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Hyperbolic-Parabolic Singular Perturbations

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RIASSUNTO: *Si studia un problema di perturbazione singolare per l'equazione dei telegrafisti con un piccolo parametro e l'equazione del calore con dati assegnati su una frontiera mobile. Si ottengono stime esplicite e rigorose, e si dimostra la convergenza uniforme.*

ABSTRACT: *We discuss a singular perturbations problem for the telegraphist equation with a small parameter and the heat equation in a problem with (some) data given on a general moving boundary. Rigorous and explicit estimates are shown and the uniform convergence is proved.*

KEY WORDS: *Hyperbolic PDE – Singular perturbations.*

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1 – Introduction

About forty years ago CATTANEO [2] derived a hyperbolic equation that could remove the paradoxe of the infinite speed of the thermals waves connected with the *heat equation* of the classical Fourier theory. A few years later, VERNOTTE [13], deduced, indipendently, the same equation, basing his arguments, as Cattaneo, on statistical mechanics. The hyperbolic equation they introduced is the well known *telegraphist equation* with an inertial small coefficient, called *material relaxation time*, and denoted by ε in this paper.

The study of the approximation of the solutions of these two equations, related to initial-boundary value problems, involves, of course, singular perturbations questions. In some works ZLAMAL [14, 15] and FULKS and GUENTHER [7] have discussed the singular perturbations problem for the Cauchy and the initial-boundary value problems when the boundary is fixed. However, no analysis seems to be for singular perturbations problems with (some) data given on a general moving boundary. The interest in the question is even increased by the possible applications to hyperbolic heat transfer models, recently developed, that could play an important role in technological problems such as targets irradiated by high intensity nuclear radiation or by power laser beams [12, 4, 8, 6, 3].

In this paper we discuss the hyperbolic-parabolic singular perturbations, related to the *telegraphist* and the *heat* equations, for an initial-boundary value problem with a moving boundary of equation $x = r(t)$. We deduce a rigorous and explicit estimate for the difference of the solutions and, as consequence, the uniform convergence. Precisely, denoting by U_ε and U the solutions of the two problems we get $|U_\varepsilon - U| \leq m\varepsilon^p$, where m is a positive constant independent of ε and p is a positive rational number.

The main difficulty arises from the forms of the solutions given by means of integral equations. Therefore, we preliminarily show basic estimates (Sec. 5) by which we can find appropriate *a priori* upper bounds for Volterra integral equations (Sec. 3). Then, by using rigorous tecniques we obtain explicit estimates for corresponding terms of the solutions (Sec. 4). In any situation, however, we furnish definite approximation formulas.

2 – The singular perturbations problem

Let us denote by α the thermal diffusivity, by ε the material relaxation time and indicate by $x = r(t)$ the equation of the moving boundary. Consider the following boundary-value problems

$$\varepsilon u_{tt} - \alpha u_{xx} + u_t = 0, \ x > r(t), \ 0 < t < T, \tag{2.1}$$

$$u = \phi(x), \ u_t = \psi(x) \text{ for } t = 0, \tag{2.2}$$

$$u = 0 \text{ for } x = r(t); \tag{2.3}$$

$$-\alpha U_{xx} + U_t = 0, \; x > r(t), \; 0 < t < T \tag{2.4}$$

$$U = \phi(x) \text{ for } t = 0, \tag{2.5}$$

$$U = 0 \text{ for } x = r(t). \tag{2.6}$$

In this paper we discuss the convergence, when ε tends to zero, of $u(x,t)$ to $U(x,t)$. Moreover, we determine explicit estimates for the remainder according to the modern formulation of the singular perturbations problems (see e.g. [5, 9, 10]).

For $x > t\sqrt{\varepsilon/\alpha}$ the hyperbolic problem does not *feel* the data on the moving boundary, that is, the solution u depends only on the initial data. In this case the analysis can be developed as in some preceding papers [7, 14, 15]. Thus, in this work we focus our attention on the region

$$\Omega = \{(x_0, t_0) \; : \; 0 < t_0 < T, \; r(t_0) < x_0 < t_0\sqrt{\alpha/\varepsilon}\}, T > 0. \tag{2.7}$$

On the data we assume that

$$\varphi(x) \in C^2([0, +\infty[), \; \varphi(0) = 0, \tag{2.8}$$

$$|\varphi(x)| < M_\varphi, |\varphi'(x)| < M'_\varphi, |\varphi''(x)| < M''_\varphi, \tag{2.9}$$

where $M_\varphi, M'_\varphi, M''_\varphi$ are positive constants,

$$\psi(x) \in C^1([0, 2T\sqrt{\alpha/\varepsilon}]), \tag{2.10}$$

$$|\psi(x)| < M_\psi, |\psi'(x)| < M'_\psi, \tag{2.11}$$

with M_ψ, M'_ψ positive constants,

$$r(t) \in C^2([0,T]),\ r(0) = 0, \tag{2.12}$$

$$|\dot r(t)| \le r_1,\ r_1 = \text{constant} < \sqrt{\alpha/\varepsilon}. \tag{2.13}$$

Finally, we introduce the *fundamental solutions* of (2.1), (2.4):

$$V(x_0 - x, t_0 - \tau) = \frac{e^{-\frac{t_0-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} I_0\left(\sqrt{\frac{(t_0-\tau)^2}{4\varepsilon^2} - \frac{(x_0-x)^2}{4\alpha\varepsilon}}\right), \tag{2.14}$$

where I_n $(n \ge 0)$ is the modified Bessel function of order n and

$$E(x_0 - x, t_0 - \tau) = \frac{e^{-\frac{(x_0-x)^2}{4\alpha(t_0-\tau)}}}{\sqrt{4\pi\alpha(t_0-\tau)}}. \tag{2.15}$$

Under the above-cited hypotheses the solution of the hyperbolic problem is given by

