Rendiconti di Matematica, Serie VII
Volume 13 , Roma (1993), 167-177

Periodic solutions of a class of Hamiltonian systems, with any prescribed minimal period less than the largest fundamental period of the linear part

I. SCARASCIA

RIASSUNTO: *In questo lavoro si prova l'esistenza di soluzioni periodiche di periodo minimo T per ogni $T \in [0, 2\pi/\omega_1)$ per un sistema Hamiltoniano di tipo (H).*

ABSTRACT: *In this paper one proves the existence of T-periodic solutions with minimal period T for any $T \in [0, 2\pi/\omega_1)$ for a Hamiltonian system of type (H).*

KEY WORDS: *Hamiltonian systems - Periodic solutions - Critical points of Mountain-Pass type.*

A.M.S. CLASSIFICATION: 70H05 - 34C25

Let us consider the following Hamiltonian system

$$(H)\begin{cases} \dot{y}_i(t) = \dfrac{\partial}{\partial x_i} H(x_1(t), \dots, x_N(t), y_1(t), \dots, y_N(t)) + \omega_i\, x_i(t) \\ -\dot{x}_i(t) = \dfrac{\partial}{\partial y_i} H(x_1(t), \dots, x_N(t), y_1(t), \dots, y_N(t)) + \omega_i\, y_i(t) \end{cases}$$

$$i = 1, \dots, N$$

where H is a strictly convex C^2-function on $\mathbb{R}^{2N}$ having a superquadratic growth and $0 < \omega_1 \leq \dots \leq \omega_N$. In [3] GIRARDI and MATZEU proved the existence of a T-periodic solution of (H) with minimal period T for any $T < \frac{2\pi}{\omega_N}$. In the following, it was shown that a T-periodic solution of the same kind, with minimal period T, can be found also for $T \in (\frac{2\pi}{\omega_j} - \varepsilon, \frac{2\pi}{\omega_j})$, $j \in \{1, \dots, N-1\}$, for $\varepsilon > 0$ sufficiently small in case that $\omega_j/\omega_i \notin \mathbb{Q}$ (see [4]), or if $\omega_j/\omega_i \notin \mathbb{N}$ and a further suitable condition on H'' is satisfied (see [5]).

In this paper we are able to state the existence of T-periodic solutions of (H), with minimal period T for any $T \in [0, \frac{2\pi}{\omega_1})$ if one assumes a condition of the following type

$$\omega_N < r(\omega_1)\omega_1$$

where $r(\omega_1)$ is a suitable function of ω_1 valued in $(1, +\infty)$ which can be precisely evaluated in dependence of the growth superquadratic coefficients of H. The result relies on two basic tools: firstly a suitable version of the duality principle by CLARKE - EKELAND [2] introduced in [3] for any period $T \neq 2k\pi/\omega_j$, which is generalized, in the present paper, for any $T > 0$; secondly, some energy estimates which are essentially consequences of the Mountain Pass nature of the solutions found by the duality method and of some properties of the Fenchel transform of a convex function.

Let us consider the following Hamiltonian system

$$(H) \qquad J\dot{z} = H'(z)$$

where $J(x, y) = (y - x) \quad \forall (x, y) \in \mathbb{R}^{2N}$, $H(z) = \frac{1}{2}\langle Q(z), z\rangle + \widehat{H}(z)$, Q is the $2N \times 2N$ matrix $\begin{pmatrix} Q_0 & 0 \\ 0 & Q_0 \end{pmatrix}$ with $Q_0 = \begin{pmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{pmatrix}$ and $0 < \omega_1 \leq \omega_2 \leq \dots \leq \omega_N$, $\widehat{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ is strictly convex and has a superquadratic behaviour.

One can state the following

THEOREM 1. *Let us suppose that $\widehat{H}$ verifies the following assumptions*

(1) $\widehat{H}(z)|z|^{-2} \longrightarrow 0 \quad as \quad |z| \longrightarrow 0$

(2) $\exists r > 0, \qquad \beta > 2 \mid \langle \widehat{H}(z), z\rangle > \beta \widehat{H}(z) \quad if \quad |z| > r$

Then for any $T > 0$ there exists a T-periodic solution of (H).

