpha^2 \frac{\partial \bar{u}^\varepsilon}{\partial z}\frac{dr}{dz} dx \le \frac{1}{2}\int_\Omega \alpha^2\Big|\frac{\partial \bar{u}^\varepsilon}{\partial z}\Big|^2 dx + \frac{1}{2}\int_\Omega \Big|\frac{dr}{dz}\Big|^2 \alpha^2(z) dx$$

and:

$$2\int_\Omega \alpha\alpha'\bar{u}^\varepsilon \frac{dr}{dz} dx \le 2\|\bar{u}^\varepsilon\|_{L^\infty(\Omega)}\Big|\int_\Omega \alpha\alpha'\frac{dr}{dz}dx\Big| \le C(\alpha)$$

We finally see that there exists a constant $C(\alpha)$ such that:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\bar{u}^\varepsilon|^2\alpha^2(z)dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla'\bar{u}^\varepsilon|^2\alpha^2(z)dx +$$
$$+\frac{1}{4}\int_\Omega \alpha^2(z)\Big|\frac{\partial \bar{u}^\varepsilon}{\partial z}\Big|^2 dx + \frac{k^\varepsilon}{\varepsilon}\int_{\Gamma_N} \gamma(\bar{u}^\varepsilon)^2\alpha^2(z)d\sigma \le C(\alpha).$$

This implies the bounds i), ii), iii), of proposition 3.2.

We can now prove the analogous of theorem 2.3 in the case $\frac{k^\varepsilon}{\varepsilon} \to +\infty$.

THEOREM 3.3. *Assume that $k^\varepsilon/\varepsilon$ tends to $+\infty$ as ε tends to 0 and that f and d satisfy hypotheses (1.6), (1.7) and (1.9). Then, for all fixed $\eta > 0, u^\varepsilon$ converges weakly to $I - r(z)$ in $L^2(0,T;H^1(\Omega^\eta))$.*

REMARK 3.4. Proposition 3.2 gives an estimate of u^ε in $L^2((0,T);H^1(\Omega^\eta))$ for $\eta > 0$ fixed. We can not hope to obtain an estimate of u^ε in $L^2((0,T);H^1(\Omega))$. Indeed, if this does occur, we should have $u^\varepsilon \longrightarrow u$ weakly in $L^2((0,T);H^1(\Omega))$ and then since $u^\varepsilon = 0$ on Γ_D, we must have $u = 0$ on Γ_D and thus $u(0) = u(\ell) = 0$. But theorem 3.3

asserts that $u \equiv I - r(z)$ and this function does not satisfy in general those boundary conditions, except if $a_0 = a_\ell = I$.

PROOF OF THEOREM 3.3. By proposition 3.2, $\bar{u}^\varepsilon$ is bounded in $L^2((0,T); H^1(\Omega^\eta))$. Therefore, there exists a subsequence $\bar{u}^{\varepsilon_k}$ and some $\hat{\bar{u}}$ such that $\bar{u}^{\varepsilon_k}$ weakly converges to $\hat{\bar{u}}$ in $L^2((0,T); H^1(\Omega^\eta))$. We will see that $\hat{\bar{u}}$ is unique and we can then drop the subscript k. We obtain from the estimate i) of proposition 3.2 that:

$$\nabla' \hat{\bar{u}} = 0 \text{ in } \Omega^\eta \times (0,T)$$

So $\hat{\bar{u}}$ can be identified to some function $\bar{u}$ which only depends on z and t;

$$\hat{\bar{u}}(x,t) = \bar{u}(z,t) \text{ on } (0,T) \times \Omega$$

Estimate iii) shows that:

$$\int_0^T \int_{\Gamma_N^\eta} |\gamma \bar{u}^\varepsilon|^2 d\sigma dt \longrightarrow 0 \qquad \text{when} \qquad \varepsilon \longrightarrow 0$$

and $\gamma(\bar{u}^\varepsilon) \rightharpoonup \gamma(\hat{\bar{u}})$ in $L^2((0,T); L^2(\Gamma_N^\eta))$ weakly, implies that $\gamma(\hat{\bar{u}}) = 0$ on $\Gamma_N^\eta \times (0,T)$.

But $\gamma(\hat{\bar{u}}(y,z,t)) = \bar{u}(z,t)$.

We then have $\bar{u}(z,t) = 0$ a.e. on $]0,\ell[\times(0,T)$

Since $\bar{u}^\varepsilon = u^\varepsilon + r(z) - I$, we obtain:

$u^\varepsilon \rightharpoonup I - r(z)$ weakly in $L^2((0,T); H^1(\Omega^\eta))$, for all fixed $\eta > 0$.