(2.16)

$$u(x_0,t_0) = \frac{1}{2} e^{-t_0/2\varepsilon}\varphi\left(x_0 + t_0\sqrt{\alpha/\varepsilon}\right) +$$

$$+ \int_0^{x_0+t_0\sqrt{\alpha/\varepsilon}} [\varepsilon\psi(x)V(x_0-x,t_0) + \varphi(x)(1+\varepsilon\partial_{t_0})V(x_0-x,t_0)]dx +$$

$$- \alpha\int_0^t w(\tau)[1 - \varepsilon\dot r^2(\tau)/\alpha]V(x_0 - r(\tau), t_0 - \tau)d\tau, \qquad (x_0,t_0) \in \Omega,$$

where t is defined by

$$t = t_0 - \sqrt{\varepsilon/\alpha}[x_0 - r(t)] \tag{2.17}$$

and

$$w(\tau) = u_x(r(\tau),\tau) \tag{2.18}$$

is solution of the following Volterra integral equation

(2.19)
$$w(t)\left[1-\dot r(t)\sqrt{\frac{\varepsilon}{\alpha}}\right]=e^{-t/2\varepsilon}\varphi'\left(r(t)+t\sqrt{\frac{\alpha}{\varepsilon}}\right)+2\varepsilon\psi(0)V(r(t),t)+$$
$$+2\int\limits_0^{r(t)+t\sqrt{\alpha/\varepsilon}}[\varepsilon\psi'(x)V(r(t)-x,t)+\varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)]dx+$$
$$+2\alpha\int\limits_0^t w(\tau)[1-\varepsilon\dot r^2(\tau)/\alpha]V_x(r(t)-r(\tau),t-\tau)d\tau,\qquad 0<t<T.$$

Moreover, from (2.4)-(2.6) we get (see e.g. [11], p. 181)

$$U(x_0,t_0)=\int\limits_0^{+\infty}\varphi(x)E(x_0-x,t_0)dx+ \tag{2.20}$$
$$-\alpha\int\limits_0^{t_0}W(\tau)E(x_0-r(\tau),t_0-\tau)d\tau,\qquad (x_0,t_0)\in\Omega,$$

with

$$W(\tau)=U_x(r(\tau),\tau) \tag{2.21}$$

solution of

$$W(t)=2\alpha\int\limits_0^t W(\tau)E_x(r(t)-r(\tau),t-\tau)d\tau+ \tag{2.22}$$
$$+2\int\limits_0^{+\infty}\varphi'(x)E(r(t)-x,t)dx,\qquad 0<t<T.$$

REMARK 2.1. We conclude this Section with two inequalities, about the fundamental solutions, that can be easily deduced from [7, Sec.2] when $t>0$ and $|x|<t\sqrt{\alpha/\varepsilon}$

$$V(x,t)\le 4E(x,t), \tag{2.23}$$

$$(2.24) \qquad \frac{e^{-\frac{t}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} \frac{I_1\left(\sqrt{\frac{t^2}{4\varepsilon^2} - \frac{x^2}{4\alpha\varepsilon}}\right)}{\sqrt{1 - \varepsilon x^2/\alpha t^2}} \leq 14E(x,t).$$

3 – Estimates for Volterra integral equations

In this Section and in the next one we make use of *basic estimates* whose proofs are given in Sec. 5.

Let us consider the integral equation (2.19) and multiply both members for $1 + \dot{r}(t)\sqrt{\varepsilon/\alpha}$. Setting

$$(3.1) \qquad H(t,\tau) = 2\alpha V_x(r(t) - r(\tau), t - \tau),$$

$$(3.2) \qquad h(t,\tau) = [1 + \dot{r}(t)\sqrt{\varepsilon/\alpha}]H(t,\tau),$$

$$(3.3) \qquad f(t) = w(t)[1 - \varepsilon\dot{r}^2(t)/\alpha],$$

(3.4)

$$g(t) = [1 + \dot{r}(t)\sqrt{\varepsilon/\alpha}]e^{-t/2\varepsilon}\varphi'\left(r(t) + t\sqrt{\alpha/\varepsilon}\right) +$$

$$+ [1 + \dot{r}(t)\sqrt{\frac{\varepsilon}{\alpha}}]2\varepsilon[\psi(0)V(r(t),t) + \int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} \psi'(x)V(r(t) - x, t)dx] +$$

$$+ [1 + \dot{r}\sqrt{\frac{\varepsilon}{\alpha}}]2 \int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} \varphi'(x)(1 + \varepsilon\partial_{t_0})V(r(t) - x, t)dx,$$

We obtain the following integral equation for $f(t)$

$$(3.5) \qquad f(t) = g(t) + \int_0^t f(\tau)h(t,\tau)d\tau, \quad 0 < t < T.$$

Moreover, let us substract (2.22) from (2.19), after multiplying the last equation for $1 + \dot{r}(t)\sqrt{\varepsilon/\alpha}$. Setting

$$(3.6) \qquad F(t) = w(t)[1 - \varepsilon\dot{r}^2(t)/\alpha] - W(t),$$

we get a new integral equation

$$F(t) = G(t) + \int_0^t F(\tau)H(t,\tau)d\tau, \; 0 < t < T, \tag{3.7}$$

with $H(t,\tau)$ defined by (3.1) and
(3.8)

$$G(t) = \sqrt{\frac{\varepsilon}{\alpha}}\dot{r}(t)e^{-t/2\varepsilon}\varphi'\left(r(t) + t\sqrt{\frac{\alpha}{\varepsilon}}\right) +$$

$$+ \sqrt{\frac{\varepsilon}{\alpha}}\dot{r}(t)2\varepsilon[\psi(0)V(r(t),t) + \int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} \psi'(x)V(r(t)-x,t)dx] +$$

$$+ \sqrt{\frac{\varepsilon}{\alpha}}\dot{r}(t)\{ \int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} 2\varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx +$$

$$+ \int_0^t f(\tau)H(t,\tau)d\tau\} + e^{-t/2\varepsilon}\varphi'\left(r(t) + t\sqrt{\frac{\alpha}{\varepsilon}}\right) +$$

$$+ 2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} \varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx +$$

$$- 2\int_0^{+\infty} \varphi'(x)E(r(t)-x,t)dx + 2\varepsilon[\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} \psi'(x)V(r(t)-x,t)dx +$$

$$+ \psi(0)V(r(t),t)] + 2\alpha\int_0^t W(\tau)[V_x(r(t)-r(\tau),t-\tau) +$$

$$- E_x(r(t)-r(\tau),t-\tau)]d\tau.$$

Equations (3.5), (3.7) can be solved, for example, by the fixed point theorem. However, we are interested only in estimates on the solutions, we are going to study in the next theorems.

