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Sasakian m-hyperbolic locally conformal Kähler manifolds

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RIASSUNTO: *Si studia una classe particolare di varietà Kähleriane localmente conformi e, come principale risultato, si dimostra che lo spazio di ricoprimento universale di tale varietà è il prodotto di una varietà c-Sasakiana con uno spazio iperbolico di dimensione dispari.*

ABSTRACT: *In this paper, we study a particular class of locally conformal Kähler manifolds and, as main result, we prove that the universal covering space of such manifolds is the product of a c-sasakian manifold with a hyperbolic space of odd dimension.*

KEY WORDS: *Locally conformal Kähler manifolds - Generalized Hopf manifolds - Sasakian manifolds - Kenmotsu manifolds - Hyperbolic space.*

A.M.S. CLASSIFICATION: 53C15 – 53C25 – 53C55

1 – Introduction

An almost Hermitian manifold V^{2n} is called locally conformal Kähler if its metric is conformally related to a Kähler metric in some neighbourhood of every point of V^{2n}. Such manifolds have been studied by various authors (see, for instance, [14], [23], [24], [25], [6], [16], [8], ...).

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Examples of locally conformal Kähler manifolds are provided by the generalized Hopf manifolds which are locally conformal Kähler manifolds with parallel Lee form (see [24] and [25]). The main non-Kähler example of such manifolds is the Hopf manifold (see [13], [23]), which is defined as the quotient

$$H_o^n = \frac{(\mathbb{C}^n - \{0\})}{\Delta_\lambda}$$

where Δ_λ is a cyclic group of transformations. Another example of a non-Kähler compact generalized Hopf manifold is the nilmanifold $N(r,1)\times S^1$, where $N(r,1) = \Gamma(r,1)\setminus H(r,1)$ is a compact quotient of the generalized Heisenberg group $H(r,1)$ by a discret subgroup $\Gamma(r,1)$ (see [6]). Examples of non-Kähler compact locally conformal Kähler manifolds with non-parallel Lee form are obtained in [22] and [1].

On the other hand, if we denote by $S^p_{c^2}$ the p-dimensional unit sphere of constant sectional curvature c^2 ($c \in \mathbb{R}, c \neq 0$) then, it is well known that the Calabi-Eckmann manifolds $V^{2n+2m} = S^{2n-1}_{c^2} \times S^{2m+1}_{c^2}$ ($n \geq 1, m \geq 0$) admit a hermitian structure (J,g), where g is the product metric (see [5]). In fact, assuming $n \geq m+1$, we have (see [5], [23] and [10]):

1. If $n = 1$ and $m = 0$ then the structure (J,g) is Kähler,
2. If $n \geq 2$ and $m = 0$ then $V^{2n+2m} = V^{2n}$ and H_o^n are diffeomorphic and (J,g) is a non-Kähler locally conformal Kähler structure and,
3. If $n \geq 2$ and $m \geq 1$ then the structure (J,g) is hermitian but it is not locally conformal Kähler.

Now, we can consider the product manifold $V^{2n+2m} = S^{2n-1}_{c^2} \times H^{2m+1}_c$, where H^{2m+1}_c is the (2m+1)-dimensional hyperbolic space of constant curvature $-c^2$ ($c \in \mathbb{R}, c \neq 0$). Then the manifold V^{2n+2m} also admits a hermitian structure (J,g), where g is the product metric. Moreover, we obtain

1. The structure (J,g) is locally conformal Kähler (see corollary 3.1).
2. There exist 2m unit 1-forms $\alpha_1, \dots, \alpha_{2m}$ on V^{2n+2m} which are independent and such that

$$(1.1) \qquad \alpha_j \circ J = \alpha_{m+j}\ , \quad \alpha_{m+j} \circ J = -\alpha_j\ , \quad \alpha_i(B) = 0$$

$$(1.2) \qquad \nabla\omega = 2c^2 \sum_{k=1}^{2m} (\alpha_k \otimes \alpha_k) \quad , \quad \nabla\alpha_i = -\frac{1}{2}(\alpha_i \otimes \omega)$$

for $i \in \{1, 2, \ldots, 2m\}$ and $j \in \{1, \ldots, m\}$, where ∇ denotes the Levi-Civita connection of the metric g and ω and B are the Lee 1-form and the Lee vector field respectively of V^{2n+2m} (see corollary 3.1).

3. The local conformal Kähler metrics are flat (see corollary 6.3).

In this paper, we study a particular class of locally conformal Kähler manifolds which we call sasakian m-hyperbolic locally conformal Kähler manifolds, with $m \in \mathbb{N}$, $m \geq 0$. These manifolds have similar properties to the locally conformal Kähler manifold $S^{2n-1}_{c^2} \times H^{2m+1}_c$. A (2n+2m)-dimensional locally conformal Kähler manifold (V^{2n+2m}, J, g) is said to be sasakian m-hyperbolic locally conformal Kähler if there exist 2m unit 1-forms $\alpha_1, \ldots, \alpha_{2m}$ on V^{2n+2m} which are independient and satisfy (1.1) and (1.2), where $c = -\frac{\|\omega\|}{2} \neq 0$ at every point. In particular, a generalized Hopf manifold is a sasakian 0-hyperbolic locally conformal Kähler manifold.