4 – Turning back to the original problem (1.1)

In the preceeding sections we have studied the convergence of the solution u^ε of (1.4) wich is posed on the fixed domain $\Omega \times (0,T)$. We are now interested in seeing in what sense the solution v^ε of the original problem (1.1) converges and what is its limit.

Recall that by definition of u^ε, we have:

$$u^\varepsilon(x,t) = v^\varepsilon(\varepsilon y, z, t) - r(z)$$

We define: $w^\varepsilon(z,t) = \frac{1}{|\omega^\varepsilon|}\int\limits_{\omega^\varepsilon} v^\varepsilon(y,z,t)dy$.

THEOREM 4.1.

i) *If* $k^\varepsilon/\varepsilon \longrightarrow C^*$, $0 \leq C^* < +\infty$, *the sequence* $w^\varepsilon(z,t)$ *converges weakly in* $L^2((0,T)\times(0,\ell))$ *to the solution* v *of problem* (1.8).

ii) *If* $k^\varepsilon/\varepsilon \longrightarrow +\infty, w^\varepsilon(z,t)$ *converges weakly in* $L^2((0,T)\times(0,\ell))$ *to* I.

PROOF. Let $\varphi(z,t) \in L^2((0,T)\times(0,\ell)$.

By the definition of $w^\varepsilon, v^\varepsilon$ and u^ε, we have:

$$\int\limits_0^T\int\limits_0^\ell w^\varepsilon(z,t)\varphi(z,t)dzdt = \frac{1}{|\omega^\varepsilon|}\int\limits_0^T\int\limits_0^\ell\int\limits_{\omega^\varepsilon} v^\varepsilon(y,z,t)\varphi(z,t)dydzdt =$$

$$=\frac{1}{|\omega|\varepsilon^2}\int\limits_0^T\int\limits_0^\ell\int\limits_{\omega^\varepsilon}(u^\varepsilon(y/\varepsilon,z,t)+r(z))\varphi(z,t)dydzdt =$$

$$=\frac{1}{|\omega|}\int\limits_0^T\int\limits_\Omega(u^\varepsilon(y',z,t)+r(z))\varphi(z,t)dy'dzdt =$$

In the case where $k^\varepsilon/\varepsilon \longrightarrow C^* < +\infty$, theorem 2.3 implies that the last term converges to:

$$\frac{1}{|\omega|}\int\limits_0^T\int\limits_\Omega(u(z,t)+r(z))\varphi(z,t)dy'dzdt =$$

$$=\int\limits_0^T\int\limits_0^\ell(u(z,t)+r(z))\varphi(z,t)dzdt.$$

But $v = u(z,t)+r(z)$ is the unique solution of problem (1.8).

In the case where $k^\varepsilon/\varepsilon \longrightarrow +\infty$, theorem 3.3 and proposition 3.2 imply that: $u^\varepsilon \longrightarrow I - r(z)$ weakly $*$ in $L^\infty(\Omega\times(0,T))$. We thus have: $\int\limits_0^T\int\limits_0^\ell w^\varepsilon(z,t)\varphi(z,t)dzdt \longrightarrow \int\limits_0^T\int\limits_0^\ell I\varphi(z,t)dzdt$. This completes the proof of theorem 4.1.

REFERENCES

[1] D. BLANCHARD - G. FRANCFORT: *Asymptotics thermoelastic behavior of flat plates*, Quarterly appl. Math. Vol 54 n°4, (1987), 645-667.

[2] H. BREZIS: *Analyse fonctionnelle, theorie et applications*, Masson, 1983.

[3] D. CAILLERIE: *Homogénéisation des équations de la diffusion stationnaire dans des domaines cylindriques aplatis*, RAIRO, Vol 15, n°4 (1981), 295-319.

[4] P.G. CIARLET: *A justification of the Von Karman equations*, Arch. Rat. Mech. Anal., Vol 73 (1980), 349-389.

[5] P.G. CIARLET: *Plates and junction in elastic multi-structures: An asymptotic analysis*, Masson, 1990

[6] P.G. CIARLET - P. DESTUYNDER: *A justification of two dimensional plate model*, J. Mécanique, Vol. 18 (1979), 315-344.

[7] L. LIONS - E. MAGENES: *Problemes aux limites non homogènes*, Dunod, 1968, Vol. 1.

Lavoro pervenuto alla redazione il 4 giugno 1992
ed accettato per la pubblicazione il 14 luglio 1992

INDIRIZZO DELL'AUTORE:

Ali Sili - Analyse Numerique, Tour 55-65 - Université P. et M. Curie - 4, place Jussieu - 75252 - Paris cedex 05