THEOREM 3.1. *Assume the hypotheses* (2.8)-(2.13) *are fulfilled. Then, there exist two constants* M_{1w} *and* M_{2w} *independent of* t *and* ε

such that

$$|w(t)|[1-\varepsilon\dot{r}^2(t)/\alpha] \leq M_{1w} + M_{2w}\sqrt{\varepsilon}. \tag{3.9}$$

PROOF. We consider Volterra integral equation (3.5) and first note that from (2.13) it follows $1+\dot{r}(t)\sqrt{\varepsilon/\alpha} < 2$ Next we discuss h given by (3.2). Recalling (2.24) we have

$$2\alpha|V_x(r(t)-r(\tau),t-\tau)| \leq 14r_1E(r(t)-r(\tau),t-\tau),$$

that is

$$|h(t,\tau)| \leq \frac{14r_1}{\sqrt{\pi\alpha(t-\tau)}}. \tag{3.10}$$

Moreover, for (2.23) and (2.24) we have

$$\begin{aligned} &2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx \leq \\ &\leq 18\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}E(r(t)-x,t)dx \leq 18. \end{aligned} \tag{3.11}$$

On the other hand, it is obvious that

$$\left|e^{-t/2\varepsilon}\varphi'\left(r(t)+t\sqrt{\alpha/\varepsilon}\right)\right| \leq M'_\varphi. \tag{3.12}$$

Therefore, using the estimate (5.20) on B'_3 and the inequalities (3.11), (3.12), for the term g of the integral equation (3.5) we get

$$|g(t)| \leq 2\left\{19M'_\varphi + \frac{M_\psi}{\sqrt{\alpha}}\left(1+14\sqrt{\frac{2}{e\pi}}\right)\varepsilon^{1/2}\right\}. \tag{3.13}$$

Applying known results, see e. g. [1 p.97], from (3.10), (3.13), we deduce the following *a priori* bound for the solution f of (3.5)

$$|f(t)| \leq M_{1w} + M_{2w}\sqrt{\varepsilon}, \tag{3.14}$$

where $M_{iw}, (i=1,2)$ are constants independent of t and ε. So, (3.9) is proved.

A similar result holds for the integral equation (3.7). Indeed, we have

THEOREM 3.2. *Assume the hypotheses* (2.8)-(2.13) *are fulfilled. Then, there exists a constant K_F independent of t and ε such that*

$$|w(t)[1-\varepsilon\dot{r}^2(t)/\alpha]-W(t)|\leq K_F\varepsilon^{q_F}, \tag{3.15}$$

where q_F is a strictly positive rational number.

PROOF. We consider Volterra integral equation (3.7) and note that, recalling (3.10), we have

$$|H(t,\tau)|\leq\frac{14r_1}{\sqrt{4\pi\alpha(t-\tau)}}. \tag{3.16}$$

Now, using the notations B', B, B_3' given, respectively, by (5.14), (5.1), (5.19), we write the function G (see (3.8)) as follows

$$\begin{aligned}G(t)=B'+2\alpha B+\sqrt{\frac{\varepsilon}{\alpha}}\dot{r}(t)\left[e^{-\frac{t}{2\varepsilon}}\varphi'\left(r(t)+t\sqrt{\frac{\alpha}{\varepsilon}}\right)+B_3'\right]+\\+\sqrt{\frac{\varepsilon}{\alpha}}\dot{r}(t)\left[2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}\varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx+\right.\\\left.+\int_0^t f(\tau)H(t,\tau)d\tau\right].\end{aligned} \tag{3.17}$$

Moreover, from the last theorem we deduce

$$\left|\int_0^t f(\tau)H(t,\tau)d\tau\right|\leq(M_{1w}+M_{2w}\sqrt{\varepsilon})14r_1\sqrt{\frac{T}{\pi\alpha}}, \tag{3.18}$$

$$\left|2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}\varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx\right|\leq 18M_\varphi'. \tag{3.19}$$

Considering (3.12), (3.18), (3.19) and the estimates (5.15), (5.2), (5.20), obtained in Sec. 5 for B', B, B_3', we achieve

$$|G(t)| \leq K_G \varepsilon^q,$$

where q is a positive rational number and K_G does not depend on ε. We can find, therefore, an *a priori* bound for the solution F of the integral equation (3.7). Since this bound depends essentially on the estimate on G , recalling the definition (3.6) of F we get the theorem.

4 – Estimates of the solutions

Let us consider the difference $u - U$ of the two solutions (2.16) and (2.20)

$$u(x_0, t_0) - U(x_0, t_0) = \tag{4.1}$$

$$= \frac{e^{-\frac{t_0}{2\varepsilon}}}{2} \varphi(x_0 + t_0\sqrt{\alpha/\varepsilon}) - \int_0^{+\infty} \varphi(x) E(x_0 - x, t_0) dx +$$

$$+ \int_0^{x_0 + t_0\sqrt{\alpha/\varepsilon}} [\varepsilon\psi(x) V(x_0 - x, t_0) + \varphi(x)(1 + \varepsilon\partial_{t_0}) V(x_0 - x, t_0)] dx +$$

$$- \alpha \int_0^t W(\tau) V(x_0 - r(\tau), t_0 - \tau) d\tau + \alpha \int_0^{t_0} W(\tau) E(x_0 - r(\tau), t_0 - \tau) d\tau +$$

$$- \alpha \int_0^t \{w(\tau)[1 - \varepsilon\dot{r}^2(\tau)/\alpha] - W(\tau)\} V(x_0 - r(\tau), t_0 - \tau) d\tau, (x_0, t_0) \in \Omega.$$