In section 2, we give some results on locally conformal Kähler, c-sasakian and c-kenmotsu manifolds. In section 3, we introduce the definition of m-hyperbolic locally conformal Kähler structure on a l.c.K. manifold. If (J, g) is a l.c.K. structure on a (2n+2m)-dimensional manifold V^{2n+2m} and $\alpha_1, \ldots, \alpha_{2m}$ are independient 1-forms on V^{2n+2m} then, we say that $(J, g, \alpha_1, \ldots, \alpha_{2m})$ is a m-hyperbolic locally conformal Kähler structure on V^{2n+2m} if

$$\alpha_j \circ J = \alpha_{m+j}\,, \qquad \alpha_{m+j} \circ J = -\alpha_j \qquad j \in \{1, \ldots, m\}$$

$$d\alpha_i = -\frac{1}{2}(\alpha_i \wedge \omega) \qquad i \in \{1, 2, \ldots, 2m\}$$

$$\alpha_i(B) = 0 \qquad i \in \{1, 2, \ldots, 2m\},$$

where ω and B are the Lee 1-form and the Lee vector field respectively of V^{2n+2m}. We prove that the product manifold of a (2n-1)-dimensional c-sasakian manifold N and a (2m+1)-dimensional c-kenmotsu manifold M admits locally a m-hyperbolic locally conformal Kähler structure (see proposition 3.3). Moreover, if the manifold M is the (2m+1)-dimensional hyperbolic space $(H^{2m+1}_c, (ds^2)_c)$ then the m-hyperbolic locally conformal Kähler structure is globally defined and the 1-forms α_i $(i = 1, \ldots, 2m)$ satisfy (1.2). In section 4, we introduce the definition of sasakian m-hyperbolic locally conformal Kähler (sasakian m-hyperbolic l.c.K.) manifold as a (2n+2m)-dimensional manifold V^{2n+2m} endowed of a m-hyperbolic l.c.K. structure $(J, g, \alpha_1, \dots, \alpha_{2m})$ such that the unit 1-forms α_i $(i = 1, \dots, 2m)$ satisfy (1.2), where $c = -\frac{\|\omega\|}{2} \neq 0$ at every point. In this section, we characterize the sasakian m-hyperbolic l.c.K. manifolds and we obtain some properties of these manifolds (see propositions 4.4 and 4.5). As consequence, we prove that a compact manifold cannot be a sasakian m-hyperbolic l.c.K. manifold with $m \geq 1$ (see corollary 4.1). In section 5, we study the Riemann curvature tensor R of a sasakian m-hyperbolic l.c.K. manifold $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$. We determine the vector fields $R(X,Y)U$, $R(X,Y)A_i$ and $R(X,Y)V$, for all vector fields X, Y on V^{2n+2m}, in terms of α_i, u, $v = -u \circ J$, A_i, U and V, where u and U are the unit Lee form and the unit Lee vector field respectively of V^{2n+2m} and A_i are the vector fields on V^{2n+2m} given by $\alpha_i(X) = g(X, A_i)$, $1 \leq i \leq 2m$ (see propositions 5.1 and 5.2). In particular, we obtain explicit formulas for the sectional curvature of a plane section containing A_i, U or V and for the Ricci curvature in the direction of these vectors (see corollaries 5.1 and 5.2).

In section 6, we prove that on a sasakian m-hyperbolic l.c.K. manifold $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ the leaves of the foliation $\mathfrak{F}$ have an induced c-sasakian structure, where $\mathfrak{F}$ is the foliation on V^{2n+2m} given by $u = 0, \alpha_i = 0$, $1 \leq i \leq 2m$. Then, we say that a sasakian m-hyperbolic l.c.K. manifold is sasakian(k) m-hyperbolic locally conformal Kähler ($k \in \mathbb{R}$) if every leaf N of the foliation $\mathfrak{F}$ is of constant φ_N-sectional curvature k, where $(\varphi_N, \xi_N, \eta_N, g_N)$ is the induced c-sasakian structure on N. Finally, using the results of the above sections, we obtain that the universal covering space $\overline{V}^{2n+2m}$ of a sasakian m-hyperbolic l.c.K. manifold $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ is the product of a (2n-1)-dimensional c-sasakian manifold $(N, \varphi_N, \xi_N, \eta_N, g_N)$ with the (2m+1)-dimensional hyperbolic space and we describe the induced sasakian m-hyperbolic l.c.K. structure $(\overline{J}, \overline{g}, \overline{\alpha}_1, \dots, \overline{\alpha}_{2m})$ on $\overline{V}^{2n+2m}$ (see theorem 6.1). Moreover, if V^{2n+2m} is a sasakian(k) m-hyperbolic l.c.K. manifold, then we determine, up to almost complex isometries, the almost Hermitian manifold $(\overline{V}^{2n+2m}, \overline{J}, \overline{g})$ (see corollary 6.4). In particular, if V^{2n+2m} is a sasakian(c^2) m-hyperbolic l.c.K. manifold then we have that the local conformal Kähler metrics are flat and the manifold $\overline{V}^{2n+2m}$ is almost complex isometric to $S^{2n-1}_{c^2} \times H^{2m+1}_c$ (see corollaries 6.3 and 6.4).

