Rendiconti di Matematica, Serie VII
Volume 13 , Roma (1993), 199-228

A functional approach to B_q-a.p. spaces and L^∞ Fourier expansions

A. AVANTAGGIATI - G. BRUNO - R. IANNACCI

RIASSUNTO: *Si introducono gli spazi di Besicovitch di funzioni quasi-periodiche B_q-a.p. come completamento dello spazio $\mathcal{P}$ dei polinomi trigonometrici rispetto alla norma definita mediante la media integrale asintotica in L^q e si identificano gli elementi di tali spazi con le loro serie di Fourier. In relazione a ciò si dimostrano criteri generali di sviluppabilità degli elementi di B_q-a.p. e teoremi di rappresentazione dei funzionali lineari e continui su B_q-a.p.. Infine si stabiliscono teoremi di immersione di certi spazi tipo Sobolev, definiti mediante i B_q-a.p., nello spazio delle funzioni quasi-periodiche classiche C^0_{ap} (= B_∞-a.p.).*

ABSTRACT: *In this paper we are concerned with the Besicovitch spaces B_q-a.p. of almost periodic functions introduced as completions of the space $\mathcal{P}$ of all trigonometric polynomials, with respect to the norm defined by the asymptotic integral mean value in L^q. We identify the elements of these spaces with their Fourier series and we give some criteria for their expansions in Fourier series. Moreover we establish representation theorems for the continuous linear functionals on the B_q-a.p. spaces. Finally we introduce a class of Sobolev type spaces, defined by means of B_q-a.p., and we state some embedding theorems of these spaces in the space C^0_{ap} (= B_∞-a.p.) of uniformly almost periodic functions.*

KEY WORDS: *Almost periodic functions – Bohr transform – Fourier series – Embedding theorems.*

A.M.S. CLASSIFICATION: 42A75 – 46E35 – 42A16

1 – Introduction

The definition of almost periodic functions, whose generalization gives the B_q-a.p. spaces, is well known, but for the reader's convenience we shall recall it.

A set E of real numbers is said to be relatively dense in $\mathbb{R}$ if there exists a real number $l > 0$ such that

$$E \cap [a, a+l] \neq \phi, \quad \forall a \in \mathbb{R}.$$

A function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ is called *almost periodic* in the classical sense or *uniformly almost periodic* (*u.a.p*) if for any $\varepsilon \in \mathbb{R}_+$ there exists a subset E_ε relatively dense in $\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon, \quad \forall \tau \in E_\varepsilon. \tag{1.1}$$

Any continuous periodic function, defined on $\mathbb{R}$, is u.a.p.. However, in general linear combinations of continuous periodic functions are not periodic, but they are always u.a.p..

The set C^0_{ap} of all u.a.p. functions is a complex vector space, complete with respect to the topology of uniform convergence (cf. [1,3]).

In BESICOVITCH [3] the generalizations of u.a.p. functions went in three different directions, by replacing the distance that occurs in (1.1) with others defined by mean values of integrals of fixed length or by asymptotic behaviours of mean values of integrals and using suitably choosen dense subsets (cf. [3], ch. II). The classes of functions defined in that way are characterized as the *completion* of the complex space of all trigonometric polynomials with respect to the corresponding metrics.

This method employed to construct classes of a.p. functions is of structural type as BESICOVITCH [3] remarks in the introduction of Ch. 2.

In this paper we will use a method that is typical of Functional Analysis.

We shall start from the complex vector space $\mathcal{P} = \mathcal{P}(\mathbb{R}; \mathbb{C})$ of all trigonometric polynomials $P(x)$ of the form

$$P(x) = \sum_{j=1}^{\nu} c_j e^{i\lambda_j x}, \quad \forall x \in \mathbb{R} \tag{1.2}$$

where $\nu \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $\lambda_j \in \mathbb{R}$, with $\lambda_j \neq \lambda_i$ for $j \neq i$, are arbitrary.

The space $C^0_{ap} = C^0_{ap}(\mathbb{R}; \mathbb{C})$ of all u.a.p. functions is the *completion* of $\mathcal{P}$ with respect to the L^∞ norm

$$\|P\|_\infty = \sup_{x\in\mathbb{R}} |P(x)|, \quad \forall P \in \mathcal{P}. \tag{1.3}$$

Analogously, it is possible to introduce $C^0_{ap}(\mathbb{R}^s, \mathbb{C})$ or more generally $C^0_{ap}(\mathbb{R}^s, X)$, where X is a functional space. Here we are interested only in the B-almost periodicity concerning the so called B_q-a.p. spaces, introduced by A.S. BESICOVITCH (cf. [9], pp. 103-108).

Fixed $q \in [1, +\infty[$, for any $P \in \mathcal{P}$ we set

$$(1.4) \qquad \|P\|_q = \lim_{T\to+\infty} \Big(\frac{1}{2T}\int_{-T}^{T} |P(x)|^q dx\Big)^{1/q},$$

which is well posed since $|P(x)|^q$ is u.a.p. and every u.a.p. function has finite mean value. Moreover (1.4) defines a norm on $\mathcal{P}$ (see Section 2).

Let B^q_{ap} or $B^q_{ap}(\mathbb{R}, \mathbb{C})$ denote the *completion* of $\mathcal{P}$ with respect to the norm defined by (1.4).

From some results due to A.S. Besicovitch, it follows that these spaces are identical with those of all functions that are B-almost periodic (cf. [3], ch. II, §6). In the sequel we will use the notation B^q_{ap} instead of the original one B_q-a.p..

It is our purpose to show that it is possible to handle the B^q_{ap} spaces without using the cumbersome definition of B-almost periodicity, by reducing it, where it is possible, to the uniformly almost periodicity. To this end we will establish a type of embedding theorems from which we deduce some regularity results.

At first, we observe that from the HÖLDER inequality for $q', q'' \in]1, +\infty[$, with $q' < q''$ it follows that

$$(1.5) \qquad C^0_{ap} := B^\infty_{ap} \hookrightarrow B^{q''}_{ap} \hookrightarrow B^{q'}_{ap} \hookrightarrow B^1_{ap}\,.$$

Moreover it is possible to associate to any $f \in B^q_{ap}$ its FOURIER series (see Section 4)

$$(1.6) \qquad \sum_{j\in\mathbb{N}} a_j e^{i\lambda_j x}\,.$$

Let us recall two of the main purposes of the theory of the a.p. functions, as well as of the periodic functions:

P.1. To establish theorems in order to decide when the FOURIER series associated with a function is convergent and the sum of the series is the same function (Expansion Criteria).

P.2. To obtain conditions which insure that a given trigonometric series is the FOURIER series of some function f (General Fischer-Riesz theorem).

On the other hand, relating to the behaviour of a FOURIER series, it is necessary to observe that there is a deep difference between the L^q spaces of periodic functions and the B^q_{ap} spaces of a.p. functions. Let us assume, for example, that the FOURIER series of a periodic function f is convergent in $L^2(]0,2\pi[)$, that is

$$\lim_{n\to\infty}\int_0^{2\pi}\left|f(x)-\sum_{k=-n}^{n} b_k e^{ikx}\right|^2 dx = 0 .$$

Then we can deduce that the sequence of the partial sums of the FOURIER series has a subsequence converging a.e. to f. On the contrary, if f is a.p. but not periodic, from the condition

$$\lim_{n\to\infty}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|f(x)-\sum_{j=1}^{n} a_j e^{i\lambda_j x}\right|^2 dx = 0 \tag{1.7}$$

we have no meaningful result concerning the pointwise convergence. Indeed substituting f by $\tilde{f} = f + h$ with $h \in L^2(\mathbb{R})$, (1.7) holds true with the same a_j. Concerning such a matter we point out the following result (relative to P.1.):

Let $f \in B^q_{ap}$ with $q \in [2,+\infty]$, if the sequence of its FOURIER coefficients is q'-summable (including $q' = 1$), then f is the sum of its FOURIER series in B^q_{ap} (see theorem 4.5).