We see that the last integral can be estimated by applying the results of Th. 3.2. Therefore, we have to discuss

$$A = \int_0^t W(\tau) V(x_0 - r(\tau), t_0 - \tau) d\tau - \int_0^{t_0} W(\tau) E(x_0 - r(\tau), t_0 - \tau) d\tau, \tag{4.2}$$

and
(4.3)

$$A' = \frac{e^{-t_0/2\varepsilon}}{2}\varphi\left(x_0 + t_0\sqrt{\alpha/\varepsilon}\right) - \int_0^{+\infty} \varphi(x)E(x_0 - x, t_0)dx +$$

$$+ \int_0^{x_0+t_0\sqrt{\alpha/\varepsilon}} [\varepsilon\psi(x)V(x_0 - x, t_0) + \varphi(x)(1 + \varepsilon\partial_{t_0})V(x_0 - x, t_0)]dx.$$

These estimates are the object of the next two theorems.

THEOREM 4.1. *Assume the hypotheses* (2.8)-(2.13) *are satisfied. Then, there exists a constant K_A independent of ε, t_0, x_0 such that*

$$|A| \le K_A \varepsilon^{q_A}, \tag{4.4}$$

where A is given by (4.2) *and q_A is a strictly positive rational number.*

PROOF. We obviously have

$$A = A_1 + A_2, \tag{4.5}$$

where

$$A_1 = \int_0^t W(\tau)[V(x_0 - r(\tau), t_0 - \tau) - E(x_0 - r(\tau), t_0 - \tau)]d\tau, \tag{4.6}$$

$$A_2 = -\int_t^{t_0} W(\tau)E(x_0 - r(\tau), t_0 - \tau)d\tau. \tag{4.7}$$

Considering the definition of t (see (2.17)), for $t < \tau < t_0$ we have $-(x_0 - r(\tau))^2/(4\alpha(t_0 - \tau)) < -(t_0 - \tau)/4\varepsilon$ and for A_2 we get

$$|A_2| \le M_W \int_t^{t_0} \frac{e^{-(t_0-\tau)/4\varepsilon}}{\sqrt{4\pi\alpha(t_0 - \tau)}}d\tau \le \frac{M_W}{\sqrt{4\pi\alpha}}\left(\frac{\varepsilon}{e}\right)^{1/4} \int_t^{t_0} \frac{d\tau}{(t_0 - \tau)^{3/4}},$$

$$|A_2| \le \frac{2M_W}{\sqrt{\alpha\pi}}\left(\frac{T}{e}\right)^{1/4} \varepsilon^{1/4}. \tag{4.8}$$

Consider, now, A_1 and assume $t \leq \varepsilon^{3/4}$. Since

$$(t_0-\tau)^{-1} < (t-\tau)^{-1}, \tag{4.9}$$

we can proceede as for B (see Th. 5.1) obtaining

$$|A_1| \leq \frac{5M_W}{\sqrt{\pi\alpha}}\varepsilon^{3/8}. \tag{4.10}$$

If, instead, $t > \varepsilon^{3/4}$, recalling (4.9) we can discuss A_1 following the arguments used for B in the same situation. So, we achieve

(4.11)
$$|A_1| \leq \frac{M_W}{\sqrt{\alpha\pi}}\left[10\left(\frac{4T\varepsilon}{e}\right)^{1/4} + 5\varepsilon^{3/8} + \left(4K\sqrt{\frac{\pi}{3e}} + \frac{8}{\sqrt{3}} + \frac{16}{e^2}\right)\varepsilon^{5/8}\right].$$

In conclusion, the theorem, that is (4.4), follows from (4.5), (4.8), (4.10), (4.11).

THEOREM 4.2. *Assume the hypotheses* (2.8)-(2.13) *are fulfilled. Then, there exists a constant* K'_A *independent of* x_0, t_0 *and* ε *such that*

$$|A'| \leq K'_A\varepsilon^{q'_A}, \tag{4.12}$$

where A' *is defined by* (4.3) *and* q'_A *is a strictly positive rational number.*

PROOF. We introduce

$$A'_1 = \frac{1}{2}e^{-t_0/2\varepsilon}\varphi\left(x_0 + t_0\sqrt{\alpha/\varepsilon}\right), \tag{4.13}$$

$$A'_2 = \int_0^{x_0+t_0\sqrt{\alpha/\varepsilon}} \varphi(x)(1+\varepsilon\partial_{t_0})V(x_0-x,t_0)dx +$$

$$- \int_0^{+\infty} \varphi(x)E(x_0-x,t_0)dx, \tag{4.14}$$

$$A'_3 = \varepsilon \int_0^{x_0+t_0\sqrt{\alpha/\varepsilon}} \psi(x)V(x_0-x,t_0)dx, \tag{4.15}$$

so that (see (4.3))

$$A' = A_1' + A_2' + A_3'. \tag{4.16}$$

The term A_3' can be computed immediately. For (2.24) we have

$$|A_3'| \le 4\varepsilon M_\psi \int_0^{x_0+t_0\sqrt{\alpha/\varepsilon}} E(x_0 - x, t_0)dx,$$

from which

$$|A_3'| \le 2\varepsilon M_\psi \frac{x_0 + t_0\sqrt{\alpha/\varepsilon}}{\sqrt{\pi\alpha t_0}} \le 4\sqrt{\frac{T}{\pi}} M_\psi \varepsilon^{1/2}. \tag{4.17}$$