2 – Preliminaries

Let V be a C^∞ almost Hermitian manifold with metric g, Riemannian connection ∇ and almost complex structure J. Denote by $\mathfrak{X}(V)$ the Lie algebra of C^∞ vector fields on V and by N_J the *Nijenhuis tensor* of V, that is,

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \tag{2.1}$$

for $X, Y \in \mathfrak{X}(V)$.

The *Kähler 2-form* Ω is given by

$$\Omega(X,Y) = g(X,JY) \tag{2.2}$$

and the *Lee* 1-*form* ω is defined by

$$\omega(X) = (\frac{1}{n-1})\delta\Omega(JX)$$

for $X \in \mathfrak{X}(V)$, where δ denotes the codifferential and dim V=2n.

An almost Hermitian manifold (V,J,g) is said to be:

Kählerian if $\nabla J = 0$; *Locally conformal Kähler (l.c.K.)* if every point $x \in V$ has an open neighbourhood U such that the structure $(J, e^{-\sigma}g)$ is Kähler on U, where $\sigma : U \longrightarrow \mathbb{R}$ is a real differentiable function on U (see [14], [23], [24], [6], ...).

Let (V,J,g) be an almost hermitian manifold with Lee form ω and ∇ the Levi-Civita connection of the metric g. Consider

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X,Y)B \tag{2.3}$$

for $X, Y \in \mathfrak{X}(V)$, where B is the *Lee vector field* of V given by $\omega(X) = g(X,B)$. $\overline{\nabla}$ is a torsionless linear connection on V, which is called the *Weyl connection* of g (see [19]). Moreover, if (V,J,g) is l.c.K. then $\overline{\nabla}$ is the Levi-Civita connection of the local metrics $e^{-\sigma}g$ (see [23]). In fact, in [23], I. Vaisman proves

Proposition 2.1. *The following are equivalent:*

1. (V,J,g) *is a l.c.K. manifold.*

2. The Lee form ω is closed and

$$\overline{\nabla}_X J = 0 \tag{2.4}$$

for all $X \in \mathfrak{X}(V)$.

3. The Lee form ω is closed and

$$(\nabla_X J)Y = \frac{1}{2}\omega(JY)X - \frac{1}{2}\omega(Y)JX - \frac{1}{2}g(X,JY)B + \frac{1}{2}g(X,Y)JB \tag{2.5}$$

for all $X, Y \in \mathfrak{X}(V)$.

4. The Lee form ω is closed and

$$d\Omega = \omega \wedge \Omega \quad , \quad N_J = 0. \tag{2.6}$$

Among the l.c.K. manifolds, those such that $\nabla\omega = 0$ are called *generalized Hopf manifolds* (see [24] and [25]).

On the other hand, let M be an almost contact metric manifold with metric g and almost contact structure (φ, ξ, η). Then we have

$$\varphi^2 = -I + \eta \otimes \xi \qquad \eta(\xi) = 1$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for $X, Y \in \mathfrak{X}(M)$, where I denotes the identity transformation (see [2] and [3]). Denote by N_φ the Nijenhuis tensor of φ, that is

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y]$$

for $X, Y \in \mathfrak{X}(M)$. The *fundamental 2-form* ϕ of M is given by

$$\phi(X,Y) = g(X, \varphi Y).$$

An almost contact metric manifold M is said to be *c-sasakian* (see [11]), with c $\in \mathbb{R}$, $c \neq 0$ if

$$N_\varphi + 2d\eta \otimes \xi = 0 \quad , \quad d\eta = c\phi \tag{2.7}$$

and it is called *c-kenmotsu* (see [11]) if

$$N_\varphi + 2d\eta \otimes \xi = 0 \quad , \quad d\phi = -2c\eta \wedge \phi \quad , \quad d\eta = 0. \tag{2.8}$$

The manifold M is said to be *sasakian* if it is 1-sasakian.

If $(M, \varphi, \xi, \eta, g)$ is a c-sasakian manifold or a c-kenmotsu manifold then

$$L_\xi \varphi = 0 \tag{2.9}$$

where L denotes the Lie derivate on M.

Let $(H_c^{2m+1}, (ds^2)_c)$ be the (2m+1)-dimensional *hyperbolic space*, i.e.,

$$H_c^{2m+1} = \{(x_1, \dots, x_{2m+1}) \in \mathbb{R}^{2m+1} / x_{2m+1} > 0\}$$

and $(ds^2)_c$ is the Riemannian metric given by

$$(ds^2)_c = \frac{1}{(cx_{2m+1})^2} \sum_{i=1}^{2m+1} (dx_i)^2 \quad , \quad (c \neq 0).$$

$(H_c^{2m+1}, (ds^2)_c)$ is a complete simply connected Riemannian manifold with constant negative curvature $-c^2$.

The vector fields E_i $(i = 1, \dots, 2m+1)$ on H_c^{2m+1} defined by

$$E_i = (cx_{2m+1}) \frac{\partial}{\partial x_i} \tag{2.10}$$

form an orthonormal basis for this space.

The dual basis of 1-forms is given by

$$\alpha_i = \frac{dx_i}{(cx_{2m+1})} \tag{2.11}$$

for $i = 1, \dots, 2m+1$.

