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# On certain sequence spaces and their Köthe-Toeplitz duals

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RIASSUNTO: *In questo lavoro si introducono spazi generali di successioni $X_v$, rispetto ad una successione v che funziona da peso, a partire da uno spazio di successioni $X$. Vengono stabiliti risultati di completezza e condizioni necessarie e sufficienti su $X$ affinché $X_v$ sia riflessivo secondo Köthe-Toeplitz o perfetto. Particolare attenzione è dedicata al caso in cui $X$ sia uno degli spazi di successioni $\ell(p), c_0(p)$ o $\ell_\infty(p)$ o $c(p)$: vengono estesi risultati precedenti di altri autori.*

ABSTRACT: *In this paper we introduce a general sequence space $X_v$, where $X$ is any sequence space and establish some inclusion relations, topological results. Furthermore we give $\alpha$- and $\beta$-duals of sequence spaces $[\ell(p)]_v, [c_0(p)]_v, [\ell_\infty(p)]_v$ and $[c(p)]_v$ together with $\alpha$-duals of sequence spaces $\ell(p), c_0(p), \ell_\infty(p)$ and $c(p)$. The perfectness of sequence spaces $[\ell(p)]_v, [\ell_\infty(p)]_v$ and $[c_0(p)]_v$ is also examined in this paper. Our results include the corresponding results of* DUTTA *([1]),* SRIVASTAVA *([9], [10])* GURUSINGH *([2]) and others.*

KEY WORDS: *Sequence spaces - $\alpha, \beta, \gamma$-duals - Perfectness.*

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## 1 – Introduction

Let $(v_k)$ be any fixed sequence of nonzero complex numbers satisfying

$$\liminf_k |v_k|^{1/k} = r\,, \qquad (0 < r \leq \infty)$$

and let $X$ be any sequence space. Then we define $X_v$ by

$$X_v = \{x = (x_k) \colon (v_k x_k) \in X\}.$$

Define a function $\Lambda \colon \mathbb{C} \longrightarrow \mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers, by

$$\Lambda(z) = \sum_k \frac{z^k}{v_k}.$$

(Throughout this paper $\sum_k$ will mean summation from $k = 1$ to $k = \infty$). Obviously $\Lambda$ is an analytic function in the disc $E_r = \{z \colon |z| \leq r\}$.

We now define

$$X\hat{} = \{f \colon f(z) = \sum_k x_k z^k \text{ such that } (v_k x_k) \in X\}.$$

It is easy to show that there exists an algebraic isomorphism between $X_v$ and $X\hat{}$ in the sense that $f \leftrightarrow x = (x_k)$ is an algebraic isomorphism. Therefore $X_v$ can be regarded as a set of analytic functions.

In this study $\ell_\infty$, $c$ and $c_0$ will denote the sequence spaces of bounded, convergent and null sequences. Throughout the paper, unless otherwise indicated, $p = (p_k)$ will denote a sequence of strictly positive numbers (not necessarily bounded in general). If we take $X = \ell_\infty(p)$ then $X_v = D\hat{}_\infty(p)$ [1], if $X = \ell(p)$ then $X_v = D\hat{}(p)$ [10], and if $X = c_0(p)$ then $X_v = D\hat{}_0(p)$ [1]. If we take $(v_k) = (k^s)$, where $s$ denotes any real number, then $X_v = E_s(X)$ [2]. Therefore this study generalizes the corresponding results of DUTTA [1], GURUSINGH [2], SRIVASTAVA et al [9,10].

For the definition of spaces $\ell_\infty(p), c_0(p), c(p)$ and $\ell(p)$, see ([8],[6],[7]). If all the terms of $(p_k)$ are constant and all equal to $p > 0$ then we have $\ell_\infty(p) = \ell_\infty, c_0(p) = c_0, c(p) = c$ and $\ell(p) = \ell_p$, where $\ell_p$ is the space of $p$-summable sequences.

In this paper we give some topological relations between $X$ and $X_v$, and also we give the $\alpha$- $\beta$- and $\gamma$-duals of $X_v$ in terms of the $\alpha$-, $\beta$- and $\gamma$-duals of $X$.