However with reference to the B^q_{ap} spaces of a.p. functions, C^0_{ap} ($= B^{\infty}_{ap}$) is the only one for which we have an useful type of convergence. For this reason we study embeddings of certain spaces of Sobolev type, constructed by means of B^q_{ap}, in C^0_{ap} or C^r_{ap}. By these embeddings we establish useful criteria for classical expansions according to P.1. purpose (see Section 6).

Furthermore we extend the well known theorems due to HAUSDORFF-YOUNG (cf. [10], ch. XII, §2) to the B^q_{ap} spaces (theorems 4.2, 4.3) and to their duals (theorems 5.5, 5.6).

This paper is organized as follows:

In Section 2 we give notations and definitions; moreover we recall some well known propositions and we prove some elementary lemmata we shall need in the sequel. Section 3 is devoted to quote two preliminary lemmata concerning the B^q_{ap} spaces with $q \geq 2$, that are the main tools in order to obtain the results of Section 4 and 5. In Section 4 we are concerned with the BOHR transformation in B^q_{ap} and its properties, and we state, among the others, a theorem of Fischer-Riesz type (P.2. purpose). This theorem allows us to get embeddings of particular subspaces of B^q_{ap} in l_p spaces and viceversa.

In Section 5 we deal with the representation of the continuous linear functionals on the B^q_{ap} spaces, with $q \geq 2$, and we state duality theorems for some particular subspaces of B^q_{ap}.

Finally, in Section 6 we introduce a class of Sobolev type spaces and we prove some embedding theorems. We deduce, for example, the following expansion criterion:

If $f, f' \in B^2_{ap}$ and the spectrum $\sigma(f) = \{\lambda_0, \lambda_1, \dots, \lambda_\nu, \dots\}$ of f verifies the condition

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} < +\infty$$

then $f \in C^0_{ap}$ and its FOURIER series converges uniformly to f on the whole real line.

2 – Notations and background material

Let $\mathcal{P}$ denote the complex vector space of all trigonometric polynomials. Observe that $\mathcal{P}$ has an algebraic basis whose cardinality is that of the continuum, because it is well known that the functions $e^{i\lambda_j x}$, $j = 1, \dots, \nu$, are linearly independent when $\lambda_1, \dots, \lambda_\nu$ are distinct, for each $\nu \in \mathbb{N}$.

For any $q \in [1, +\infty[$ let us consider the map $\|\cdot\|_q \colon \mathcal{P} \to \mathbb{R}^+ \cup \{0\}$, defined by

$$(2.1) \qquad \|P\|_q := \lim_{T \to +\infty} \left(\frac{1}{2T} \int_{-T}^{T} |P(x)|^q dx \right)^{1/q}, \quad \forall P \in \mathcal{P}.$$

For the sake of clearness of exposition we will now prove the following lemmata.

LEMMA 2.1. *Let $q \in [1, +\infty]$. If $P \in \mathcal{P}$ verifies the condition $\|P\|_q = 0$ then P is the polynomial identically zero.*

PROOF. Let $P(x)$ be given by (1.1). Using the Hölder inequality and $|e^{-i\lambda_j x}| = 1$, we have

$$|c_j| \leq \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left|P(x)e^{-i\lambda_j x}\right| dx \leq \|P\|_q = 0\,.$$

From this fact if follows that all the coefficients of $P(x)$ are zero and the thesis holds.

LEMMA 2.2. *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of trigonometric polynomials such that $\lim_{n\to\infty} \|P_n\|_q = 0$. For any $\lambda \in \mathbb{R}$, setting*

$$a(\lambda; P_n) := \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} P_n(x)e^{-i\lambda x} dx, \tag{2.2}$$

one has uniformly with respect to $\lambda \in \mathbb{R}$

$$\lim_{n\to\infty} a(\lambda; P_n) = 0.$$

PROOF. The thesis follows from $|a(\lambda; P_n)| \leq \|P_n\|_q$.

DEFINITION 2.1. *For any fixed $q \in [1, +\infty[$ we shall denote by B^q_{ap} the completion of $\mathcal{P}$ with respect to the norm $\|\cdot\|_q$ defined by* (2.1). *These spaces are called Besicovitch spaces (of almost periodic functions).*

Furthermore according to the definition given in the Introduction we set $B^{\infty}_{ap} := C^0_{ap}$.

Observe that for any $f \in B^q_{ap}$ there exists an $\tilde{f} \in L^q_{loc}(\mathbb{R})$ such that if $(P_n)_{n\in\mathbb{N}}$ is a Cauchy sequence defining f we have $\|P_n - \tilde{f}\|_q \to 0$ as $n \to \infty$. Moreover if g is an arbitrary element of $L^q(\mathbb{R})$, then $g + \tilde{f}$ has the same property.

This definition introduced by ZAANEN [9] is equivalent to that due to BESICOVITCH [3, theorem 1, p. 95]. For the properties of B^q_{ap} spaces the reader is referred to [1, 3, 5, 9].

Nevertheless, for the reader's convenience, we report some propositions we will use in the sequel.

PROPOSITION 2.1. *For each $f \in B^q_{ap}$ and for each $\lambda \in \mathbb{R}$ there exists*

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\mathrm{e}^{-i\lambda x}dx =: a(\lambda; f) \in \mathbb{C}. \tag{2.3}$$

PROPOSITION 2.2. *If $(P_n)_{n\in\mathbb{N}}$, with $P_n \in \mathcal{P}$, is such that $P_n \longrightarrow f$ in B^q_{ap} then*

2a) there exists

$$\lim_{T\to+\infty} \left(\frac{1}{2T}\int_{-T}^{T} |f(x)|^q dx\right)^{1/q} =: \|f\|_q \tag{2.4}$$

and the following relation holds true

$$\|f\|_q = \lim_{n\to\infty} \|P_n\|_q . \tag{2.5}$$

2b) one has uniformly with respect to $\lambda \in \mathbb{R}$

$$\lim_{n\to\infty} a(\lambda; P_n) = a(\lambda; f). \tag{2.6}$$

PROPOSITION 2.3. *Let $f \in B^q_{ap}$, $g \in B^{q'}_{ap}$, with $\frac{1}{q}+\frac{1}{q'}=1$. Assume that $\|P_n - f\|_q$ and $\|Q_n - g\|_{q'}$ tend to zero, where P_n, $Q_n \in \mathcal{P}$ for any $n \in \mathbb{N}$. Then there exists*

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\overline{g(x)}dx =: (f|g) \tag{2.7}$$

and it results

$$(f|g) = \lim_{n\to\infty}(P_n|Q_n), \tag{2.8}$$

$$|(f|g)| \le \|f\|_q \, \|g\|_{q'} . \tag{2.9}$$

Observe that if $P(x)$ is given by (1.2) then

$$a(\lambda; P) = \begin{cases} c_j & \text{if } \lambda = \lambda_j,\ j = 1, \dots, \nu \\ 0 & \text{if } \lambda \in \mathbb{R} \setminus \{\lambda_1, \dots, \lambda_\nu\} \end{cases}$$

From Proposition 2.1, it follows that $a(\lambda; f)$ is well defined for each $\lambda \in \mathbb{R}$ and for each $f \in B^q_{ap}$.

DEFINITION 2.2. *The map $\lambda \longrightarrow a(\lambda; f)$ is called the* BOHR *transform of the almost periodic function f (cf. [1], p. 22).*

By means of this definition, one can read the Lemmata and the Propositions of this section as properties of the BOHR transform.