Then, it is not difficult to prove that

$$\begin{cases} \nabla \alpha_{2m+1} = -c \sum_{i=1}^{2m} \alpha_i \otimes \alpha_i \\ \nabla \alpha_i = c\alpha_i \otimes \alpha_{2m+1} \end{cases} \tag{2.12}$$

for $i \in \{1, \ldots, 2m\}$, where ∇ is the Levi-Civita connection of the metric $(ds^2)_c$.

Let $(\varphi_{H_c^{2m+1}}, \xi_{H_c^{2m+1}}, \eta_{H_c^{2m+1}}, g_{H_c^{2m+1}})$ be the almost contact metric structure on H_c^{2m+1} defined by

$$
\begin{aligned}
&\varphi_{H_c^{2m+1}} = \sum_{i=1}^{m}(E_i \otimes \alpha_{m+i} - E_{m+i} \otimes \alpha_i) \ , \ \xi_{H_c^{2m+1}} = E_{2m+1} \\
&\eta_{H_c^{2m+1}} = \alpha_{2m+1} \ , \ g_{H_c^{2m+1}} = (ds^2)_c \, .
\end{aligned} \tag{2.13}
$$

Then (see [12], [7]), the almost contact metric structure $(\varphi_{H_c^{2m+1}}, \xi_{H_c^{2m+1}}, \eta_{H_c^{2m+1}}, g_{H_c^{2m+1}} = (ds^2)_c)$ on H_c^{2m+1} is c-kenmotsu.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and x a point of M. A plane section π in the tangent space to M at x, T_xM, is called a φ-section if there exists a unit vector X in T_xM orthogonal to ξ such that $\{X, \varphi X\}$ is an orthonormal basis of π. Then the sectional curvature $K_{X\varphi X} = g(R(X, \varphi X)\varphi X, X)$ is called a φ-sectional curvature.

A c-sasakian manifold is said to be a c-sasakian space form if M has constant φ-sectional curvature. Examples of sasakian space forms are provided by the manifolds S^{2n-1}, $\mathbb{R}^{2n-1}$ and $\mathbb{R} \times CD^{n-1}$. In fact, the unit sphere S^{2n-1} has a sasakian structure of constant φ-sectional curvature k, for all $k > -3$ (see [20] and [21]); the real (2n-1)-dimensional number space $\mathbb{R}^{2n-1}$ is a sasakian space form with $k = -3$ [18]; and the product manifold $\mathbb{R} \times CD^{n-1}$, where CD^{n-1} is a simply connected bounded complex domain in C^{n-1} with negative constant holomorphic sectional curvature, has a sasakian structure of constant φ-sectional curvature k, for all $k < -3$ [21].

Let $(M, \varphi, \xi, \eta, g)$ be a sasakian manifold with constant φ- sectional curvature k. Put

$$
\varphi' = \varphi \ , \ \xi' = c\xi \ , \ \eta' = \frac{1}{c}\eta \ , \ g' = \frac{1}{c^2}g
$$

where $c \in \mathbb{R}$, $c \neq 0$. Then, $(M, \varphi', \xi', \eta', g')$ is a c-sasakian space form of constant φ-sectional curvature kc^2. We denote by $M(c, kc^2)$ the c-sasakian manifold with this structure.

In [21], Tanno proves that if $(M, \varphi, \xi, \eta, g)$ and $(M', \varphi', \xi', \eta', g')$ are (2n-1)-dimensional complete simply connected sasakian manifolds of constant φ-sectional curvature k, then, M is almost contact isometric to M', i.e., there exists an isometry F of M into M' such that $F_* \circ \varphi = \varphi' \circ F_*$ and $F_*\xi = \xi'$. Therefore, by using this result, we deduce

PROPOSITION 2.2. *Let M be a (2n-1)-dimensional complete simply connected c-sasakian manifold with constant φ-sectional curvature k.*

1. *If $k > -3c^2$, then M is almost contact isometric to $S^{2n-1}(c,k)$.*
2. *If $k = -3c^2$, then M is almost contact isometric to $\mathbb{R}^{2n-1}(c,-3c^2) = \mathbb{R}^{2n-1}(c)$.*
3. *If $k < -3c^2$, then M is almost contact isometric to $(\mathbb{R}\times CD^{n-1})(c,k)$.*

REMARK. It is clear that the manifold $S^{2n-1}(c,c^2)$ is $S^{2n-1}_{c^2}$ (see section 1).

All the manifolds considered in this paper are assumed to be connected.

3 – m-Hyperbolic locally conformal Kähler structures

In this section, we study a particular class of structures on a l.c.K. manifold which we call m-hyperbolic locally conformal Kähler structures.

First, we describe the local structure of a c-kenmotsu manifold (see [12] and [15]). For this purpose, we examine the following example:

Let M be the product manifold $L \times V$, where L is an open interval (a,b), $-\infty \leq a < b \leq \infty$, and (V,J',G) is a $2m$-dimensional Kählerian manifold. Let E be a nowhere vanishing vector field on L, E^* its dual field of 1-forms and σ a positive function on L such that $d(\ln\sigma) = -2cE^*$, with $c \in \mathbb{R}$, $c \neq 0$. Put

$$(3.1)\qquad \begin{cases} \varphi(a'E,X) = (0,J'X) , \\ \xi = (E,0) \quad , \quad \eta = (E^*,0) \\ g((a'E,X),(b'E,Y)) = \sigma G(X,Y) + a'b' , \end{cases}$$

where a' and b' are differentiable functions on M, and $X,Y \in \mathfrak{X}(V)$. Then it is not difficult to check that (M,φ,ξ,η,g) is a c-kenmotsu manifold.