## 2 – Some topological properties of $X_v$

In this section we give some relations between $X_v$ and $X$, and we discuss some properties of $X_v$.

THEOREM 1. *If $X$ is a complete paranormed space, then $X_v$ is also a complete paranormed space.*

PROOF. Since $(0) \in X_v, X_v \neq \emptyset$. It is easy to check that $X_v$ is a linear space. And also it is clear that the function defined by

$$g^*(x) = g(vx),$$

where $g$ is the paranorm in $X$, satisfies that $g^*(0) = 0, g^*(x) = g^*(-x)$ and $g^*(x+y) \leq g^*(x)+g^*(y)$. Now clearly $\lambda_n \longrightarrow \lambda$ in $\mathbb{C}$ and $g^*(x^n-x) \longrightarrow 0$ as $n \longrightarrow \infty$ imply that $g^*(\lambda_n x^n - \lambda x) \longrightarrow 0$ as $n \longrightarrow \infty$, where $x^n = (x_k^n)_k = (x_1^n, x_2^n, x_3^n, \dots, x_k^n, \dots)$ and $x = (x_k)$, therefore $g^*$ is a paranorm in $X_v$.

To show that $X_v$ is complete, let $(x^n)$ be a Cauchy sequence in $X_v$, where $x^n = (x_1^n, x_2^n, \dots) \in X_v$. Then $(vx^n) = ((v_k x_k^1), (v_k x_k^2), \dots)$ is a Cauchy sequence in $X$. Since $X$ is complete, it converges to $(z_k)$ say. Let $z_k = v_k x_k$, so that $x_k = v_k^{-1} z_k$. Then $(vx^n)$ converges to $(v_k x_k)$ in $X$.

Hence

$$g((v_k x_k^n) - (v_k x_k)) = g(v(x^n - x)) \longrightarrow 0$$

as $n \longrightarrow \infty$, which implies that

$$g^*(x^n - x) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore $(x^n)$ is convergent, consequently $X_v$ is a complete paranormed space.

COROLLARY 1. *If $X$ is a Banach space then so is $X_v$. Here the norm in $X_v$ is defined by*

$$\|x\|_v = \|(v_k x_k)\|,$$

*where $\|\ \|$ is the norm in $X$.*

COROLLARY 2.

(i) $(\ell_\infty)_v$ *is a BK-space with the norm*

$$\|x\| = \sup_k |v_k x_k| . \tag{2.1}$$

(ii) $(c_0)_v$ *and* $c_v$ *are BK-spaces with norm* (2.1).

(iii) $(\ell_p)_v$ $(1 \le p < \infty)$ *is a BK-space with the norm defined by*

$$\|x\| = \left( \sum_k |v_k x_k|^p \right)^{1/p} .$$

COROLLARY 3.

(i) $[c_0(p)]_v$ *is a complete paranormed space with the paranorm defined by*

$$g_1(x) = \sup_k |v_k x_k|^{p_k/M} \tag{2.2}$$

*where* $M = \max\{1, \sup p_k\}$.

(ii) $[\ell_\infty(p)]_v$ *and* $[c(p)]_v$ *are complete paranormed spaces with the paranorm* (2.2) *if* $\inf p_k > 0$.

(iii) $[\ell(p)]_v$ *is a complete paranormed space with the paranorm defined by*

$$g_2(x) = \left[ \sum_k |v_k x_k|^{p_k} \right]^{1/M} .$$

PROOF. Proof follows from Theorem 1. A direct proof of (iii) may be found in [10] and for $[c_0(p)]_v$, $[\ell_\infty(p)]_v$ see [1].

LEMMA 1.

(i) *If* $X \subset Y$ *then* $X_v \subset Y_v$ *and* $X_{v^*} \subset Y_{v^*}$, *where* $X_{v^*} = \{x = (x_k) : (x_k v_k^{-1}) \in X\}$.

(ii) $(\cup_i X_i)_v = \cup_i (X_i)_v$,

(iii) $(\cap_i Y_i)_v = \cap_i (Y_i)_v$.

PROOF. It is easy and hence omitted.