DEFINITION 2.3. *We will call spectrum of the function $f \in B^q_{ap}$ the subset of real numbers defined by*

$$\sigma(f) := \{\lambda \in \mathbb{R} / a(\lambda; f) \neq 0\} \tag{2.12}$$

Hence, in particular, when f is the polynomial P given by (1.2), we have

$$\sigma(P) \subseteq \{\lambda_1, \dots, \lambda_\nu\}.$$

Easily by proposition 2.2 it follows

LEMMA 2.3. *For each $f \in B^q_{ap}$, if $P_n \longrightarrow f$ in B^q_{ap} we have*

$$\sigma(f) \subset \bigcup_{k \in \mathbb{N}} \bigcap_{n \geq k} \sigma(P_n) = \liminf_{n \to \infty} \sigma(P_n).$$

Observe now that, while $\mathcal{P}$ is a normed space by means of the norm $\|\cdot\|_q$, certainly there exist functions f not identically zero, for which $\|f\|_q = 0$. Hence, recalling a result of A.S. BESICOVITCH (cf. [3], ch. II, § 8, th. 6), we have

PROPOSITION 2.4. *The null element of the B^q_{ap} space is the object which has the spectrum ϕ, that is the element ω such that*

$$a(\lambda; \omega) = 0, \quad \forall \lambda \in \mathbb{R}.$$

REMARK 2.1. By means of Proposition 2.4, using Lemma 2.2, we have

$$\lim_{n\to\infty} \|P_n\|_q = 0 \implies P_n \to \omega \text{ in } B^q_{ap}.$$

Since the trigonometric polynomial space is *dense* in B^q_{ap}, meaning that the BOHR transform is extended from $\mathcal{P}$ to B^q_{ap} by continuity, we have

PROPOSITION 2.5. *Assume* $f, g \in B^q_{ap}$ *then*

$$a(\lambda; f) = a(\lambda; g), \forall \lambda \in \mathbb{R} \implies f \equiv g \textit{ in } B^q_{ap}.$$

REMARK 2.2. According to (1.5), the metric of C^0_{ap} is stronger than the metric of B^q_{ap} for each $q \in [1, +\infty[$. From this fact it follows that C^0_{ap} is *dense* in any B^q_{ap}, since $\mathcal{P}$ is dense in B^q_{ap} with respect to the metric of B^q_{ap}, for any $q \in [1, +\infty[$.

3 – Auxiliary lemmata

In this section we quote two lemmata we will use in the sequel.

LEMMA 3.1. *Let* $q' \in]1, +\infty[$. *If* $(g_n)_{n\in\mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ *is a Cauchy sequence in* $B^{q'}_{ap}$ *then* $(\operatorname{sign} g_n(x)|g_n(x)|^{q'-1})_{n\in\mathbb{N}} \in (C^0_{ap})^{\mathbb{N}}$ *is a Cauchy sequence in* B^q_{ap} *of a.p. functions, where obviously* $\frac{1}{q} = 1 - \frac{1}{q'}$.

For the proof the reader is referred to [2, Lemma 4.1].

Let $\Lambda = \{\lambda_1, \dots, \lambda_n, \dots\}$ be a fixed sequence of distinct real numbers, containing zero; assume for simplicity $\lambda_1 = 0$. Let us introduce now the set V^q of all functions $f \in B^q_{ap}$ satisfying the condition

$$\sigma(f) \cap \Lambda \neq \phi, \tag{3.1}$$

Any trigonometric polynomial $P_n(x)$ of the form

$$P_n(x) = \sum_{j=1}^{n} c_j e^{i\lambda_j x} \tag{3.2}$$

with $n \in \mathbb{N}$, belongs to V^q provided that it is not identically zero, i.e. $(c_1, \dots, c_n) \neq (0, \dots, 0)$.

For any fixed $n \in \mathbb{N}$ we consider the n-tuple $(\lambda_1, \dots, \lambda_n)$ and to any $f \in V^q$ we associate the trigonometric polynomial

$$f_n(x) = \sum_{j=1}^{n} b_j e^{i\lambda_j x} \tag{3.3}$$

where, for each $j = 1, \dots, n, b_j$ is defined by

$$b_j = a(\lambda_j; f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda_j x} dx\,. \tag{3.4}$$

Obviously some b_j may be zero and f_n may be identically zero.

If P_n is given by (3.2) we put

$$S_q(P_n) = \Big(\sum_{j=1}^{n} |c_j|^q\Big)^{1/q}. \tag{3.5}$$

We are going now to quote two inequalities that are the main tools for the proofs of the results concerning the properties of the BOHR transform and the BOHR transformation. These properties are the analogous ones given, in the context of trigonometric series, by HAUSDORFF-YOUNG (cf. [10], XII, 2).

The first inequality is the natural extension to the trigonometric polynomials not necessary periodic, of the inequality for periodic polynomials due to HARDY and LITTLEWOOD, whose tecnique of proof we follow (cf. [8], ch. 4, §4). The second inequality is the dual reverse one for a.p. functions.

LEMMA 3.2. *Given $\Lambda = \{\lambda_1, \dots, \lambda_n, \dots\}$ and fixed $n \in \mathbb{N}$, assume $1 < q \leq 2$ and*

$$\frac{1}{q} + \frac{1}{q'} = 1\,. \tag{3.6}$$

Then for any trigonometric polynomial P_n of the form (3.2) *one has*

$$\|P_n\|_{q'} \leq S_q(P_n)\,, \tag{3.7}$$

Moreover for any $f \in V^q$ *one has*

$$S_{q'}(f_n) \le \|f\|_q, \tag{3.8}$$

where f_n *is the trigonometric polynomial associated with* f *by means of relations* (3.3) *and* (3.4).

For the proof the reader is referred to [2, Lemma 4.2].

4 – The Bohr transformation

In Section 2 we introduced the BOHR transform of any $f \in B^q_{ap}$, for each $q \in [1, +\infty]$, by setting

$$a(\lambda; f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x} dx \tag{4.1}$$

Observe that for each fixed $f \in B^q_{ap}$, $a(\cdot, f)$ is a complex function of the real variable λ, defined on the whole of $\mathbb{R}$.

Proposition 2, 2b) of Section 2 allows us to claim that the BOHR transform of the functions f belonging to B^1_{ap} is the extension by continuity of the BOHR transform of the trigonometric polynomials $P \in \mathcal{P}$, since $B^1_{ap} \supset B^q_{ap}, \forall q \in [1, +\infty]$.

THEOREM 4.1. *For any* $f \in B^1_{ap}$, *the function* $a(\lambda; f)$, *defined on the whole of* $\mathbb{R}$, *takes at most an enumerably infinite set of values different from nought and moreover satisfies the condition*

$$\lim_{|\lambda| \to +\infty} a(\lambda; f) = 0. \tag{4.2}$$

PROOF. The first statement of Theorem 4.1 follows from Lemma 2.3.

We obtain (4.2) considering a sequence $(P_n)_{n \in \mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ converging to f in B^1_{ap} and taking into account (2.6) and $\lim_{|\lambda| \to +\infty} a(\lambda; P_n) = 0$ for any $n \in \mathbb{N}$.

Hence the BOHR transform of any $f \in B^q_{ap}$ is a function defined on the whole of $\mathbb{R}$, with range at most enumerably and infinitesimal to infinity.