The converse holds locally, i.e.,

PROPOSITION 3.1. [15] *If $(M^{2m+1},\varphi,\xi,\eta,g)$ is a $(2m+1)$-dimensional c-kenmotsu manifold, then the manifold M^{2m+1} is locally the product*

$(a,b)\times V^{2m}$, where (a,b) is an open interval and V^{2m} is a $2m$-dimensional Kählerian manifold, on which the structure (φ,ξ,η,g) is given as in (3.1).

Let $(N,\varphi_N,\xi_N,\eta_N,g_N)$ be a c-sasakian manifold and $(M,\varphi_M,\xi_M,\eta_M,g_M)$ a $(2m+1)$-dimensional c-kenmotsu manifold, with $c\in\mathbb{R}$, $c\neq 0$. Let us consider the product manifold $V=N\times M$ with the almost hermitian structure (J,g) defined by:

$$(3.2)\qquad \begin{cases} J(X,X')=(\varphi_N X-\eta_M(X')\,\xi_N,\ \varphi_M X'+\eta_N(X)\,\xi_M)\\ g((X,X'),(Y,Y'))=g_N(X,Y)+g_M(X',Y')\end{cases}$$

where $X,Y\in\mathfrak{X}(N)$ and $X',Y'\in\mathfrak{X}(M)$.

PROPOSITION 3.2. *The almost Hermitian manifold (V,J,g) is a l.c.K. manifold with Lee form*

$$\omega=-2c\pi_M^*\eta_M$$

where $\pi_M : N\times M\longrightarrow M$ is the canonical projection onto the second factor.

PROOF. Let X,Y be vector fields on N and X',Y' vector fields on M. Then:

$$\begin{aligned}
&N_J((X,X'),(Y,Y'))=\\
&=\big(N_{\varphi_N}(X,Y)+2d\eta_N(X,Y)\,\xi_N-2d\eta_M(X',\varphi_M Y')\,\xi_N-\\
&\quad -2d\eta_M(\varphi_M X',Y')\,\xi_N+\eta_M(Y')\,(L_{\xi_N}\varphi_N)X-\eta_M(X')\,(L_{\xi_N}\varphi_N)Y+\\
&\quad +2\eta_N(X)\,d\eta_M(Y',\xi_M)\,\xi_N+2\eta_N(Y)\,d\eta_M(\xi_M,X')\,\xi_N\ ,\\
&\quad N_{\varphi_M}(X',Y')+2d\eta_M(X',Y')\,\xi_M+2d\eta_N(\varphi_N X,Y)\,\xi_M+\\
&\quad +2d\eta_N(X,\varphi_N Y)\,\xi_M+\eta_N(X)\,(L_{\xi_M}\varphi_M)Y'-\eta_N(Y)\,(L_{\xi_M}\varphi_M)X'-\\
&\quad -2\eta_M(X')\,d\eta_N(\xi_N,Y)\,\xi_M+2\eta_M(Y')\,d\eta_N(\xi_N,X)\,\xi_M\big)
\end{aligned}$$

where N_J, N_{φ_N} and N_{φ_M} denote the Nijenhuis tensors of J, φ_N and φ_M respectively and L denotes the Lie derivate operator on N and M. Thus, from (2.7), (2.8) and (2.9), we obtain that $N_J((X,X'),(Y,Y'))=0$.

On the other hand, using (2.2) and (3.2), the Kähler 2-form Ω of the almost Hermitian manifold (V, J, g) is given by

$$\Omega = \pi_N^* \phi_N + \pi_M^* \phi_M + 2(\pi_M^* \eta_M \wedge \pi_N^* \eta_N) \tag{3.3}$$

where ϕ_N and ϕ_M denote the fundamental 2-forms of N and M respectively and where $\pi_N : V = N \times M \longrightarrow N$ is the projection of V onto the first factor. Then, from (2.7), (2.8) and (3.3), we have that:

$$d\Omega = -2c(\pi_M^* \eta_M) \wedge \Omega.$$

Consequently, since η_M is a closed 1-form, we deduce that the almost hermitian manifold (V, J, g) is l.c.K. with Lee form $\omega = -2\,c\pi_M^* \eta_M$. □

Next, we shall study the l.c.K. structure (J, g) on the product manifold $N \times M$.

PROPOSITION 3.3. *Let (J, g) be the l.c.K. structure given by* (3.2) *on the product manifold $N \times M$. Then, for every point $(p, q) \in N \times M$ there exists an open neighbourhood U of q in M and $2m$ independent 1-forms $\alpha_1, \dots, \alpha_{2m}$ on U, such that:*

$$\begin{cases} \pi_U^* \alpha_j \circ J = \pi_U^* \alpha_{m+j}, & \pi_U^* \alpha_{m+j} \circ J = -\pi_U^* \alpha_j \quad j \in \{1, \dots, m\} \\ d(\pi_U^* \alpha_i) = -\frac{1}{2} \pi_U^* \alpha_i \wedge \omega, & (\pi_U^* \alpha_i)(B) = 0 \qquad\qquad i \in \{1, \dots, 2m\} \end{cases} \tag{3.4}$$

where $\pi_U : N \times U \longrightarrow U$ is the projection onto the second factor and ω and B are the Lee 1-form and the Lee vector field respectively of $N \times M$.