THEOREM 2. *Let $X$ be a complete paranormed space and let $Z$ be a closed subset of $X$. Then $Z_v$ is a closed subset of $X_v$.*

PROOF. Since $Z \subset X$, $Z_v \subset X_v$ by Lemma 1. Now let $x \in \overline{(Z_v)}$, then there exists a sequence $(x^n) \subset Z_v$ such that $(x^n)$ converges to $x$. This implies that

$$g^*(x^n - x) = g^*((x_k^n) - (x_k)) \longrightarrow 0$$

as $n \longrightarrow \infty$ in $Z_v$. Thus

$$g((v_k x_k^n) - (v_k x_k)) \longrightarrow 0$$

as $n \longrightarrow \infty$ in $Z$. Hence $(v_k x_k)$ is the limit of a sequence of points in $Z$. Therefore $(v_k x_k) \in \overline{Z}$ which gives that $x \in (\overline{Z})_v$. Conversely if $x \in (\overline{Z})_v$ then $x \in \overline{(Z_v)}$. Since $Z$ is closed $\overline{Z} = Z$. Therefore $\overline{(Z_v)} = (\overline{Z})_v = Z_v$, hence $Z_v$ is closed in $X_v$.

COROLLARY 4. *Let $X$ be a Banach space and $Z$ be a closed subset of $X$. Then $Z_v$ is a closed subset of $X_v$.*

THEOREM 3. *If $X$ is a separable space then so is $X_v$.*

PROOF. Let $X$ be a separable space. Then there exists a countable subset $Z$ of $X$ such that $\overline{Z} = X$. Then $\overline{(Z_v)} = X_v$ by Theorem 2. Hence $Z_v$ is dense in $X_v$. Let us define $f\colon Z_v \longrightarrow Z$ by $f(x) = (v_k x_k)$. It is clear that $f$ is bijective. Since $Z$ is countable, $Z_v$ is also a countable subset of $X_v$. Hence $X_v$ is separable.

THEOREM 4. *If $X$ is a Hilbert space then $X_v$ is also a Hilbert space.*

PROOF. Let $X$ be a Hilbert space. If we define the inner product $\langle \cdot, \cdot \rangle_v$ in $X_v$ by

$$\langle x, y \rangle_v = \langle (v_k x_k), (v_k y_k) \rangle, \qquad (x, y \in X_v)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $X$, then $X_v$ becomes an inner product space. It is easy to show that $\langle \cdot, \cdot \rangle_v$ satisfies the conditions of inner product. And also it is clear that $X_v$ is complete. Hence $X_v$ is a Hilbert space.

REMARK: $X_v$ needs not to be a sequence algebra if $X$ is so. Indeed, it is known that $c_0$ is a sequence algebra. But $(c_0)_v$ is not a sequence algebra for $(v_k) = (1/k)$. Let $x = (\sqrt{k})$ and $y = (\lambda\sqrt{k})$, where $\lambda \in \mathbb{C}$ is a constant. Then $x, y \in (c_0)_v$, but $z \notin (c_0)_v$, where $z = (x_k y_k)$.

## 3 – Köthe-Toeplitz duals of $X_v$

In this section first we give the $\alpha$-duals of $\ell_\infty(p)$, $c_0(p)$, $c(p)$ and $\ell(p)$, and we discuss the second duals and perfectness of some sequence spaces. Then we characterize the $\alpha$-, $\beta$- and $\gamma$-duals of $X_v$ in terms of the $\alpha$-, $\beta$- and $\gamma$-duals of $X$, respectively. We also discuss the second duals and perfectness of $X_v$ for various sequence spaces $X$. It may be noted here that $\beta$-duals of $\ell_\infty(p)$, $c_0(p), c(p)$ and $\ell(p)$ have been characterized in [4] and [6].

DEFINITION 1. *Let $X$ be a sequence space and define*
(i) $X^\alpha = \{a = (a_k)\colon \sum_k |a_k x_k| < \infty$ *for all* $x \in X\}$,
(ii) $X^\beta = \{a = (a_k)\colon \sum_k a_k x_k$ *converges for all* $x \in X\}$,
(iii) $X^\gamma = \{a = (a_k)\colon \sup_n |\sum_{k=1}^n a_k x_k| < \infty$ *for all* $x \in X\}$.