Let us consider the set l_0 defined by

$$l_0 = \left\{ h \in \mathbb{C}^{\mathbb{R}} |\ h \text{ is bounded, } \operatorname{card} h(\mathbb{R}) \leq \operatorname{card} \mathbb{N} \text{ and } \lim_{|\lambda| \to +\infty} h(\lambda) = 0 \right\}$$

and equipped with the norm

$$\|h\|_{l_0} = \sup_{\lambda \in \mathbb{R}} |h(\lambda)|\,.$$

In the following part of this paper we will call the BOHR *transformation* the map $\mathcal{B}\colon B^1_{ap} \longrightarrow l_0$, $f \to a(\cdot, f)$. It is evident that $\mathcal{B}$ is a linear transformation which is defined on the whole B^1_{ap} and that takes values in l_0. Moreover it is continuous, because

$$\|a(\cdot, f)\|_{l_0} \leq \|f\|_1\,.$$

By Lemma 3.2 we derive the following

THEOREM 4.2. *Let $f \in B^q_{ap}$ and $\sigma(f) = \{\lambda_1, \dots, \lambda_n, \dots\}$; one has*

$$\left(\sum_{j=1}^{\infty} |a(\lambda_j; f)|^{q'} \right)^{1/q'} \leq \|f\|_q \qquad \text{if } q \in]1,2] \tag{4.3}$$

$$\|f\|_q \leq \left(\sum_{j=1}^{\infty} |a(\lambda_j; f)|^{q'} \right)^{1/q'} \qquad \text{if } q \in [2, +\infty[\tag{4.4}$$

and the series occurring in (4.4) *may be divergent.*

From this theorem a property of the BOHR transformation follows

COROLLARY 4.1. *If $q \in]1,2]$, the restriction of $\mathcal{B}$ to B^q_{ap} takes values in the Banach space of the sequences that are q'-summable and it is continuous as a map from B^q_{ap} into $l_{q'}$.*

The following result is complementary to this corollary and it is, in some sense, related to the FISCHER-RIESZ theorem.

THEOREM 4.3. *For any fixed sequence $(\gamma_j)_{j\in\mathbb{N}}$ of complex numbers which is q'-summable with $q' \in]1,2]$ and for any fixed enumerable set of real numbers $\Lambda = \{\lambda_1, \dots, \lambda_n, \dots\}$ there exists a function $f \in B^q_{ap}$ verifing $\sigma(f) \subseteq \Lambda$ and $\gamma_j = a(\lambda_j; f)$ for $j \in \mathbb{N}$, furthermore such f is the sum in B^q_{ap} of its* FOURIER *series. Moreover one has*

$$\|f\|_q \le \left(\sum_{j=1}^{\infty} |\gamma_j|^{q'}\right)^{\frac{1}{q'}}. \tag{4.5}$$

The claim holds true also for $q' = 1$ and $q = \infty$.

PROOF. Let us consider the sequence of partial sums $P_n(x) = \sum_{j=1}^{n} \gamma_j e^{i\lambda_j x}$. Taking into account that $q' \in]1,2]$, by Lemma 3.2 we get

$$\|P_{n+s} - P_n\|_q \le \left(\sum_{j=n+1}^{n+s} |\gamma_j|^{q'}\right)^{1/q'}.$$

This fact implies that P_n is convergent in B^q_{ap} to some $f \in B^q_{ap}$. By using Proposition 2.2 and Lemma 3.2, we get inequality (4.5).

THEOREM 4.4. *For any $q \in [2, +\infty[$ and for any $f \in B^q_{ap}$ one has*

$$\left(\sum_{j=1}^{\infty} |a(\lambda_j; f)|^2\right)^{1/2} = \|f\|_2 \le \|f\|_q. \tag{4.6}$$

PROOF. The thesis follows from theorems 4.2, 4.3 and the Hölder inequality.

REMARK 4.1. By means of Theorems 4.2 and 4.4 we can claim that fixed $f \in B^q_{ap}$, the sequence of its FOURIER coefficients is always summable with a suitable exponent; more precisely we have

$$\sum_{j=1}^{\infty} \left|a(\lambda_j; f)\right|^{q'} \leq \|f\|_q^{q'} < +\infty \quad \text{for } q \in]1, 2]$$

and

$$\sum_{j=1}^{\infty} \left|a(\lambda_j; f)\right|^{2} = \|f\|_2^2 \leq \|f\|_q^2 \quad \text{for } q \in]2, +\infty] .$$

The arguments used in the proof of Theorem 4.3 yield to get the following result, very interesting in our opinion.

THEOREM 4.5. *Let $q \in]2, +\infty[$. If $f \in B^q_{ap}$ has the sequence of its* FOURIER *coefficients that is q'-summable, then f is the sum of its* FOURIER *series in the B^q_{ap} metric.*

REMARK 4.2. Recalling (1.5) and observing that any sequence that is q-summable is also $q+\eta$-summable for each $\eta > 0$, we have the following inclusions chain

$$l_1 \subset l_{2-\xi} \subset l_2 \subset l_{2+\epsilon} \subset \bigcup_{q>2} l_q \subset l_\infty$$

and the following embeddings chain

$$C^0_{ap} = B^\infty_{ap} \hookrightarrow B^{2+\eta}_{ap} \hookrightarrow B^2_{ap} \hookrightarrow B^{2-\eta}_{ap} \hookrightarrow B^1_{ap} .$$

Theorem 4.2 gives an embedding φ of B^q_{ap} into $l_{q'}$ for $q \in]1, 2]$.

Fixed now a sequence of distinct real numbers $\Lambda = \{\lambda_1, \dots, \lambda_n, \dots\}$, let us consider the space $B^q_{ap}(\Lambda)$ of all functions $f \in B^q_{ap}$ verifing $\sigma(f) \subseteq \Lambda$. Theorem 4.3 gives an embedding ψ of $l_{q'}$ into $B^q_{ap}(\Lambda)$ for $q' \in]1, 2]$. Moreover it is well known that $B^2_{ap}(\Lambda)$ and l_2 are isomorphic.

$$\begin{array}{ccccccccccc}
l_1 & \subset & l_{2-\xi} & \subset & l_2 & \subset & l_{2+\epsilon} & \subset & \bigcup_{q>2} l_q & \subset & l_\infty \\
 & & \psi \updownarrow & & \updownarrow & & \updownarrow \varphi & & & & \\
 & & B^\infty_{ap}(\Lambda) \hookrightarrow & B^{2+\eta}_{ap}(\Lambda) \hookrightarrow & B^2_{ap}(\Lambda) \hookrightarrow & & B^{2-\eta}_{ap}(\Lambda) \hookrightarrow & & B^1_{ap}(\Lambda) & & .
\end{array}$$

Fixed Λ, let us consider the space $B^q_{q'}(\Lambda)$ of all functions $f \in B^q_{ap}(\Lambda)$ such that the sequence of its FOURIER coefficients is q'-summable. From Theorem 4.5 it follows that f is the sum of its FOURIER series in B^q_{ap} if $q \geq 2$. Therefore we have that φ is invertible on $B^q_{q'}(\Lambda)$ and

$$l_{q'} \simeq B^q_{q'}(\Lambda) \subset B^q_{ap}(\Lambda) \qquad \text{for } q' \in]1,2] .$$

5 – Duality properties of B^q_{ap} spaces

THEOREM 5.1. *Assume $q \in [1,+\infty[$. Each function g in $B^{q'}_{ap}$ defines a linear continuous functional G on B^q_{ap} by*

$$(5.1) \qquad G(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}dx .$$

We have

$$(5.2) \qquad \|G\| \leq \|g\|_{q'} .$$

PROOF. We will consider only the case $q \in]1,+\infty[$, since the case $q = 1$ ($q' = +\infty$) can be proved with similar and simpler arguments.

Since $f \in B^q_{ap}$ and $g \in B^{q'}_{ap}$ are limits of trigonometric polynomials in the sense of the B^q_{ap} norm and of the $B^{q'}_{ap}$ norm respectively, we can apply Proposition 2.3 and so $G(f)$ is well defined and we have

$$|G(f)| \leq \|f\|_q \|g\|_{q'} ,$$

which completes the proof of Theorem 5.1.

THEOREM 5.2. *Assume $q' \in]1,+\infty[$ then the functional G defined by (5.1) verifies*

$$\|G\| = \|g\|_{q'} .$$

PROOF. Since $g \in B^{q'}_{ap}$ there exists $(g_n)_{n\in\mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ such that, for $n \to \infty$, $g_n \longrightarrow g$ in $B^{q'}_{ap}$; hence by Lemma 3.1 we have that $f_n(x) = \operatorname{sign} g_n(x)|g_n(x)|^{q'-1}$ is u.a.p. for any $n \in \mathbb{N}$ and f_n converges to some f_* in B^q_{ap}.