PROOF. If $u = (p, q)$ is a point of the product manifold $V = N \times M$ then, using proposition 3.1, we deduce that there exists an open neighbourhood $U' = (a, b) \times V$ of q, a positive function σ and a nowhere vanishing vector field E on (a, b) such that

$$d(\ln \sigma) = -2c\eta_M \quad , \quad \xi_M = E, \tag{3.5}$$

and the almost contact structure $(\varphi_M, \xi_M, \eta_M, g_M)$ on U' is given by (3.1), where (V, J', G) is a 2m-dimensional Kählerian manifold and (a, b) is an open interval, $-\infty \leq a < b \leq \infty$.

Suppose that $q = (l, v)$ with $l \in L$ and $v \in V$. Since (V, J', G) is a Kählerian manifold there exists a coordinate neighbourhood W of v in V, with coordinates $(x_1, \dots, x_{2m})$, such that:

$$J'\frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x^{m+i}} \quad , \quad J'\frac{\partial}{\partial x^{m+i}} = \frac{\partial}{\partial x^i} \tag{3.6}$$

for $i \in \{1, \dots, m\}$.

Let U be the open neighbourhood of q in M given by $U = (a, b) \times W$. From (3.1), (3.5) and using proposition 3.2, we have that:

$$\omega = \pi_U^*(d(ln\sigma)) \quad , \quad B = -2c\xi_M\,. \tag{3.7}$$

Now, define on U the 1-forms α_i by

$$\alpha_i = \frac{\sqrt{\sigma}}{c}dx^i \tag{3.8}$$

$i \in \{1, \dots, 2m\}$. Then, from (3.6), (3.7) and (3.8), we obtain (3.4). □

The above results suggests us to consider the following particular class of l.c.K. structure:

DEFINITION 3.1. *Let (V, J, g) be a $(2n + 2m)$-dimensional l.c.K. manifold with Lee form ω and Lee vector field B, and let $\alpha_1, .., \alpha_{2m}$ be independent 1-forms on V, with $m \geq 0$. We say that $(J, g, \alpha_1, \dots, \alpha_{2m})$ is a* ***m-hyperbolic locally conformal Kähler (m-hyperbolic l.c.K.) structure*** *on V if*

$$\begin{array}{lll} \alpha_j \circ J = \alpha_{m+j} & \alpha_{m+j} \circ J = -\alpha_j & j \in \{1, \dots, m\} \\ d\alpha_i = -\frac{1}{2}(\alpha_i \wedge \omega) & & i \in \{1, 2, \dots, 2m\} \\ \alpha_i(B) = 0 & & i \in \{1, 2, \dots, 2m\}. \end{array} \tag{3.9}$$

REMARK. If $(N, \varphi_N, \xi_N, \eta_N, g_N)$ is a c-sasakian manifold and $(M, \varphi_M, \xi_M, \eta_M, g_M)$ is a (2m+1)-dimensional c-kenmotsu manifold, with $c \in \mathbb{R}$, $c \neq 0$, then, from proposition 3.3, we deduce that for every point $(p, q) \in N \times M$, there exists an open neighbourhood U of q in M and 2m 1-forms $\alpha_1, \dots, \alpha_{2m}$ on U, such that $(J, g, \pi_U^*\alpha_1, \dots, \pi_U^*\alpha_{2m})$ is a m-hyperbolic l.c.K. structure on $N \times U$, where (J, g) is the l.c.K. structure given by (3.2) on the manifold $N \times M$ and $\pi_U : N \times U \longrightarrow U$ is the projection onto the second factor.

Now, let H_c^{2m+1} be the $(2m+1)$-dimensional hyperbolic space. Denote by $\alpha_1, \dots, \alpha_{2m}$ the 1-forms on H_c^{2m+1} given by (2.11) and by $(\varphi_{H_c^{2m+1}}, \xi_{H_c^{2m+1}}, \eta_{H_c^{2m+1}}, g_{H_c^{2m+1}})$ the c-kenmotsu structure on H_c^{2m+1} given by (2.13). Then, if N is a c-sasakian manifold and $\pi_{H_c^{2m+1}}$: $N \times H_c^{2m+1} \longrightarrow H_c^{2m+1}$ is the projection onto the second factor, we obtain that

COROLLARY 3.1. *The almost Hermitian structure (J, g) given by (3.2) onto the product manifold $N \times H_c^{2m+1}$ is l.c.K. with Lee form*

$$\omega = -2c\pi^*_{H_c^{2m+1}}\eta_{H_c^{2m+1}}.$$

Moreover, $(J, g, \pi^*_{H_c^{2m+1}}\alpha_1, \dots, \pi^*_{H_c^{2m+1}}\alpha_{2m})$ is a m-hyperbolic l.c.K. structure on $N \times H_c^{2m+1}$ and we have that

$$\begin{aligned} \nabla\omega &= 2c^2 \sum_{j=1}^{2m} (\pi^*_{H_c^{2m+1}}\alpha_j) \otimes (\pi^*_{H_c^{2m+1}}\alpha_j) \\ \nabla\pi^*_{H_c^{2m+1}}\alpha_i &= -\frac{1}{2}(\pi^*_{H_c^{2m+1}}\alpha_i) \otimes \omega \end{aligned} \tag{3.10}$$

for $i \in \{1, \dots, 2m\}$, where ∇ is the Levi-Civita connection of the Riemannian metric g.