Then $X^\alpha$, $X^\beta$, and $X^\gamma$ are called the $\alpha$–, $\beta$- and $\gamma$-dual spaces of $X$, respectively. $X^\alpha$ is also called Köthe-Toeplitz dual space and $X^\beta$ is also called generalized Köthe-Toeplitz dual space. It is easy to show that $\emptyset \subset X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$ then $Y^\eta \subset X^\eta$ for $\eta = \alpha, \beta$ or $\gamma$. Also for a sequence space $X$ it is clear that $X \subset (X^\eta)^\eta = X^{\eta\eta}$, where $\eta = \alpha, \beta$ or $\gamma$.

DEFINITION 2. *For a sequence space $X$, if $X = X^{\eta\eta}$ then $X$ is called an $\eta$-space, where $\eta = \alpha, \beta$ or $\gamma$. In particular an $\alpha$-space is called Köthe space or a perfect sequence space.*

THEOREM 5. *Let $\eta$ denote $\alpha, \beta$ or $\gamma$. Then*
(i) *If the $\eta$-dual $X^\eta$ exists, then $(X_v)^\eta$ exists and*

$$(X_v)^\eta = \left\{a = (a_k)\colon \left(\frac{a_k}{v_k}\right) \in X^\eta\right\} = (X^\eta)_{v^*},$$

(ii) *If $X^{\eta\eta}$ exists, then $(X_v)^{\eta\eta}$ exists and*

$$(X_v)^{\eta\eta} = \left\{x = (x_k) : (v_k x_k) \in X^{\eta\eta}\right\} = (X^{\eta\eta})_v .$$

PROOF. (i) Let $\eta = \alpha$ and $D = \{a = (a_k) : (a_k/v_k) \in X^\alpha\}$. We show that $(X_v)^\alpha = D$. Let $a \in (X_v)^\alpha$ then $\sum_k |a_k x_k| < \infty$ for every $x \in X_v$ so that

$$\sum_k \left|\frac{a_k}{v_k} v_k x_k\right| = \sum_k |a_k x_k| < \infty .$$

Since $(v_k x_k) \in X$ it follows that $(a_k/v_k) \in X^\alpha$ which implies that $a \in D$. Hence $(X_v)^\alpha \subset D$.

Conversely, if $a \in D$ and $x \in X_v$, then $(a_k/v_k) \in X^\alpha$ and $(v_k x_k) \in X$ so that

$$\sum_k |a_k x_k| = \sum_k \left|\frac{a_k}{v_k} v_k x_k\right| < \infty .$$

Since $x \in X_v$ it follows that $a \in (X_v)^\alpha$. Hence $D \subset (X_v)^\alpha$. Consequently $(X_v)^\alpha = (X^\alpha)_{v^*}$.

For $\eta = \beta$ and $\eta = \gamma$ the proofs are similar; therefore we omit them.

(ii) Let $\eta = \alpha$ and let $X^{\alpha\alpha}$ exist. Then, by (i):

$$(X_v)^{\alpha\alpha} = [(X_v)^\alpha]^\alpha = [(X^\alpha)_{v^*}]^\alpha = (X^{\alpha\alpha})_v .$$

For $\eta = \beta$ and $\eta = \gamma$ the proof is same.

THEOREM 6. *$X_v$ is an $\eta$-space if and only if $X$ is an $\eta$-space, where $\eta = \alpha, \beta$ or $\gamma$.*

PROOF. Let $X$ be an $\eta$-space. Then $X^{\eta\eta} = X$. Now $(X_v)^{\eta\eta} = (X^{\eta\eta})_v$ by Theorem 5(ii), and hence $(X_v)^{\eta\eta} = X_v$. Thus $X_v$ is an $\eta$-space.

Conversely if $X_v$ is an $\eta$-space then $(X_v)^{\eta\eta} = X_v$ which implies that $(X^{\eta\eta})_v = X_v$ by Theorem 5(ii). From Lemma 1 it follows that $X^{\eta\eta} = X$, that is, $X$ is an $\eta$-space.