PROOF. The proof of the existence of T-periodic solutions of (H) for $T \neq \frac{2k\pi}{\omega_i}$, $k \in \mathbb{N}$, $i \in \{1, \dots, N\}$ was shown in [3]. It is based on the use of a suitable version of the dual principle by CLARKE - EKELAND [2] related to the consideration of the critical points of a "dual" functional given by

$$F_T(v) = \int_0^T \widehat{G}(v) - \frac{1}{2}\int_0^T \langle L_T^{-1} v, v\rangle$$

where $L_T = J\frac{d}{dt} - Q \colon H^{1,\alpha}_{\neq} \longrightarrow L^\alpha$ with

$$H^{1,\alpha}_{\neq} = \left\{ z \in H^{1,\alpha}_{\neq}(\mathbb{R}^{2N}\mathbb{R}) \mid z(0) = z(T) \right\} \qquad \alpha = \frac{\beta}{\beta - 1}$$
$$L^\alpha = L^\alpha(0, T; \mathbb{R}^{2N})$$

and $\widehat{G}$ is the Fenchel transform of $\widehat{H}$ that is

$$\widehat{G}(v) = \sup\left\{ v \cdot z - \widehat{H}(z) \mid z \in \mathbb{R}^{2N} \right\}$$

In case that $T = \frac{2k\pi}{\omega_i}$ F_T is not well defined, since L_T is not invertible. Neverthless, in this case, one can consider the (proper) subspace of $H^{1,\alpha}_{\neq}$ defined as $\widetilde{H}^{1,\alpha}_{\neq} = H^{1,\alpha}_{\neq} \cap R(L_T)$, where $R(L_T)$= range of $L_T = (\mathrm{Ker}(L_T))^{\perp}$ and the restriction $\widetilde{L}_T$ of L_T on $\widetilde{H}^{1,\alpha}_{\neq}$, so $\widetilde{L}_T$ is a bijection from $\widetilde{H}^{1,\alpha}_{\neq}$ into $R(L_T)$. Then one defines the "dual" functional $\widetilde{F}_T$ on $R(L_T)$ as

$$\widetilde{F}_T(v) = \int_0^T \widehat{G}(v) - \frac{1}{2}\int_0^T \langle \widetilde{L}_T^{-1} v, v\rangle$$

One can check that a duality principle holds for the functional $\widetilde{F}_T$ in the sense that, if $u \in R(L_T)$ and $D\widetilde{F}_T(u) = 0$, then, for a suitable $u_0 \in \mathrm{Ker}(L_T)$ one has that

$$z = \widetilde{L}_T^{-1} u + u_0 \tag{3}$$

is a T-periodic solution of (H).

[Indeed the criticality of u for $\widetilde{F}_T$ means that

$$\langle \widetilde{L}_T^{-1} u, v \rangle = \langle \widehat{G}(u), v \rangle \quad \forall v \in R(L_T)$$

that is, for a suitable $u_0 \in \mathrm{Ker}(L_T)$, one has

$$\widehat{G}'(u) = \widetilde{L}_T^{-1} u + u_0$$

so, using the very definition of $\widetilde{L}_T$ and the coniugacy property between $\widehat{H}$ and $\widehat{G}$, one verifies that z given by (3) is a T-periodic solution of (H)].

At this point, as in [3], one suitably modifies the Hamiltonian $\widehat{H}$ (let us still call $\widehat{H}$ the "modified" Hamiltonian function) in such a way that $\widehat{H}$ satisfies some further suitable conditions of superquadratic growth and such that any T-periodic solution of the "modified" Hamiltonian system is a T-periodic solution of (H) too (for details see §5 of [3]).

One can verify that, as in the case $T \neq \dfrac{2k\pi}{\omega_i}$ for any $k \in \mathbb{N}$, $i \in \{1, \dots, N\}$, with L_T replaced by $\widetilde{L}_T$, the functional $\widetilde{F}_T$ verifies all the assumptions of the Mountain Pass Theorem by AMBROSETTI and RABINOWITZ, namely

MP1) $\widetilde{F}_T(0) = 0$ and there exists ρ such that $\widetilde{F}_T(v) > 0$ if $\|v\| < \rho$

MP2) $\widetilde{F}_T$ verifies the Palais-Smale condition

MP3) $\widetilde{F}_T$ is negative at some point of $R(L_T)$

The proof is completely analogous to that related to the case $T \neq \dfrac{2k\pi}{\omega_i}$ $\forall k \in \mathbb{N}$, $i \in \{1, \dots N\}$, the main difference is in the proof of (MP3) which is based on the consideration of the eigenvalues of $\widetilde{L}_T^{-1}$. They are given by

$$\lambda_{h,j} = \frac{T}{2h\pi - T\omega_j} \qquad \text{where} \qquad k \in \mathbb{N}, \qquad j \in \{1, \dots, N\} \tag{4}$$

with the obvious exception of the pair $(h,j)=(k,i)$: this is the unique difference with the case $T \neq \frac{2k\pi}{\omega_i}$. □