Therefore

$$
\begin{aligned}
G(f_*) = \lim_{n\to\infty} G(f_n) = \lim_{n\to\infty} \Big\{ \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f_n(x)\overline{g_n(x)}dx + \\
+ \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f_n(x)\Big[\overline{g(x)} - \overline{g_n(x)}\Big]dx \Big\}.
\end{aligned} \tag{5.3}
$$

We observe now that from $|f_n(x)|^q = |g_n(x)|^{q'}$, $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f_n(x)\overline{g_n(x)}dx = \lim_{T\to+\infty} \left(\frac{1}{2T}\int_{-T}^{T} |g_n(x)|^{q'}dx\right)^{\frac{1}{q}+\frac{1}{q'}} = \\
&= \lim_{T\to+\infty} \left(\frac{1}{2T}\int_{-T}^{T} |f_n(x)|^{q}dx\right)^{\frac{1}{q}} \left(\frac{1}{2T}\int_{-T}^{T} |g_n(x)|^{q'}dx\right)^{\frac{1}{q'}} = \|f_n\|_q \|g_n\|_{q'}\,;
\end{aligned} \tag{5.4}
$$

on the other hand, using the HÖLDER inequality, we have

$$
\lim_{T\to+\infty} \frac{1}{2T} \left| \int_{-T}^{T} f_n(x)\Big(\overline{g(x)} - \overline{g_n(x)}\Big)dx \right| \le \|f_n\|_q \|g - g_n\|_{q'}\,;
$$

therefore

$$
\lim_{n\to\infty} \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f_n(x)\Big(\overline{g(x)} - \overline{g_n(x)}\Big)dx = 0\,. \tag{5.5}
$$

Taking into account that $\|f_n\|_q \to \|f_*\|_q$ and $\|g_n\|_{q'} \longrightarrow \|g\|_{q'}$, from (5.3), (5.4) and (5.5) it follows that

$$
G(f_*) = \|f_*\|_q \|g\|_{q'}
$$

and the proof is complete.

COROLLARY 5.1. *Assume* $q' \in]1, +\infty[$. *Then* $B^{q'}_{ap}$ *is isometrically embedded in the dual of* B^q_{ap}, *that is*

$$B^{q'}_{ap} \hookrightarrow (B^q_{ap})^* .$$

THEOREM 5.3. *Let* $q \in]1, 2]$. *Assume that* $g \in B^{q'}_{ap}$ *satisfies the condition*

$$\sum_{j=1}^{\infty} |a(\lambda_j, g)|^q < +\infty , \tag{5.6}$$

then the functional G *defined by* (5.1) *has the following representation*

$$G(f) = \sum_{j=1}^{\infty} a(\lambda_j; f)\overline{a(\lambda_j; g)} \quad , \qquad \forall f \in B^q_{ap} . \tag{5.7}$$

PROOF. By Theorem 4.2, we know that the series of the FOURIER coefficients of f is q'-summable. From this fact and (5.6) it follows that the series (5.7) is convergent by the HÖLDER inequality.

Denoting by $Q_n(x)$ the n-th partial sum of the FOURIER series of g, we get

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{Q_n(x)}dx = \sum_{j=1}^{n} a(\lambda_j; f)\overline{a(\lambda_j; g)} . \tag{5.8}$$

Therefore we obtain the thesis by passing to the limit in (5.8) for $n \to \infty$. The following result completes the above one.

THEOREM 5.4. *Given* $\Lambda = \{\lambda_1, \dots, \lambda_n, \dots\}$, *for any sequence* $(\gamma_j)_{j \in \mathbb{N}}$ *q-summable with* $q \in]1, 2]$, *the formula*

$$G(f) := \sum_{j=1}^{\infty} \bar{\gamma}_j a(\lambda_j; f) \tag{5.9}$$

defines a continuous linear functional on B^q_{ap} *and one has*

$$\|G\| \leq \Big(\sum_{j=1}^{\infty} |\gamma_j|^q\Big)^{1/q}. \tag{5.10}$$

The functional defined by (5.9) *can be written in the form* (5.1) *and* $\|G\| = \|g\|_{q'}$ *where* g *is given by*

$$g(x) = \sum_{j=1}^{\infty} \gamma_j e^{i\lambda_j x}. \tag{5.11}$$

PROOF. It's enough to use Theorems 4.3, 5.2.

To any continuous linear functional G on B^q_{ap} we can associate its sequence of FOURIER coefficients, setting for every $G \in (B^q_{ap})^*$

$$G(e^{i\lambda x}) = a(\lambda; G).$$

THEOREM 5.5. *Assume* $q \in]1, +\infty[$. *Let* $G \in (B^q_{ap})^*$ *be fixed. Then for any* $f \in B^q_{ap}$ *which is the sum in* B^q_{ap} *of its* FOURIER *series one has*

$$G(f) = \sum_{j=1}^{\infty} a(\lambda_j; f) a(\lambda_j; G). \tag{5.12}$$

When $q \in]1, 2]$ *then* $\|G\| \leq \Big(\sum_{j=1}^{\infty} |a(\lambda_j; G)|^q\Big)^{1/q}$.

PROOF. Let us introduce the n-th partial sum of the FOURIER series of f

$$P_n(x) = \sum_{j=1}^{n} a(\lambda_j; f) e^{i\lambda_j x}.$$

Observe that by continuity of G on B^q_{ap} and by the convergence of P_n to f in B^q_{ap} we deduce

$$G(f) = \lim_{n\to\infty} G(P_n) = \sum_{j=1}^{\infty} (\lambda_j; f) a(\lambda_j; G).$$

Let $q \in]1,2]$. For any $f \in B^q_{ap}$ we have

$$\left(\sum_{j=1}^{\infty} |a(\lambda_j; f)|^{q'}\right)^{\frac{1}{q'}} \le \|f\|_q .$$

Therefore applying (5.12) to P_n we get

$$|G(P_n)| \le \left(\sum_{j=1}^{n} |a(\lambda_j; f)|^{q'}\right)^{\frac{1}{q'}} \left(\sum_{j=1}^{n} |a(\lambda_j; G)|^{q}\right)^{\frac{1}{q}} \le$$
$$\le \|f\|_{q'} \left(\sum_{j=1}^{\infty} |a(\lambda_j; G)|^{q}\right)^{\frac{1}{q}}$$

from which the conclusion follows.

We will call the *spectrum* of G the set $\sigma(G)$ defined by

$$\sigma(G) := \{\lambda \in \mathbb{R} / a(\lambda; G) \neq 0\} .$$

We have the following

THEOREM 5.6. *For each $q \in [1, +\infty[$ and for any continuous linear functional G on B^q_{ap}, card $\sigma(G) \le$ card $\mathbb{N}$ and the series of its* FOURIER *coefficients is always summable with some exponent; moreover*

$$\text{(5.13)} \qquad \begin{aligned} &\left(\sum_{j=1}^{\infty} |a(\lambda_j; G)|^{q}\right)^{1/q} \le \|G\| \qquad \textit{for} \quad q \in [2, +\infty[,\\ &\left(\sum_{j=1}^{\infty} |a(\lambda_j; G)|^{2}\right)^{1/2} \le \|G\| \qquad \textit{for} \quad q \in [1, 2] . \end{aligned}$$

PROOF. Let $q \ge 2$. By the continuity of G on B^q_{ap} and by Lemma 3.2 we will have for any $P_n \in \mathcal{P}$

$$\text{(5.14)} \qquad \left|\sum_{j=1}^{\infty} a(\lambda_j; G) a(\lambda_j; P_n)\right| = \left|G(P_n)\right| \le \|G\| \left(\sum_{j=1}^{n} |a(\lambda_j; P_n)|^{q'}\right)^{1/q'} .$$

Let us consider now $\tilde{P}_n(x) = \sum_{j=1}^{n} |a(\lambda_j; G)|^{q-1}\Big(\operatorname{sign} \overline{a(\lambda_j; G)}\Big) e^{i\lambda_j x}$. Using (5.14) for $G(\tilde{P}_n)$ and by easy calculations we derive

$$(5.15) \qquad \Big(\sum_{j=1}^{n} |a(\lambda_j; G)|^q\Big)^{1-\frac{1}{q'}} = \Big(\sum_{j=1}^{n} |a(\lambda_j; G)|^q\Big)^{1/q} \leq \|G\|.$$

By the same argument employed for u.a.p. functions, (see [5]), we can deduce that card $\sigma(G) \leq$ card $\mathbb{N}$. Let assume now $q \in [1, 2[$. Since $B^2_{ap} \hookrightarrow B^q_{ap}$, we have $(B^q_{ap})^* \hookrightarrow (B^2_{ap})^*$. Then we can use the preceeding proof for $q = 2$. It is easy now to derive (5.13).