THEOREM 7. *Let $p_k > 1$ for all $k$. Then*
(i) *$\ell^\alpha(p) = M(p)$, where*

$$M(p) = \bigcup_{N>1} \left\{ a = (a_k) \colon \sum_k |a_k|^{q_k} N^{-q_k/p_k} < \infty \right\},$$

(ii) $$[(\ell(p))_v]^\eta = \bigcup_{N>1} \left\{ a = (a_k) \colon \sum_k |\tfrac{a_k}{v_k}|^{q_k} N^{-q_k/p_k} < \infty \right\},$$
*where $\eta = \alpha$ or $\beta$, and $p_k^{-1} + q_k^{-1} = 1$.*
(iii) *The following statements are equivalent:*
(1) *$\ell(p)$ is perfect,*
(2) *$[\ell(p)]_v$ is perfect,*
(3) *$p \in \ell_\infty$.*

PROOF. (i) Let $a \in M(p)$ and $x \in \ell(p)$. From the inequality

$$|x_k y_k| \leq |x_k|^{q_k} + |y_k|^{p_k}$$

(see [6]) we obtain

$$|a_k x_k| \leq |a_k|^{q_k} N^{-q_k/p_k} + N|x_k|^{p_k} \text{ for some } N > 1,$$

whence $\sum_k |a_k x_k| < \infty$ and so that $a \in \ell^\alpha(p)$. Hence $M(p) \subset \ell^\alpha(p)$. Since $\ell^\alpha(p) \subset \ell^\beta(p)$ it follows that $\ell^\alpha(p) = M(p)$ by Theorem 4 in [6].

(ii) Proof follows from (i), Theorem 5(i) and Theorem 1 in [6].

(iii) (1) is equivalent to (2) by Theorem 6. Therefore we only show that (1) is equivalent to (3). Let $p \in \ell_\infty$. Since $\ell^\alpha(p) = \ell^\beta(p)$, then $\ell(p) \subset \ell^{\alpha\alpha}(p) \subset \ell^{\beta\beta}(p)$ which implies that $\ell^{\alpha\alpha}(p) = \ell(p)$. For $\ell^{\beta\beta}(p) = \ell(p)$, if $p \in \ell_\infty$, by Theorem 4(i) in [4]. Conversely if $\ell(p)$ is perfect, then it is a linear space and therefore $p \in \ell_\infty$.

THEOREM 8. *Let $0 < p_k \leq 1$ for every $k$. Then the following statements are equivalent:*
(1) *$\ell(p)$ is perfect,*
(2) *$[\ell(p)]_v$ is perfect,*
(3) *$\ell(p) = \ell_1$.*

PROOF. (1) is equivalent to (2) by Theorem 6. We show that (1) is equivalent to (3). It is easy to show that (3) implies (1). Now suppose that (1) holds, that is $\ell^{\alpha\alpha}(p) = \ell(p)$. Since $\ell^{\alpha}(p) = \ell_{\infty}(p)$ then $\ell^{\alpha\alpha}(p) = \ell^{\alpha}(p) = M_{\infty}(p)$. We shall show that $M_{\infty}(p) = \ell(p)$ implies $\inf p_k > 0$. Suppose that $M_{\infty}(p) = \ell(p)$ but $\inf p_k = 0$. Then there exists a strictly increasing sequence $(k_i)$ of positive integers such that $p_{k_i} < i^{-1}$. We put

$$a_k = \begin{cases} 0 & \text{if } k \neq k_i \\ i^{-1/p_k} & \text{if } k = k_i \end{cases} (i = 1, 2, \dots).$$

Then for every $N > 1$ we have, for $i > 2N$, $|a_k|^{p_k} = i^{-1}$ and $|a_k| N^{1/p_k} < i^{-1}$ where $k = k_i$. Therefore $a \in M_{\infty}(p) - \ell(p)$, contrary to the assumption that $M_{\infty}(p) = \ell(p)$. Hence $\inf p_k > 0$, which gives us $\ell(p) = \ell_1$. It is easy to check that $M_{\infty}(p) = \ell_1$ if and only if $\inf p_k > 0$.