For A_1' given by (4.13) we get

$$|A_1'| \le M_\varphi \varepsilon / t_0. \tag{4.18}$$

Moreover, the integral A_2' can be analyzed by applying the same methodology developed in Th. 5.1 with reference to B_2'. So, we obtain

$$|A_2'| \le 336\sqrt{2} M_\varphi \frac{\varepsilon}{t_0} + M_\varphi \left(\frac{4K}{3\sqrt{e}} + \frac{2}{\sqrt{\pi}} + \frac{2\alpha}{e^2\sqrt{\pi}}\right)\sqrt{\frac{\varepsilon}{t_0}}.$$

From this last estimate and from (4.17), (4.18) we see that (4.12) is satisfied if $t_0 > \varepsilon^{3/4}$. Therefore, we have to discuss the case

$$t_0 \le \varepsilon^{3/4} \tag{4.19}$$

for A_1' and A_2', since the inequality (4.17) involving A_3' does not depend on t_0. As, $\varphi(0) = 0$, we also have

$$|\varphi(x)| \le M_\varphi' x, \; x > 0. \tag{4.20}$$

Thus, for A_1' we deduce

$$|A_1'| \le M_\varphi' \frac{x_0 + t_0\sqrt{\alpha/\varepsilon}}{2} \le M_\varphi' \sqrt{\alpha}\varepsilon^{1/4}. \tag{4.21}$$

Now, we estimate A_2' using (4.19). Considering (2.23) and (2.24)and setting $y = (x - x_0)/\sqrt{4\alpha t_0}$, we get

$$|A_2'| \leq 5M_\varphi' \int\limits_0^{x_0+\sqrt{\alpha t_0/\varepsilon^{1/4}}} \frac{x dx}{\sqrt{\pi \alpha t_0}} + 10M_\varphi \int\limits_{1/\sqrt{4\varepsilon^{1/4}}}^{+\infty} \frac{e^{-y^2}}{\sqrt{\pi}} dy.$$

Finally, from this last result we achieve

$$|A_2'| \leq 10M_\varphi' \sqrt{\frac{\alpha}{\pi}} \varepsilon^{1/8} + 40\sqrt{2} M_\varphi \varepsilon^{1/4},$$

that proves completely the theorem.

We are, now, ready to study the *uniform convergence* of the solution u of the hyperbolic equation to the solution U of the corresponding parabolic one.

THEOREM 4.3. *Assume the hypotheses* (2.8)-(2.13) *are fulfilled. Then, there exists a constant M independent of x_0, t_0 and ε such that for the solutions u and U given by* (2.16) *and* (2.20) *we have*

$$|u - U| \leq M\varepsilon^p, \ (x_0, t_0) \in \Omega, \tag{4.22}$$

where p is a strictly positive rational number.

PROOF. Using the definitions (4.3), (4.2) of A', A the difference $u - U$ can be rewritten as follows

$$u - U = A' - \alpha A - \alpha \int\limits_0^t F(\tau) V(x_0 - r(\tau), t_0 - \tau) d\tau,$$

where $F(t)$, defined by (3.6), satisfies inequality (3.15). Moreover, for (2.23) we get

$$\alpha \int\limits_0^t V(x_0 - r(\tau), t_0 - \tau) d\tau \leq 4\sqrt{\frac{\alpha}{\pi}} \int\limits_0^t \frac{d\tau}{\sqrt{4(t-\tau)}} \leq 4\sqrt{\frac{\alpha T}{\pi}}.$$

In conclusion, from this last relation and from the estimates (4.12), (4.4) on A', A, we obtain

$$|u-U| \le K'_A \varepsilon^{q'_A} + \alpha K_A \varepsilon^{q_A} + 4K_F \varepsilon^{q_F} \sqrt{T\alpha/\pi}$$

that proves the theorem.

5 – Basic estimates

We discuss some basic estimates useful to estimate the difference of the solutions. So, setting

$$B = \int_0^t W(\tau)[V_x(r(t)-r(\tau), t-\tau) - E_x(r(t)-r(\tau), t-\tau)]d\tau, \tag{5.1}$$

we show

THEOREM 5.1. *Assume the hypotheses (2.8)-(2.13) are satisfied. Then, there exists a constant K_B independent of ε, x_0, t_0 such that*

$$|B| \le K_B \varepsilon^{q_B}, \tag{5.2}$$

where q_B is a strictly positive rational number.

PROOF. For the solution $W(t)$ of integral equation (2.22) we have (see e.g. [1, p.97]) $|W(t)| \le M_W, 0 < t < T$, where M_W is a constant that, obviously, does not depend on ε. Therefore, if

$$t \le \varepsilon^{3/4}, \tag{5.3}$$

using (2.13) and (2.24), for B we have

$$|B| \le M_W r_1 \frac{15}{2} \int_0^t \frac{d\tau}{\sqrt{4\pi\alpha(t-\tau)}} \le M_W r_1 \frac{15}{2} \sqrt{\frac{t}{\pi\alpha}},$$

and, hence, for (5.3)

$$|B| \le 15 M_W r_1 (4\pi\alpha)^{-1/2} \varepsilon^{3/8}. \tag{5.4}$$

If $t > \varepsilon^{3/4}$, we introduce

$$t_1 = t - \varepsilon^{3/4}, \tag{5.5}$$

$$D_1 = \left\{0 < \tau < t_1 : \frac{\varepsilon}{\alpha}\left(\frac{r(t) - r(\tau)}{t - \tau}\right)^2 \leq \frac{1}{4}\right\},$$

$$D_2 = \left\{0 < \tau < t_1 : \frac{\varepsilon}{\alpha}\left(\frac{r(t) - r(\tau)}{t - \tau}\right)^2 > \frac{1}{4}\right\}.$$

Then, we note that

$$B \leq B_1 + B_2 + B_3, \tag{5.6}$$

where

$$B_1 = M_W \int_{D_1} |V_x(r(t) - r(\tau), t - \tau) - E_x(r(t) - r(\tau), t - \tau)| d\tau,$$

$$B_2 = M_W \int_{D_2} |V_x(r(t) - r(\tau), t - \tau) - E_x(r(t) - r(\tau), t - \tau)| d\tau,$$

$$B_3 = M_W \int_{t_1}^{t} |V_x(r(t) - r(\tau), t - \tau) - E_x(r(t) - r(\tau), t - \tau)| d\tau.$$