REMARK 5.1. For any fixed $q \in [1, +\infty[$ and for any $G \in (B^q_{ap})^*$ we can define the FOURIER series of G and we set

$$G \sim \sum_{j=1}^{\infty} a(\lambda_j; G) e^{i\lambda_j x},$$

where $\sigma(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$. When the FOURIER series of G converges to some g in $B^{q'}_{ap}$ then we can claim that G is of the form (5.1). For $q \geq 2$, the sequence of the FOURIER coefficients of G is q-summable; but when $q > 2$ we can not say if the FOURIER series of G is converging in $B^{q'}_{ap}$. If $q \in]1, 2[$, we know nothing concerning the q-summability of the sequence of the FOURIER coefficients of G; however if it is q-summable then the FOURIER series of G converges to some $g \in B^{q'}_{ap}$ and G is of the form (5.1).

6 – B^2_{ap}-Sobolev spaces and regularity results

For any two trigonometric polynomials P and Q, let us consider the inner product defined by

$$(6.1) \qquad (P|Q) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} P(x)\overline{Q(x)}dx.$$

Since $\mathcal{P}$ is dense in B^2_{ap} equipped with the norm $\|\cdot\|_2$, this inner product may be extended to B^2_{ap} by continuity and gives the $\|\cdot\|_2$ norm of B^2_{ap} exactly. Easily, integrating by parts (6.1), one gets

$$(6.2) \qquad (P|Q') = -(P'|Q)$$

and more generally iterating the process one has for each integer k

$$(P|Q^{(k)}) = (-1)^k (P^{(k)}|Q)\,. \tag{6.3}$$

Formula (6.3) for integration by parts will be fundamental for the following part of the paper.

DEFINITION 6.1. *Denote by H^r the functions space obtained by completion of $\mathcal{P}$ with respect to the norm*

$$\|P\|_{r,2} = \left(\sum_{k=0}^{r} \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |P^{(k)}(x)|^2 dx\right)^{\frac{1}{2}}. \tag{6.4}$$

According to definition (6.1) an element of H^2 is a function f defined by means of a sequence $(P_n(x))_{n\in\mathbb{N}}$ of trigonometric polynomials converging to f in B^2_{ap} and such that $(P'_n(x))_{n\in\mathbb{N}}$ and $(P''_n(x))_{n\in\mathbb{N}}$ are convergent in B^2_{ap} too. Generally

$$\begin{aligned} f\in H^r \iff \exists (P_n)_{n\in\mathbb{N}},\ &\text{with } P_n\in\mathcal{P},\ \text{such that}\\ &P_n \longrightarrow f \text{ in } B^2_{ap} \text{ and for each } k=1,2,\dots,r\\ &(P_n^{(k)})_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } B^2_{ap}\,. \end{aligned}$$

Since the space B^2_{ap} is complete, we can set

$$f_k := \lim_{n\to+\infty} P_n^{(k)} \tag{6.5}$$

and we will call f_k the strong k-derivative of the function f, putting by convenction $f^{(k)} = f_k$, according to FICHERA [6]. We are interested to this definition because for the strong derivatives of the functions belonging to H^r the integration by parts formula holds.

REMARK 6.1. If we consider $f\in H^1$, i.e. f and its strong derivative are the limits of sequences of trigonometric polynomials with respect to the metric of B^2_{ap}, we always choose these sequences as those of the partial sums of the FOURIER series

$$f(x) \sim \sum_{j=1}^{\infty} c_j(f) e^{i\lambda_j x}\,, \tag{6.6}$$

$$f'(x) \sim \sum_{j=1}^{\infty} i\lambda_j c_j(f) e^{i\lambda_j x}\,. \tag{6.7}$$

PROPOSITION 6.1. *For any $f, g \in H^r$ and for each integer k with $k \leq r$, one has*

$$(f|g^{(k)}) = (-1)^k (f^{(k)}|g)\,.$$

PROOF. There exist sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ of trigonometric polynomials converging to f and g respectively; using (6.3), one gets

$$(6.8) \qquad (P_n|Q_n^{(k)}) = (-1)^k (P_n^{(k)}|Q_n)\,, \quad \forall n \in \mathbb{N}\,.$$

Now take into account that

$$P_n \longrightarrow P, P_n^{(k)} \longrightarrow f^{(k)}, Q_n \longrightarrow Q \text{ and } Q_n^{(k)} \longrightarrow g^{(k)} \text{ in } B^2_{ap}\,.$$

Passing to the limit in (6.6) and using the continuity of the inner product defined by (6.1) with respect to the metric of B^2_{ap}, the thesis follows.

PROPOSITION 6.2. *Let $f \in H^r$. Putting*

$$\sigma(f) = \{\lambda_1, \dots, \lambda_\nu, \dots\},$$

the following Parseval equalities hold

$$(6.9) \qquad \left(f^{(k)}|f^{(k)}\right) = \sum_{j=1}^{\infty} |c_j|^2 \lambda_j^{2k}\,, \quad \forall k \in \{0, 1, \dots, r\}.$$

PROOF. For $k = 0$, formula (6.7) becomes the well known PARSEVAL equality. For $k = 1$ it is enough to observe that if $\{c_j^{(n)}\}_{n\in\mathbb{N}}$ are the FOURIER coefficients of a trigonometric polynomial P_n then $\{i\lambda_j c_j^{(n)}\}_{n\in\mathbb{N}}$ are the FOURIER coefficients of P_n', also in the case that some characteristic exponents are zero. For $1 < k \leq r$, we can go on by recurrence.

By (6.9) it is easy to obtain

PROPOSITION 6.3. *The norm defined by* (6.4) *in H^r, can be written in the following form*

$$(6.10) \qquad \|f\|_{r,2} = \left(\sum_{j=0}^{\infty} (1 + \lambda_j^2 + \lambda_j^4 + \dots + \lambda_j^{2r})|c_j|^2\right)^{1/2}.$$

The use of formula (6.10) is related to the behaviour of the sequence $(\lambda_j)_{j\in\mathbf{N}}$ as $j \to \infty$. We observe that in the case of periodic functions the norm written in the form (6.10) is equivalent to the norm of the standard SOBOLEV space $\overset{\circ}{H}{}^r([0,2\pi])$ with j instead of λ_j (cf. [6], p.20). On the contrary the embedding theorems in the spaces of almost periodic functions depend on this behaviour strictly. Now we would like to exhibit how the embedding theorems in the spaces H^r depend on the behaviour of the sequences $(\lambda_j)_{j\in\mathbf{N}}$, beginning with the simplest one.