THEOREM 9. *For every $p = (p_k)$ we have*

(i) $c_0^{\alpha}(p) = M_0(p)$, *where*

$$M_0(p) = \bigcup_{N>1} \left\{ a = (a_k) \colon \sum_k |a_k| N^{-1/p_k} < \infty \right\},$$

(ii) $$[(c_0(p))_v]^{\eta} = \bigcup_{N>1} \left\{ a = (a_k) \colon \sum_k \left| \frac{a_k}{v_k} \right| N^{-1/p_k} < \infty \right\}$$
*where $\eta = \alpha$ or $\beta$.*

(iii) $c_0^{\alpha\alpha}(p) = E_0$, *where*

$$E_0 = \bigcap_{N>1} \left\{ x = (x_k) \colon \sup_k |x_k| N^{1/p_k} < \infty \right\},$$

(iv) $$[(c_0(p))_v]^{\eta\eta} = \bigcap_{N>1} \left\{ x = (x_k) \colon \sup_k |v_k x_k| N^{1/p_k} < \infty \right\},$$
*where $\eta = \alpha$ or $\beta$.*

(v) *The following conditions are equivalent:*

(1) $c_0(p)$ *is perfect,*

(2) $[c_0(p)]_v$ *is perfect,*

(3) $p \in c_0$.

PROOF. (i) Let $a \in M_0(p)$ and $x \in c_0(p)$. Then $\sum_k |a_k| N^{-1/p_k} < \infty$ for some $N > 1$ and $|x_k|^{p_k} < N^{-1}$ for sufficiently large $k$; whence for such $k$ it follows that

$$|a_k x_k| \leq |a_k| N^{-1/p_k}$$

so $\sum_k |a_k x_k| \leq \sum_k |a_k| N^{-1/p_k} < \infty$ and hence $M_0(p) \subset c_0^\alpha(p)$. Since $c_0^\alpha(p) \subset c_0^\beta(p)$ it follows that $c_0^\alpha(p) = M_0(p)$ by Theorem 6 in [6].

(ii) Proof follows from (i), Theorem 5(i) and Theorem 6 in [6].

(iii) Let $a \in E_0$ and $x \in c_0^\alpha(p)$. Then for every $N > 1$, $|a_k| N^{1/p_k} \leq K$ for all $k$ and for some $K > 0$, and $\sum_k |x_k| N^{1/p_k} < \infty$ for some $N > 1$. Hence

$$|a_k x_k| \leq K |x_k| N^{-1/p_k}$$

which implies that

$$\sum_k |a_k x_k| \leq K \sum_k |x_k| N^{-1/p_k} < \infty,$$

consequently $a \in c_0^{\alpha\alpha}(p)$, whence $E_0 \subset c_0^{\alpha\alpha}(p)$. Since $c_0^\alpha(p) = c_0^\beta(p)$ by (i), it follows that $c_0^{\alpha\alpha}(p) \subset c_0^{\beta\beta}(p)$. Then we have $c_0^{\alpha\alpha}(p) = E_0$, since $c_0^{\beta\beta}(p) = E_0$ by Theorem 2 in [4].

Theorem 2 in [4], (iii) and Theorem 5(ii) give us(iv).

(v). (1) is equivalent to (2) by Theorem 6. Since $c_0(p)$ is a $\beta$-space if and only if $p \in c_0$ by Theorem 8 in [4] and since $c_0^{\alpha\alpha}(p) = c_0^{\beta\beta}(p)$ by (iv) the equivalence of (1) and (3) is immediate.

THEOREM 10. *For every $p = (p_k)$ we have*

(i) $\ell_\infty^\alpha(p) = M_\infty(p)$, *where*

$$M_\infty(p) = \bigcap_{N>1} \left\{ a = (a_k) : \sum_k |a_k| N^{1/p_k} < \infty \right\},$$

(ii) $[(\ell_\infty(p))_v]^\eta = \bigcap_{N>1} \left\{ a = (a_k) : \sum_k \left| \frac{a_k}{v_k} \right| N^{1/p_k} < \infty \right\}$
*where $\eta = \alpha$ or $\beta$.*

(iii) $\ell_\infty^{\alpha\alpha}(p) = E_\infty$, *where*

$$E_\infty = \bigcup_{N>1} \left\{ x = (x_k) : \sup_k |x_k| N^{-1/p_k} < \infty \right\},$$

(iv) $[(\ell_\infty(p))_v]^{\eta\eta} = \bigcup_{N>1} \left\{x = (x_k) \colon \sup_k |v_k x_k| N^{-1/p_k} < \infty\right\}$

*where* $\eta = \alpha$ *or* $\beta$.