We, first, compute this last integral obtaining

$$B_3 \leq 15 M_W r_1 \int_{t_1}^{t} \frac{d\tau}{\sqrt{16\pi\alpha(t - \tau)}} \leq 15 M_W r_1 \sqrt{\frac{t - t_1}{4\pi\alpha}},$$

that is

$$B_3 \leq 15 r_1 M_W (4\pi\alpha)^{-1/2} \varepsilon^{3/8}. \tag{5.7}$$

For B_2, recalling (2.24) and the definition of D_2, we have

$$|B_2| \leq \frac{15 r_1}{2} M_W \int_{D_2} E(r(t) - r(\tau), t - \tau) d\tau \leq \frac{15 r_1 M_W}{4\sqrt{\pi\alpha}} \int_0^{t_1} \frac{e^{-\frac{t-\tau}{16\varepsilon}}}{\sqrt{t - \tau}} d\tau;$$

hence,

$$|B_2| \le \frac{15 r_1 M_W}{4\sqrt{\pi\alpha}} \left(\frac{4\varepsilon}{e}\right)^{1/4} \int_0^{t_1} \frac{d\tau}{(t-\tau)^{3/4}} \le \tag{5.8}$$

$$\le \frac{15 r_1 M_W}{\sqrt{\pi\alpha}} \left(\frac{4T}{e}\right)^{1/4} \varepsilon^{1/4}.$$

In order to discuss B_1 we divide this term into three integrals

$$B_1 \le B_{11} + B_{12} + B_{13} \tag{5.9}$$

by setting

$$B_{11} = M_W \frac{r_1}{2} \int_{D_1} \frac{e^{-\frac{t-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} \left| \frac{I_1\left(\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2}\right)}{\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2\right]^{1/2}} + \right.$$

$$\left. - \frac{e^{\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2}}}{\sqrt{\pi(t-\tau)/\varepsilon}\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2\right]^{3/4}} \right| d\tau,$$

$$B_{12} = \frac{M_W r_1}{2} \int_{D_1} \frac{e^{-\frac{t-\tau}{2\varepsilon}} e^{\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2}}}{\sqrt{4\pi\alpha(t-\tau)}} \cdot$$

$$\left| \frac{1}{\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2\right]^{3/4}} - 1 \right| d\tau,$$

$$B_{13} = \frac{r_1 M_W}{2} \int_{D_1} \left| \frac{e^{-\frac{t-\tau}{2\varepsilon}} e^{\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2}}}{\sqrt{4\pi\alpha(t-\tau)}} - E(r(t)-r(\tau), t-\tau) \right| d\tau.$$

For B_{11}, recalling the known property for the modified Bessel functions [16]

$$(5.10)\qquad |I_n(z) - e^z/\sqrt{2\pi z}| \leq K/z,\ K = \text{constant}, z > 0,$$

we get

$$B_{11} \leq \frac{K}{2} r_1 M_W \int_{D_1} \frac{e^{-\frac{t-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} \frac{2\varepsilon}{t-\tau} \left[1 - \frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2\right]^{-1} d\tau,$$

from which, for the definition of D_1 and for (5.5)

$$(5.11)\qquad B_{11} \leq \frac{2r_1 M_W K\varepsilon}{3\sqrt{e\alpha}} \int_0^{t_1} \frac{d\tau}{(t-\tau)^{3/2}} \leq \frac{4r_1 M_W K}{3\sqrt{e\alpha}} \varepsilon^{5/8}.$$

Moreover, for B_{12} we deduce

$$B_{12} \leq r_1 M_W \left(\frac{4}{3}\right)^{3/4} \int_{D_1} \frac{e^{-\frac{(r(t)-r(\tau))^2}{4\alpha(t-\tau)}}}{\sqrt{\pi\alpha}(t-\tau)^{3/2}} \frac{\varepsilon(r(t)-r(\tau))^2}{4\alpha(t-\tau)} d\tau.$$

Hence

$$(5.12)\qquad B_{12} \leq \frac{2r_1 M_W}{\sqrt{\alpha\pi}} \left(\frac{4}{3}\right)^{3/4} \varepsilon^{5/8}.$$

Finally, for B_{13} we achieve

$$B_{13} \leq \frac{r_1 M_W}{2} \int_{D_1} E(r(t)-r(\tau), t-\tau) \frac{t-\tau}{2\varepsilon} \cdot$$
$$\cdot \left\{1 - \frac{\varepsilon}{2\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2 - \sqrt{1 - \frac{\varepsilon}{\alpha}\left(\frac{r(t)-r(\tau)}{t-\tau}\right)^2}\right\} d\tau,$$

and, therefore,

$$B_{13} \leq \frac{r_1 M_W \varepsilon}{2} \int_{D_1} E(r(t)-r(\tau), t-\tau) \frac{t-\tau}{4\alpha^2} \left(\frac{r(t)-r(\tau)}{t-\tau}\right)^4,$$

that is

$$B_{13} \leq \frac{2r_1 M_W}{e^2\sqrt{\pi\alpha}} \varepsilon^{5/8}. \tag{5.13}$$

Now, we note that the inequality (5.2) easily follows from (5.6)-(5.13). Thus, the theorem is proved.

We define

(5.14)

$$B' = e^{-\frac{t}{2\varepsilon}} \varphi'\left(r(t) + t\sqrt{\frac{\alpha}{\varepsilon}}\right) + 2\varepsilon\psi(0)V(r(t),t) +$$

$$-2\int_0^{+\infty} \varphi'(x)E(r(t)-x,t)dx +$$

$$+2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} [\varepsilon\psi'(x)V(r(t)-x,t) + \varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)]dx,$$

and state

THEOREM 5.2. *Assume the hypotheses (2.8)-(2.13) are fulfilled. Then, there exists a constant K'_B independent of x_0, t_0 and ε such that*

$$|B'| \leq K'_B \varepsilon^{q'_B}, \tag{5.15}$$

where q'_B is a strictly positive rational number.