Now let us investigate about T-periodic solutions of (H) having minimal period T. In [3] it was shown that, for any $T < \frac{2\pi}{\omega_N}$, there exists a T-periodic solution of (H) with minimal period T. Under suitable assumptions on ω_i's it is possible to state that for any $T < \frac{2\pi}{\omega_1}$ there exists a T-periodic solution of (H) with minimal period T. More precisely one has the following

THEOREM 2. *Let $\widehat{H}$ verify* (1), (2) *and*

(5) $$\widehat{H}(z) \geq a_1 |z|^\beta \quad \forall z \in \mathbb{R}^{2N}, \qquad a_1 > 0$$

(6) $$\widehat{H}(z) \leq a_2 |z|^\beta \quad \forall z \in \mathbb{R}^{2N}, \qquad a_2 > 0$$

(7) $$\langle \widehat{H}'(z), z \rangle \geq \beta \widehat{H}(z) \quad \forall z \in \mathbb{R}^{2N}$$

(8) $$\widehat{H}'(z)| \leq a_4 |z|^{\beta-1} \quad \forall z \in \mathbb{R}^{2N}, \qquad a_4 > 0$$

(9) $$\langle \widehat{H}''(z)\xi, \xi \rangle \leq a_3 |z|^{\beta-2} \quad \forall z \in \mathbb{R}^{2N}, \quad \forall\, \xi \in \mathbb{R}^{2N}, \quad |\xi| = 1, \quad a_3 > 0$$

Moreover let

(10) $$\omega_N < r(\omega_1)\omega_1$$

where $r(\omega_1) = \min\{2, (1-c)^{-1}\}$ *with* $c = f(a_1, a_2, a_3, a_4, \beta)\left(\frac{\omega_1}{2(a_2+\omega_1)}\right)^{\frac{\beta}{2}-1}$ *and f is a suitable positive function of the coefficients $a_1, \dots, a_4, \beta$. Then for any $T < \frac{2\pi}{\omega_1}$ there exists a T-periodic solution of (H) having minimal period T.*

REMARK 1. From (10) it follows that in the interval $\left(\frac{2\pi}{\omega_N}, \frac{2\pi}{\omega_1}\right)$ there are points of the type $\frac{2k\pi}{\omega_i}$ only for $k = 1$.

PROOF OF THEOREM 2. First of all we recall that the result was already shown in case that $T < \frac{2\pi}{\omega_N}$ in [3], so we can limit ourselves to the case

$$T \geq \frac{2\pi}{\omega_N} \tag{11}$$

Let z be the solution given by Theorem 1 and let $u = L_T(z)$ (with L_T replaced by $\widetilde{L}_T$ if $T = \frac{2\pi}{\omega_i}$ for some $i \in \{1, \dots, N\}$) the corresponding critical point of Mountain Pass for the dual functional. From now on we use the same notation L_T for all T. We proceed by steps.

STEP 1. *An estimate from below for* $\langle \widehat{G}''(x)y, y\rangle$ $x, y \in \mathbb{R}^{2N}, x \neq 0$

First of all let us prove some properties of $\widehat{G}$.

By taking into account (5) an easy calculation yields

$$\widehat{G}(v) \leq a_6|v|^\alpha \quad \forall v \in \mathbb{R}^{2N}, \qquad \alpha = \frac{\beta}{\beta - 1} \tag{12}$$

with a_6 given by

$$a_6 = \frac{a_1(\beta - 1)}{(a_1\beta)^{\beta/\beta-1}} \tag{13}$$

and an analogous argument shows that from (6) it follows

$$\widehat{G}(v) \geq a_7|v|^\alpha \quad \forall v \in \mathbb{R}^{2N} \tag{14}$$

with a_7 given by

$$a_7 = \frac{a_2(\beta - 1)}{(a_2\beta)^{\beta/\beta-1}} \tag{15}$$

Moreover it is easy to check that from (7) it follows

$$\langle \widehat{G},(v), v\rangle \leq \alpha\widehat{G}(v) \quad \forall v \in \mathbb{R}^{2N} \tag{16}$$