From the PARSEVAL equality one gets

$$(6.11)\quad \|f\|_2^2 = \sum_{j=1}^{\infty} |c_j|^2, \quad \|f'\|_2^2 = \sum_{j=1}^{\infty} |c_j|^2\lambda_j^2, \lim_{n\to\infty}\left\| f - \sum_{j=1}^{n} c_j e^{i\lambda_j x}\right\|_2 = 0.$$

The regularity of f as an element of B^2_{ap} depends on the behaviour of the sequence of its FOURIER coefficients.
For example if the sequence of the FOURIER coefficients is 1-summable, i.e $\sum_{j=1}^{\infty} |c_j| < +\infty$, then the FOURIER series of f is convergent in L^∞, since

$$(6.12)\qquad \left|\sum_{j=n+1}^{n+p} c_j e^{i\lambda_j x}\right| \le \sum_{j=n+1}^{n+p} |c_j| .$$

Hence we can claim that there exists an u.a.p. function f^* that is the sum of the FOURIER series of f and we can write in symbol

$$(6.13)\qquad f^*(x) = \sum_{j=1}^{\infty} c_j e^{i\lambda_j x} .$$

But by (6.12) one gets that f^* is the limit in B^2_{ap} of the FOURIER series of f too, since obviously

$$\left(\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|f^*(x) - \sum_{j=1}^{n} c_j e^{i\lambda_j x}\right|^2\right)^{1/2} \le \sup_{x\in\mathbf{R}} \left|f^*(x) - \sum_{j=1}^{n} c_j e^{i\lambda_j x}\right| .$$

Therefore f can be identified , with respect to the B^2_{ap} metric, with f^* that is an almost periodic function according to the classical definition due to BOHR. It means that condition (6.13) allowed us to *regularize* f: an element belonging to B^2_{ap} turned to an element of C^0_{ap}. It is evident that this conclusion is very important. It seems an obvious result for the spaces defined here.

Now we can introduce the space C^r_{ap} as the completion of the space of trigonometric polynomials with respect to the metric

$$\|P\|_{C^r} = \sum_{k=0}^{r} \|P^{(k)}\|_{L_\infty} = \sum_{k=0}^{r} \sup |P^{(k)}(x)| \,. \tag{6.14}$$

So that $f \in C^r_{ap}$ means that f, together with its derivatives of order less than or equal to r, is the uniform limit of trigonometric polynomials, $f \in C^r_{ap} \iff f, f', \dots, f^{(r)}$ are u.a.p. functions.

By means these remarks, we are now in a position to give embedding theorems of the spaces H^r into the spaces C^s_{ap}, where $s \leq r$.

We would like now to remark that the regularity of the elements of $B^q_{ap}(\Lambda)$ depends on the sequence $\Lambda = (\lambda_j)_{j\in\mathbb{N}}$.

In this paper we assume that the set Λ has a single limit point. Preliminarly we shall consider the case that Λ has the limit point at infinity and, for the sake of simplicity of notations, we suppose that $\lambda_0 = 0$.

THEOREM 6.1. *Let $H^1(\Lambda)$ be the subspace of H^1 of functions f such that $\sigma(f) \subseteq \Lambda = \{\lambda_1, \dots, \lambda_\nu, \dots\}$. If assumption*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} < +\infty \tag{6.15}$$

holds then

$$H^1(\Lambda) \hookrightarrow C^0_{ap}\,;$$

moreover for any $f \in H^1(\Lambda)$ one has

$$\|f\|_{C^0} \leq \left(\max\left\{1, \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}\right\}\right)^{1/2} \|f\|_{H^1} \,.$$

PROOF. If $f \in H^1$, let us consider the FOURIER series of f and f'; we have

$$\sum_{j=0}^{\infty} |c_j|^2 = \|f\|_2^2 < +\infty \text{ and } \sum_{j=1}^{\infty} |c_j|^2 \lambda_j^2 = \|f'\|_2^2 < +\infty .$$

Therefore using the HÖLDER inequality, we obtain

$$\sum_{j=0}^{\infty} |c_j| = |c_0| + \sum_{j=1}^{\infty} |c_j||\lambda_j| \frac{1}{|\lambda_j|} \le |c_0| + \Big(\sum_{j=1}^{\infty} |c_j|^2 \lambda_j^2\Big)^{1/2} \Big(\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}\Big)^{1/2} .$$

Recalling the foregoing remark, the thesis follows.

THEOREM 6.2. *For any integer $r \ge 1$, if Λ satisfies hypothesis* (6.15) *then $H^r(\Lambda)$ is continuously embedded into C_{ap}^{r-1}:*

$$H^r(\Lambda) \hookrightarrow C_{ap}^{r-1} .$$

EXAMPLE 6.1. If $(\lambda_j)_{j\in\mathbb{N}}$ verifies the following condition

$$\exists K, \beta \in \mathbb{R}_+ \quad \text{s.t.} \quad |\lambda_j| \ge K\sqrt{j}(\log(1+j))^\beta$$

for some $\beta > \frac{1}{2}$ then

$$H^1(\Lambda) \hookrightarrow C_{ap}^0 .$$

Clearly it is possible to get several theorems dealing with different assumptions on the sequence $(\lambda_j)_{j\in\mathbb{N}}$.

Let us consider now the space $C_{ap}^{0,\alpha}(\mathbb{R})$ of the almost periodic functions in the classical sense that are hölderianen too, i.e.

$$C_{ap}^{0,\alpha}(\mathbb{R}) = \left\{ f \in C_{ap}^0 \ s.t. \sup_{x' \ne x''} \frac{|f(x') - f(x'')|}{|x' - x''|^\alpha} < +\infty \right\}$$

equipped with the norm defined by

$$\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + [f]_\alpha , \tag{6.16}$$

where

$$[f]_\alpha := \sup_{x' \neq x''} \frac{|f(x') - f(x'')|}{|x' - x''|^\alpha}. \tag{6.17}$$

We have the following embeddings

THEOREM 6.3. *For any given* $\Lambda = \{\lambda_1, \dots, \lambda_\nu, \dots\}$ *such that*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{2(1-\alpha)}} < +\infty, \tag{6.18}$$

one has $H^1(\Lambda) \hookrightarrow C^{0,\alpha}_{ap}$.

PROOF. Since (6.18) implies (6.15) we have that any $f \in H^1(\Lambda)$ is the pointwise sum of its FOURIER series $\sum_{j=0}^{\infty} c_j e^{i\lambda_j x}$. We will use the following inequality

$$[e^{i\lambda(\cdot)}]_\alpha \leq 2^{1-\alpha}|\lambda|^\alpha, \quad \forall\lambda \in \mathbb{R}, \forall\alpha \in]0,1]. \tag{6.19}$$

From which by the HÖLDER inequality we get

$$\begin{aligned} [f]_\alpha \leq & \sum_{j=1}^{\infty} |c_j| [e^{i\lambda_j(\cdot)}]_\alpha \leq 2^{1-\alpha} \sum_{j=1}^{\infty} |c_j||\lambda_j|^\alpha \leq \\ \leq & 2^{1-\alpha} \Big(\sum_{j=1}^{\infty} |c_j|^2 |\lambda_j|^2 \Big)^{1/2} \Big(\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{2(1-\alpha)}} \Big)^{1/2}. \end{aligned} \tag{6.20}$$

So that, from hypothesis (6.18) we have $[f]_\alpha < +\infty$. On the other hand there exists $K > 0$ s.t. one has

$$\|f\|_{L^\infty} \leq \sum_{j=0}^{\infty} |c_j| \leq K \sum_{j=1}^{\infty} |c_j||\lambda_j|^\alpha + |c_0|, \tag{6.21}$$

because by (6.18) $\frac{1}{|\lambda_j|}$ tends to zero; hence $|\lambda_j| > 1$ definitively.

From (6.21) and (6.20) we derive, for some $H > 0$,

$$\|f\|_{C^{0,\alpha}} \leq |c_0| + (K + 2^{1-\alpha}) \Big(\sum_{j=1}^{\infty} |c_j|^2 |\lambda_j|^2 \Big)^{1/2} \Big(\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{2(1-\alpha)}} \Big)^{1/2} \leq H \|f\|_{H^1}.$$

These inequalities complete the proof.