PROOF. From (5.14) we immediately have

$$B' = B'_1 + B'_2 + B'_3, \tag{5.16}$$

where

(5.17)
$$B_1' = e^{-t/2\varepsilon}\varphi'\left(r(t) + t\sqrt{\alpha/\varepsilon}\right),$$

(5.18)
$$B_2' = 2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} \varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx + \\ -2\int_0^{+\infty}\varphi'(x)E(r(t)-x,t)dx,$$

(5.19)
$$B_3' = 2\varepsilon\left[\psi(0)V(r(t),t) + \int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}\psi'(x)V(r(t)-x,t)dx\right].$$

The term B_3' is the simplest to compute. Indeed, by integrating by parts and using (2.24) we get

$$|B_3'| \le M_\psi\left[\sqrt{\varepsilon/\alpha} + \frac{14\varepsilon}{\alpha t}\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} |r(t)-x|E(r(t)-x,t)dx\right].$$

Hence,

(5.20)
$$|B_3'| \le \frac{M_\psi}{\sqrt{\alpha}}\left[14\sqrt{2/e\pi}+1\right]\varepsilon^{1/2}.$$

Consider, now, B_2' and note that

(5.21)
$$|B_2'| \le 2M_\varphi'\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} |(1+\varepsilon\partial_{t_0})V(r(t)-x,t) + \\ -E(r(t)-x,t)|dx + 2M_\varphi'\int_{r(t)+t\sqrt{\alpha/\varepsilon}}^{+\infty} E(r(t)-x,t)dx.$$

If $r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}<0$, setting

$$B'_{21}=2M'_\varphi \int\limits_{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}}^{r(t)+\frac{t}{2}\sqrt{\alpha/\varepsilon}} |(1+\varepsilon\partial_{t_0})V(r(t)-x,t)-E(r(t)-x,t)|dx,$$

$$B'_{22}=22M'_\varphi \int\limits_{r(t)+\frac{t}{2}\sqrt{\alpha/\varepsilon}}^{+\infty} E(r(t)-x,t)dx,$$

from (5.18) one has

$$|B'_2|\le B'_{21}+B'_{22}.$$

If, instead, $r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}\ge 0$ and, therefore $r(t)/\sqrt{4t\alpha}\ge\sqrt{t/16\varepsilon}$, then

$$|B'_2|\le B'_{21}+B'_{22}+B'_{23}, \tag{5.22}$$

where

$$B'_{23}=2M'_\varphi \int\limits_{0}^{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}} |(1+\varepsilon\partial_{t_0})V(r(t)-x,t)-E(r(t)-x,t)|dx.$$

We evaluate the last integral and, recalling (2.23), (2.24), we obtain

$$B'_{23}=20M'_\varphi \int\limits_{0}^{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}} E(r(t)-x,t)|dx\le$$

$$\le\frac{20}{\sqrt{\pi}}M'_\varphi \int\limits_{\sqrt{t/16\varepsilon}}^{r(t)/\sqrt{4t\alpha}} e^{-p^2}dp,$$

with $p=(r(t)-x)/\sqrt{4\alpha t}$. Hence,

$$B'_{23}\le\frac{20}{\sqrt{\pi}}M'_\varphi e^{-t/32\varepsilon}\int\limits_{0}^{+\infty} e^{-p^2/2}\le 320M'_\varphi\sqrt{2}\varepsilon/t. \tag{5.23}$$

Similarly, the term B'_{22} can be computed

$$B'_{22} \le 342 M'_\varphi \sqrt{2}\varepsilon/t. \tag{5.24}$$

To estimate B'_{21} is a little more difficult. First, we divide B'_{21} into three terms

$$|B'_{21}| \le B'_{211} + B'_{212} + B'_{213} \tag{5.25}$$

by setting

$$B'_{211} = M'_\varphi \int\limits_{r(t)-\frac{t}{2}\sqrt{\frac{\alpha}{\varepsilon}}}^{r(t)+\frac{t}{2}\sqrt{\frac{\alpha}{\varepsilon}}} 2\Bigg|(1+\varepsilon\partial_{t_0})V(r(t)-x,t)+$$

$$-\frac{e^{-t/2\varepsilon}}{\sqrt{4\pi\alpha t}}\frac{e^{\frac{t}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2}}}{\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2\right]^{1/4}} - \frac{e^{-t/2\varepsilon}}{\sqrt{4\pi\alpha t}}\frac{e^{\frac{t}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2}}}{\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2\right]^{3/4}}\Bigg| dx,$$

$$B'_{212} = M'_\varphi \int\limits_{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}}^{r(t)+\frac{t}{2}\sqrt{\alpha/\varepsilon}} e^{\frac{t}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2}}\frac{e^{-t/2\varepsilon}}{\sqrt{4\pi\alpha t}}\Bigg|\frac{1}{\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2\right]^{1/4}}+$$

$$+\frac{1}{[1-\frac{\varepsilon}{\alpha}(\frac{r(t)-x}{t})^2]^{3/4}} - 2\Bigg| dx,$$

$$B'_{213} = 2M'_\varphi \int\limits_{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}}^{r(t)+\frac{t}{2}\sqrt{\alpha/\varepsilon}} \Bigg|\frac{e^{-t/2\varepsilon}}{\sqrt{4\pi\alpha t}}e^{\frac{t}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2}} - E(r(t)-x,t)\Bigg| dx.$$

Then, recalling (5.10), for B'_{211} one has

$$B'_{211} \le KM'_\varphi \int\limits_{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}}^{r(t)+\frac{t}{2}\sqrt{\alpha/\varepsilon}} \frac{e^{-\frac{t}{2\varepsilon}}}{t}\sqrt{\frac{\varepsilon}{\alpha}}\left[\frac{1}{\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2}}+\frac{1}{1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2}\right] dx.$$