At this point using (5) and (7) one has $|\widehat{H}'(z)| \geq \beta a_1 |z|^{\beta-1}$ from which, taking into account that $\widehat{G}' = (\widehat{H}')^{-1}$, it follows

$$|\widehat{G}'(v)| \leq a_5 |v|^{\alpha-1} \tag{17}$$

with a_5 given by

$$a_5 = (\beta a_1)^{1/\beta-1} \tag{18}$$

Now let us estimate $\langle \widehat{G}''(x)y, y\rangle$ for $x, y \in \mathbb{R}^{2N}$, $x \neq 0$. Let $z = \widehat{G}'(x)$. The relation $\widehat{G}''(x) = (\widehat{H}'')^{-1}(\widehat{H}'(x))$ $\forall x \in \mathbb{R}^{2N}$ and (9) yield

$$\langle \widehat{G}''(x)y, y\rangle \geq \frac{1}{a_3}|z|^{2-\beta} = \frac{1}{a_3}|\widehat{G}'(x)|^{2-\beta} \tag{19}$$

and from (17) it follows $|\widehat{G}'(x)|^{2-\beta} \geq \dfrac{1}{(a_5|x|^{\alpha-1})^{\beta-2}}$ so (19) becomes

$$\langle \widehat{G}''(x)y, y\rangle \geq \frac{1}{a_3}\frac{1}{(a_5)^{\beta-2}|x|^{(\alpha-1)(\beta-2)}} = (a_3)^{-1}(a_5)^{2-\beta}|x|^{\alpha-2}$$

and putting $a_8 = (a_3)^{-1}(a_5)^{2-\beta}$ one obtain

$$\langle \widehat{G}''(x)y, y\rangle \geq a_8 |x|^{\alpha-2}|y|^2 \quad \forall x, y \in \mathbb{R}^{2N}, \quad x \neq 0, \quad a_8 > 0 \tag{20}$$

STEP 2. *An estimate from below for $\int_0^s \langle \widehat{G}''(u)v, v\rangle$,*

$$v \in L^2(0,s), \qquad s \in [0,T)$$

Using (8), (17) and arguing as in [5] one has

$$\begin{aligned} |u(t)| &\leq a_4 (2/\omega_1)^{(\beta-1)/2}\Big(a_2 + (\omega_N/2)\Big)^{(\beta-1)/2} \frac{a_5^{\beta-1}}{T}\int_0^T |u(t)| = \\ &= \frac{a_4 a_5^{\beta-1}}{T}\left(\frac{2a_2 + \omega_N}{\omega_1}\right)^{(\beta-1)/2}\int_0^T |u(t)| \end{aligned} \tag{21}$$

so, by Hölder's inequality,

$$\sup_{t\in[0,T]} |u(t)| \leq \frac{a_4 a_5^{\beta-1}}{T}\left(\frac{2a_2+\omega_N}{\omega_1}\right)^{(\beta-1)/2} T^{1/\beta}\left(\int_0^T |u(t)|^\alpha\right)^{1/\alpha} \tag{22}$$

At this point, taking into account the Mountain Pass nature of u, the fact that the maximum eigenvalue of L_T^{-1} is given by $\dfrac{T}{2\pi - T\omega_j}$ for a suitable $j \in \{1,\dots,N\}$ and (14), (16), one has

$$\int_0^T |u(t)|^\alpha \leq \frac{1}{(1-\alpha/2)a_7}\left(\frac{2-\alpha}{\alpha}\right)\left(\frac{\alpha a_6}{2}\right)^{\frac{2}{(2-\alpha)}}\left(\frac{(2\pi - T\omega_j)^{\frac{\alpha}{(2-\alpha)}}}{T^{\frac{2(\alpha-1)}{(2-\alpha)}}}\right) \tag{23}$$

so, putting

$$d = \frac{1}{(1-\alpha/2)a_7}\left(\frac{2-\alpha}{\alpha}\right)\left(\frac{\alpha a_6}{2}\right)^{2/(2-\alpha)}, \tag{24}$$

(22) and (23), taking into account that $\alpha < 2$, give

$$\sup_{t\in[0,T]} |u(t)|^{\alpha-2} \geq \left(\frac{a_4 a_5^{\beta-1}}{T^{(\beta-1)/\beta}}\right)^{\alpha-2}\left(\frac{2a_2+\omega_N}{\omega_1}\right)^{(\beta-1)(\alpha-2)/2} d^{(\alpha-2)/\alpha}\cdot \tag{25}$$

$$\left(\frac{(2\pi - T\omega_j)^{\alpha(\alpha-2)/(2-\alpha)}}{T^{2(\alpha-1)(\alpha-2)/(2-\alpha)}}\right) \geq$$

$$\geq f(a_1,a_2,a_3,a_4,\beta)\left(\frac{\omega_1}{2a_2+\omega_N}\right)^{\beta/2-1}\frac{T}{2\pi - T\omega_j}$$

where f can be evaluated taking into account (13), (15), (18) and (24). At this point from (20) it follows that, putting $b = a_8 f(a_1,a_2,a_3,a_4,\beta)$ $\left(\dfrac{\omega_1}{2a_2+\omega_N}\right)^{\beta/2-1}$, one has

$$\int_0^s \langle \widehat{G}''(u)v, v\rangle \geq \frac{bT}{2\pi - T\omega_j}\|v\|^2_{L^2(0,s)} \tag{26}$$