COROLLARY 6.1. *If* $\Lambda = \{\lambda_1, \dots, \lambda_n, \dots\}$ *satisfies the assumption*

$$\liminf_{n\to\infty} \frac{|\lambda_j|}{j} = l > 0, \tag{6.22}$$

then for any $\alpha \in]0, \frac{1}{2}[$ *one has*

$$H^1(\Lambda) \hookrightarrow C^{0,\alpha}_{ap}.$$

PROOF. By hypothesis, there exists $\nu_\varepsilon \in \mathbb{N}$ such that for each $j \in \mathbb{N}$ with $j > \nu_\varepsilon$, $\frac{|\lambda_j|}{j} > \frac{l}{2}$. This fact implies that (6.18) holds true; taking into account that $\alpha \in]0, \frac{1}{2}[$ and using Theorem 6.3, we get the thesis and the proof is complete.

Assume now that Λ has an unique limit point $\lambda^* \neq \infty$. We may suppose without loss of generality that $\lambda^* = 0$, since otherwise we could consider $\delta_j = \lambda_j - \lambda^*$, i.e. $\Delta = \{\lambda_1 - \lambda^*, \lambda_2 - \lambda^*, \dots, \lambda_\nu - \lambda^*, \dots\}$ modifing suitably the function f.

THEOREM 6.4. *Let* $\Lambda = \{\lambda_1, \dots, \lambda_n, \dots\} \subset \mathbb{R}$ *verify the condition*

$$\sum_{j=1}^{\infty} \lambda_j^2 < +\infty. \tag{6.23}$$

Then for any $f \in B^2_{ap}(\Lambda)$ *that has the partial sums of its* FOURIER *series equibounded in* $\mathbb{R}$, *one has* $f \in C^\infty_{ap}$.

PROOF. Let us introduce the FOURIER series of f and its partial sums

$$P_n(x) = \sum_{j=1}^{n} c_j e^{i\lambda_j x} \qquad [c_j = a(\lambda_j; f)].$$

The PARSEVAL equality gives

$$\sum_{j=1}^{\infty} |c_j|^2 = \|f\|_2^2 < +\infty. \tag{6.24}$$

We claim that $P_n'(x) = \sum_{j=1}^{n} ic_j\lambda_j e^{i\lambda_j x}$, $n \in \mathbb{N}$, is a CAUCHY sequence in C^0_{ap}. In fact from (6.23) and (6.24), using the HÖLDER inequality, we have

$$\sum_{j=n+1}^{n+p} |c_j|\,|\lambda_j| \le \Big(\sum_{j=n+1}^{n+p} |c_j|^2\Big)^{1/2} \Big(\sum_{j=n+1}^{n+p} \lambda_j^2\Big)^{1/2}. \tag{6.25}$$

Then the series $\sum_{j=1}^{\infty} ic_j\lambda_j e^{i\lambda_j x}$ is uniformly convergent in $\mathbb{R}$ and its sum $g(x)$ is an u.a.p. function having this series as its FOURIER series, i.e. we have uniformly

$$g(x) = \sum_{j=1}^{\infty} ic_j\lambda_j e^{i\lambda_j x}, \qquad \forall\, x \in \mathbb{R}.$$

Let us introduce the primitive G of g defined by

$$G(x) = \int_0^x g(t)\,dt = \int_0^x \lim_{n\to\infty} \Big(\sum_{j=1}^{n} ic_j\lambda_j e^{i\lambda_j t}\Big) dt.$$

From the uniform convergence it follows that

$$G(x) = \lim_{n\to\infty} \int_0^x P_n'(x)\,dx = \lim_{n\to\infty} (P_n(x) - P(0)), \qquad \forall\, x \in \mathbb{R}.$$

On the other hand by hypothesis there exists $M \in \mathbb{R}^+$ such that

$$|P_n(x)| \le M, \qquad \forall\, x \in \mathbb{R}, \quad \forall\, n \in \mathbb{N}, \tag{6.26}$$

which implies

$$|G(x)| \le 2M, \qquad \forall\, x \in \mathbb{R}.$$

Therefore, by the BOLH-BOHR theorem we get $G \in C^0_{ap}$. Consequently there exists finite

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(x)\,dx = a(0; G). \tag{6.27}$$

Let us introduce now the almost periodic function defined by

$$\tilde{G}(x) = G(x) - a(0;G) + a(0;f)\,.$$

It is easy to verify

$$a(0;\tilde{G}) = a(0;f)\,. \tag{6.28}$$

We shall prove now that

$$a(\lambda_j;\tilde{G}) = a(\lambda_j;f)\,, \qquad \forall\, j \in \mathbb{N}\,. \tag{6.29}$$

By (6.28) we have to consider only $\lambda_j \neq 0$. For $\lambda_j \neq 0$, we can write

$$a(\lambda_j;\tilde{G}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \tilde{G}(x)e^{-i\lambda_j x}dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} G(x)e^{-i\lambda_j x}dx\,.$$

Furthermore, by easy calculations we obtain

$$\begin{aligned} &\frac{1}{2T}\int_{-T}^{T} G(x)e^{-i\lambda_j x}dx = \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda_j x}dx\int_{0}^{x} g(\tau)d\tau = \\ &= i\frac{G(T)e^{-i\lambda_j T} - G(-T)e^{-i\lambda_j T}}{2T\lambda_j} + \frac{1}{2T}\int_{-T}^{T} g(x)\frac{e^{-i\lambda_j x}}{i\lambda_j}dx\,. \end{aligned} \tag{6.30}$$

Taking into account that $G(x)$ is bounded in $\mathbb{R}$, from (6.30) relations (6.29) follow.

Hence, since $\tilde{G}$ and f have equal FOURIER series we get that $\|\tilde{G} - f\|_2 = 0$. Therefore we can identify f to $\tilde{G}$, and we obtain the thesis observing that the following relation holds uniformly in $\mathbb{R}$

$$\tilde{G}^{(k)}(x) = \sum_{j=1}^{\infty}(i\lambda_j)^k c_j e^{i\lambda_j x}\,,$$

for each $k \in \mathbb{N}$.

REFERENCES

[1] L. AMERIO - G. PROUSE: *Almost periodic functions and functional equations*, Van Nostrand Reinhold Company, 1971.

[2] A. AVANTAGGIATI - G. BRUNO - R. IANNACCI: *Classical and new results on Besicovitch spaces of almost periodic functions and their duals*, Quaderno del Dip. Me.Mo.Mat., Univ. "La Sapienza", Roma, 1993.

[3] A.S. BESICOVITCH: *Almost periodic functions*, Cambridge, 1932.

[4] S. CINQUINI: *Funzioni quasi periodiche*, Quad. Mat. Scuola Norm. Sup. Pisa, Anno 1948-49.

[5] C. CORDUNEANU: *Almost periodic functions*, Interscience publishers, 1968.

[6] G. FICHERA: *Linear elliptic differential systems and eigenvalue problems*, Lect. Notes in Math., n.8, Springer-Verlag, 1965.

[7] A.M. FINK: *Almost periodic differential equations*, Lect. Notes in Math. n.377, Springer-Verlag, 1974.

[8] E.C. TITCHMARSH: *Introduction to the theory of Fourier integrals 2end*, ed. Oxford, Clarendom Don Press., 1948.

[9] A.C. ZAANEN: *Linear analysis*, North-Holland, Publ. Co-Amsterdam P. Noordhoff N.V. - Groningen, 1964.

[10] A. ZYGMUND: *Trigonometric series*, Cambridge University Press, 1968.

Lavoro pervenuto alla redazione il 1 ottobre 1992
ed accettato per la pubblicazione il 3 marzo 1993

INDIRIZZO DEGLI AUTORI:

A. Avantaggiati - G. Bruno - R. Iannacci - Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate - Università di Roma "La Sapienza" - Via A. Scarpa, 10 - 00161 Roma - Italia