(v) *The following conditions are equivalent:*

(1) $p \in \ell_\infty$,

(2) $\ell_\infty(p)$ *is perfect,*

(3) $[\ell_\infty(p)]_v$ *is perfect.*

PROOF.

(i) It is similar to the proof of Theorem 9(i), since $\ell_\infty^\beta(p) = M_\infty(p)$ by Theorem 2 in [5], therefore we omit it.

(i), Theorem 5(i) and Theorem 2 in [5] give us (ii). The proof of (iii) is similar to the proof of Theorem 9(iii), since $\ell_\infty^{\beta\beta}(p) = E_\infty$ by Theorem 3 in [4]. Theorem 3 in [4], (iii) and Theorem 5(ii) give us (iv). Using Theorem 5 in [4] and (iv) the proof of (v) is similar to the proof of Theorem 9(v).

THEOREM 11. *For every* $p = (p_k)$ *we have*

(i) $c^\alpha(p) = c_0^\alpha(p) \cap \ell_1$,

(ii) $[(c(p))_v]^\alpha = [M_0(p)]_{v^*} \cap (\ell_1)_{v^*}$,

(iii) $[(c(p))_v]^\beta = [M_0(p)]_{v^*} \cap \gamma_{v^*}$,

*where* $\gamma = \{a = (a_k) \colon \sum_k a_k$ *converges*$\}$.

PROOF. (i) Let $a \in c^\alpha(p) \cap \ell_1$ and $x \in c(p)$, $|x_k - \ell|^{p_k} \to 0$ (say). Then $\sum_k |a_k| < \infty$ and since $x \in c(p)$, $(x_k - \ell) \in c_0(p)$ and hence $\sum_k |a_k(x_k - \ell)| < \infty$. Now from the inequality

$$|a_k x_k| \le |a_k(x_k - \ell)| + |\ell a_k|$$

we obtain that $\sum_k |a_k x_k| < \infty$. Therefore $a \in c^\alpha(p)$.

Since $c_0(p) \subset c(p)$ it follows that $c^\alpha(p) \subset c_0^\alpha(p)$. Let $a \in c^\alpha(p)$, since $e = (1, 1, \dots) \in c(p)$, it follows that $\sum_k |a_k| < \infty$, so that $a \in \ell_1$. Hence $a \in c_0^\alpha(p) \cap \ell_1$. This completes the proof of (i).

(i) and Theorem 5(i) give us (ii); Theorem 5(i) and Theorem 1 in [4] give us (iii).

COROLLARY 5. (i). $[(c_0)_v]^\eta = (c_v)^\eta = [(\ell_\infty)_v]^\eta = (\ell_1)_{v^*}$,

(ii) $[(\ell_p)_v]^\eta = (\ell_q)_{v^*}$ $(1 < p < \infty, p^{-1} + q^{-1} = 1)$,

(iii) $[(\ell_p)_v]^\eta = (\ell_\infty)_{v^*}$ $(0 < p \le 1)$

*where* $\eta = \alpha, \beta$ *or* $\gamma$.

PROOF. Use $X^\eta$ for $X = c_0, c, \ell_\infty, \ell_p$ (see [3]) and apply Theorem 5(i).

COROLLARY 6. (i). $[(c_0)_v]^{\eta\eta} = (c_v)^{\eta\eta} = [(\ell_\infty)_v]^{\eta\eta} = (\ell_\infty)_v$,
(ii) $[(\ell_p)_v]^{\eta\eta} = (\ell_p)_v$ $(1 < p < \infty)$
(iii) $[(\ell_p)_v]^{\eta\eta} = (\ell_1)_v$ $(0 < p \leq 1)$
*where* $\eta = \alpha, \beta$ *or* $\gamma$.

PROOF. Use $X^{\eta\eta}$ for $X = c_0, c, \ell_\infty, \ell_p$ and apply Theorem 5(ii).


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