STEP 3. *The form $Q_s(v)$.*

Let us fix s in $(0, T/2]$ and consider the form $Q_s(v)$ defined as

$$Q_s(v) = \int_0^s \langle \widehat{G}''(u(t))v, v\rangle - \frac{1}{2}\int_0^s \langle L_s^{-1}v, v\rangle \quad \forall v \in L^2(0,s)$$

where u is a critical point of Mountain-Pass type for the functional F_T and L_s^{-1} has the same definition of L_T^{-1} on the space $L^2(0,s)$. Note that Q_s is well defined on the whole space $L^2(0,s)$ since $s < \frac{T}{2} < \frac{1}{2}\frac{2\pi}{\omega_1} < \frac{2\pi}{\omega_N}$.

We claim that the following condition

$$Q_s(v) > 0 \quad \forall s \in (0, T/2]\,. \tag{27}$$

is sufficient to guarantee that u has minimal period T, so the solution z itself has minimal period T.

Indeed a possible other period T/m, $m \in \mathbb{N}$, $m \geq 2$, cannot exist for u, since, as a general fact, if s is a period for u, then Q_s has a nontrivial kernel, which is not allowed by condition (27). Therefore the thesis of Theorem 2 will be performed if (27) will be shown.

One can check that the maximum eigenvalue of L_s^{-1}, for $s \in (0, T/2]$, is given by

$$\frac{s}{2\pi - s\omega_N} \tag{28}$$

(look at (4) and take into account that $s < \frac{2\pi}{\omega_N}$) so

$$\int_0^s \langle L_s^{-1}v, v\rangle \leq \frac{s}{2\pi - s\omega_N}\int_0^s |v|^2 \leq \frac{T/2}{2\pi - T/2\omega_N}\int_0^s |v|^2 \tag{29}$$

From (26) and (29) it follows that the form Q_s, for $s \in (0, T/2]$, verifies the estimate

$$Q_s(v) \geq \frac{bT}{2\pi - T\omega_1}\|v\|^2_{L^2(0,s)} - \\ -\frac{T/2}{2\pi - T/2\omega_N}\|v\|^2_{L^2(0,s)} \quad \forall v \in L^2(0,s)$$

then a sufficient condition in order that Q_s is positive definite is $\frac{bT}{2\pi - T\omega_1} - \frac{T/2}{2\pi - T/2\omega_N} > 0$ that is

$$T\left(b\frac{\omega_N}{2} - \frac{\omega_1}{2}\right) < 2\pi\left(b - \frac{1}{2}\right) \tag{30}$$

Now we analyse the two possible alternatives $b > \frac{1}{2}$, or $b \leq \frac{1}{2}$

1^{st} *case*: $b > \frac{1}{2}$

Obviously (30) is verified if $b \leq \frac{\omega_1}{\omega_N}$. If $b > \frac{\omega_1}{\omega_N}$, one has

$$T\left(b\frac{\omega_N}{2} - \frac{\omega_1}{2}\right) < 2\pi\left(\frac{b}{2}\frac{\omega_N}{\omega_1} - \frac{1}{2}\right)$$

then still (30) is verified since $\omega_1 > \frac{1}{2}\omega_N$.

2^{nd} *case*: $b \leq \frac{1}{2}$

The relation (30) can be written as

$$T\left(\frac{\omega_1}{2} - b\frac{\omega_N}{2}\right) > 2\pi\left(\frac{1}{2} - b\right) \tag{31}$$

By (11) one has

$$T\left(\frac{\omega_1}{2} - b\frac{\omega_N}{2}\right) \geq 2\pi\left(\frac{\omega_1}{2\omega_N} - \frac{b}{2}\right)$$

then (31) is verified if $\omega_1 > (1-b)\omega_N$.

We can conclude that a sufficient condition in order that $Q_s(v) > 0$ in $(0, T/2]$ is

$$\begin{cases} \frac{\omega_1}{\omega_N} > \frac{1}{2} & \text{if} \quad b > \frac{1}{2} \\ \frac{\omega_1}{\omega_N} > 1 - b & \text{if} \quad b \leq \frac{1}{2} \end{cases} \qquad \text{that is}$$

$$\omega_1 > \omega_N\left(\max\left\{\frac{1}{2}, 1 - b\right\}\right) \tag{32}$$