PROOF. The first part of this corollary follows from proposition 3.2.

Let B be the Lee vector field of the product manifold $N \times H_c^{2m+1}$. Then, using (3.2) and proposition 3.2 we have that

$$B = -2cE_{2m+1} \tag{3.11}$$

where E_{2m+1} is the vector field on H_c^{2m+1} given by (2.10).

Therefore, from (2.11), (2.13), (3.2) and (3.11) we obtain that $(J, g, \pi^*_{H_c^{2m+1}}\alpha_1, \dots, \pi^*_{H_c^{2m+1}}\ \alpha_{2m})$ is a m-hyperbolic l.c.K. structure on $N \times H_c^{2m+1}$

Finally, using (2.12), (2.13) and (3.2), we deduce (3.10). □

REMARK. In proposition 3.1 we described the local structure of a c-kenmotsu manifold. It is not difficult to prove that in the particular case of the c-kenmotsu manifold $(H_c^{2m+1}, \varphi_{H_c^{2m+1}}, \xi_{H_c^{2m+1}}, \eta_{H_c^{2m+1}}, g_{H_c^{2m+1}})$ such a proposition is globally true. In fact, $H_c^{2m+1} = \mathbb{R}^{2m} \times (0,\infty)$ and thus it is sufficient to take in (3.1), (J', G) the usual Kählerian structure on $\mathbb{R}^{2m}$ and

$$\sigma = \frac{1}{(x_{2m+1})^2} \quad , \quad E = (cx_{2m+1})\frac{\partial}{\partial x_{2m+1}} \tag{3.12}$$

where x_{2m+1} is the coordinate on the interval $(0,\infty)$. Consequently, from (2.11), (3.8) and (3.12), we also deduce that $(J, g, \pi^*_{H_c^{2m+1}}\alpha_1, \ldots$ $\ldots, \pi^*_{H_c^{2m+1}}\alpha_{2m})$ is a m-hyperbolic l.c.K. structure on the product manifold $N \times H_c^{2m+1}$.

Now, denote by N_i $(i = 1,2,3)$ the following $(2n-1)$-dimensional c-sasakian manifolds of constant φ-sectional curvature k (see proposition 2.2),

$$N_1 = S^{2n-1}(c,k) \quad , \quad N_2 = \mathbb{R}^{2n-1}(c) \quad , \quad N_3 = (\mathbb{R} \times CD^{n-1})(c,k).$$

Let (J_i, g_i) be the almost Hermitian structure on $N_i \times H_c^{2m+1}$ (i=1,2,3) given by (3.2). Then, from corollary 3.1, we deduce that

COROLLARY 3.2. *The almost Hermitian structure (J_i, g_i) onto the product manifold $N_i \times H_c^{2m+1}$ $(i = 1,2,3)$ is l.c.K. with Lee form*

$$\omega = -2c\pi^*_{H_c^{2m+1}}\eta_{H_c^{2m+1}}.$$

Moreover, $(J_i, g_i, \pi^*_{H_c^{2m+1}}\alpha_1, \ldots, \pi^*_{H_c^{2m+1}}\alpha_{2m})$ is a m-hyperbolic l.c.K. structure on $N_i \times H_c^{2m+1}$ satisfying (3.10).

4 – Sasakian m-hyperbolic locally conformal Kähler manifolds

The results obtained in corollary 3.1 suggest us to introduce the following definition.

DEFINITION 4.1. *Let $(J, g, \alpha_1, \ldots, \alpha_{2m})$ be a m-hyperbolic l.c.K. structure on a manifold V^{2n+2m} of dimension $(2n+2m)$, such that $\alpha_1, \ldots$*

$\dots, \alpha_{2m}$ are unit 1-forms. We say that V^{2n+2m} is a ***sasakian m-hyperbolic locally conformal Kähler (sasakian m-hyperbolic l.c.K.) manifold*** *if*

$$\left\{ \begin{array}{l} \nabla\omega = \frac{l^2}{2}\sum_{j=1}^{2m} \alpha_j \otimes \alpha_j \\ \nabla\alpha_i = -\frac{1}{2}\alpha_i \otimes \omega \end{array} \right. \tag{4.1}$$

for $i \in \{1, \dots, 2m\}$, where ω is the Lee form of V^{2n+2m}, ∇ is the Levi-Civita connection of the metric g and $l = \|\omega\| \neq 0$ at every point.

If $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ is a sasakian m-hyperbolic l.c.K. manifold then V^{2n+2m} is said to have a *sasakian m-hyperbolic l.c.K. structure* $(J, g, \alpha_1, \dots, \alpha_{2m})$.

We remark that the above definition generalizes the notion of generalized Hopf manifold. In fact, a generalized Hopf manifold is a sasakian 0-hyperbolic l.c.K. manifold.