Hence, using the inequality $[1-\varepsilon(r(t)-x)^2/(\alpha t^2]^{-1} < 4/3$ that immediately follows from $r(t)-\sqrt{\alpha/\varepsilon}\,t/2 < x < r(t)+\sqrt{\alpha/\varepsilon}\,t/2$, we get

$$B'_{211} \leq \frac{8KM'_\varphi}{3\sqrt{e}}\sqrt{\varepsilon/t} \tag{5.26}$$

For B'_{212} we note that

$$B'_{212} \leq M'_\varphi \int\limits_{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}}^{r(t)+\frac{t}{2}\sqrt{\alpha/\varepsilon}} E(r(t)-x,t)\left\{\frac{1-\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2\right]^{1/4}}{\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2\right]^{1/4}}+\right.$$
$$\left.+\frac{1-\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2\right]^{3/4}}{\left[1-\frac{\varepsilon}{\alpha}\left(\frac{r(t)-x}{t}\right)^2\right]^{3/4}}\right\}dx,$$

and, therefore

$$B'_{212} \leq \frac{4M'_\varphi}{\sqrt{\pi}}\left(\frac{4}{3}\right)^{3/4}\sqrt{\varepsilon/t}. \tag{5.27}$$

Moreover, by applying to B'_{213} the same arguments used for B_{13} one obtains

$$B'_{213} \leq 2M'_\varphi \int\limits_{r(t)-\frac{t}{2}\sqrt{\alpha/\varepsilon}}^{r(t)+\frac{t}{2}\sqrt{\alpha/\varepsilon}} E(r(t)-x,t)\frac{t\varepsilon}{4\alpha^2}\left(\frac{r(t)-x}{t}\right)^4 dx,$$

from which

$$B'_{213} \leq \frac{8M'_\varphi}{e^2\sqrt{\pi}}\sqrt{\varepsilon/t}. \tag{5.28}$$

Finally, for B'_1 we easily have

$$B'_1 \leq 2M'_\varphi\varepsilon/t.$$

From this last inequality and from (5.14), (5.16), (5.20), (5.22)-(5.28) we deduce

$$|B'| \leq h'_B\varepsilon/t + k'_B\sqrt{\varepsilon/t} + l'_B\sqrt{\varepsilon}.$$

Consequently, if we assume $t > \varepsilon^{3/4}$, (5.15) is fulfilled and the theorem is proved. But, it holds also if

(5.29) $$t \leq \varepsilon^{3/4}.$$

Indeed, let

(5.30) $$B'_1 + B'_2 = B'_{01} + B'_{02} + B'_{03}.$$

where, $(\varphi(x) = -\varphi(-x),\ x < 0)$,

$$B'_{01} = e^{-t/2\varepsilon}\varphi'\left(r(t) + t\sqrt{\frac{\alpha}{\varepsilon}}\right) - \varphi'(0) + \\ + \int_{r(t)}^{r(t)+t\sqrt{\alpha/\varepsilon}} \varphi'(x)2(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx,$$

$$B'_{02} = 2\int_{r(t)}^{+\infty}[\varphi'(0) - \varphi'(x)]E(r(t)-x,t)dx,$$

$$B'_{03} = 2\int_0^{r(t)} \varphi'(x)[(1+\varepsilon\partial_{t_0})V(r(t)-x,t) - E(r(t)-x,t)]dx.$$

For B'_{01} we achieve

$$B'_{01} = e^{-\frac{t}{2\varepsilon}}\varphi'\left(r(t) + t\sqrt{\frac{\alpha}{\varepsilon}}\right) - \varphi'(0) + \\ + e^{-\frac{t}{2\varepsilon}}\varphi'(\bar{x})[\sinh(t/2\varepsilon) + \cosh(t/2\varepsilon) - 1],$$

with

(5.31) $$r(t) < \bar{x} < r(t) + t\sqrt{\alpha/\varepsilon}.$$

Hence, for (2.13) and (5.29), we get

$$|B'_{01}| \leq 6M''_{\varphi}t\sqrt{\alpha/\varepsilon} \leq 6M''_{\varphi}\sqrt{\alpha}\varepsilon^{1/4}. \tag{5.32}$$

Considering that hypothesis (5.29) implies $|r(t)| \leq \sqrt{t\alpha/\varepsilon^{1/4}}$, for B'_{02} one has

$$|B'_{02}| \leq 2 \int_{r(t)}^{r(t)+\sqrt{t\alpha/\varepsilon^{1/4}}} |\varphi'(x) - \varphi'(0)|E(r(t)-x),t)dx +$$
$$+\, 2 \int_{r(t)+\sqrt{t\alpha/\varepsilon^{1/4}}}^{+\infty} |\varphi'(x) - \varphi'(0)|E(r(t)-x),t)dx,$$

from which, setting $z = (x - r(t))/\sqrt{4t\alpha}$,

$$|B'_{02}| \leq M''_{\varphi} \int_{r(t)}^{r(t)+\sqrt{t\alpha/\varepsilon^{1/4}}} \frac{|x|}{\sqrt{t\pi\alpha}}dx + M'_{\varphi}\frac{4}{\sqrt{\pi}} \int_{1/\sqrt{4\varepsilon^{1/4}}}^{+\infty} e^{-z^2}dz,$$

and therefore

$$|B'_{02}| \leq M''_{\varphi}\frac{5}{2}\sqrt{\alpha/\pi}\varepsilon^{1/8} + 16\sqrt{2}M'_{\varphi}\varepsilon^{1/4}. \tag{5.33}$$

Moreover, using (2.23), (2.24), we obtain

$$|B'_{03}| \leq 10M'_{\varphi}r_1\sqrt{\frac{t}{\pi\alpha}} \leq \frac{10}{\sqrt{\pi\alpha}}M'_{\varphi}r_1\varepsilon^{3/8}. \tag{5.34}$$

From (5.30), (5.32)-(5.34) we see that (5.15) is still satisfied. Thus, the theorem is completely proved.

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