In this section, our intention is to obtain information about the structure of the sasakian m-hyperbolic l.c.K. manifolds and we begin by introducing some of their properties.

Let $(V^{2n+2m}, J, g, \alpha_1, .., \alpha_{2m})$ be a sasakian m-hyperbolic l.c.K. manifold and denote by A_i, with $1 \leq i \leq 2m$, the vector fields on V^{2n+2m} given by

$$\alpha_i(X) = g(X, A_i) \tag{4.2}$$

for all $X \in \mathfrak{X}(V^{2n+2m})$. From (3.9) and (4.2), we obtain that

$$JA_i = -A_{m+i} \quad , \quad JA_{m+i} = A_i \tag{4.3}$$

for $i \in \{1, \dots, m\}$. Moreover,

PROPOSITION 4.1. *On a sasakian m-hyperbolic l.c.K. manifold V^{2n+2m} the vector fields A_i and A_j, with $i \neq j$, are orthogonal.*

PROOF. If B is the Lee vector field of V^{2n+2m} then, from (3.9) and (4.2), we have that

$$(\nabla_{A_i}\alpha_i)B = -(\nabla_{A_i}\omega)A_i$$

and thus, using (4.1), we deduce that

$$-\left(\frac{l^2}{2}\right) = -\left(\frac{l^2}{2}\right)\sum_{\substack{k=1\\k\neq i}}^{2m}(\alpha_k(A_i))^2 - \left(\frac{l^2}{2}\right). \tag{4.4}$$

Consequently, from (4.4) and since $l \neq 0$ at every point, we obtain that $\alpha_j(A_i) = 0$.

This completes the proof. □

We also have,

PROPOSITION 4.2. *On a sasakian m-hyperbolic l.c.K. manifold the Lee* 1*-form has constant norm.*

PROOF. Let $(V^{2n+2m}, J, g, \alpha_1, .., \alpha_{2m})$ be a sasakian m-hyperbolic l.c.K. manifold with Lee 1-form ω and Lee vector field B and let X be a vector field on V^{2n+2m}. Denote by $l = \|\omega\|$. Then, using (4.1) and (3.9), we get

$$(\nabla_X\omega)B = 0.$$

On the other hand

$$(\nabla_X\omega)B = l\,dl(X)$$

and thus, since $l \neq 0$ at every point, we have that $dl(X) = 0$.

Therefore, we deduce that $dl = 0$ which implies that l is constant. □

Let $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ be a sasakian m-hyperbolic l.c.K. manifold with Lee vector field B and Lee form ω. Then, in the rest of this paper, we shall use the following notation

$$l = \|\omega\| \quad , \quad u = \frac{\omega}{l} \quad , \quad U = \frac{B}{l} \quad , \quad v = -u \circ J \quad , \quad V = JU. \tag{4.5}$$

From (3.9), (4.3) and (4.5) we obtain that

$$\begin{aligned} &u(V) = v(U) = u(A_i) = v(A_i) = 0 \\ &\alpha_i(U) = \alpha_i(V) = 0 \end{aligned} \tag{4.6}$$

for $i \in \{1, \ldots, 2m\}$.

Moreover, if Ω is the Kähler 2-form of V^{2n+2m} then, using that Ω is nondegenerate and (4.6), we have that

PROPOSITION 4.3. *On a sasakian m-hyperbolic l.c.K. manifold* V^{2n+2m}

$$\Omega = \psi + 2(\sum_{j=1}^{m}(\alpha_j \wedge \alpha_{m+j}) + v \wedge u)$$

where ψ *is a 2-form of rank* $(2n-2)$ *such that:*

$$\psi^{n-1} \wedge u \wedge v \wedge \alpha_1 \wedge \ldots \wedge \alpha_{2m} \neq 0$$
$$\psi(X, A_i) = \psi(X, U) = \psi(X, V) = 0$$

for $i \in \{1, \ldots, 2m\}$.

Next, we give some characterizations of sasakian m-hyperbolic l.c.K. manifold.

PROPOSITION 4.4. *Let* $(J, g, \alpha_1, .., \alpha_{2m})$ *be a* m*-hyperbolic l.c.K. structure on a manifold* $(2n+2m)$*-dimensional* V^{2n+2m} *such that* $\alpha_1, \ldots$ $\ldots, \alpha_{2m}$ *are unit* 1*-forms and the Lee form* $\omega \neq 0$ *at every point. Then,* $(V^{2n+2m}, J, g, \alpha_1, .., \alpha_{2m})$ *is a sasakian* m*-hyperbolic l.c.K. manifold if and only if* $l = \|\omega\|$ *is constant and one of the following relations holds*

(i) $$\nabla u = \frac{l}{2}\sum_{j=1}^{2m} \alpha_j \otimes \alpha_j \qquad \nabla \alpha_i = -\frac{l}{2}\alpha_i \otimes u$$

(ii) $$\nabla U = \frac{l}{2}\sum_{j=1}^{2m} \alpha_j \otimes A_j \qquad \nabla A_i = -\frac{l}{2}\alpha_i \otimes U$$

(iii) $$\nabla V = -\frac{l}{2}\Big[J + v \otimes U - u \otimes V + \sum_{j=1}^{m}(\alpha_j \otimes A_{m+j} - \alpha_{m+j} \otimes A_j)\Big) \qquad \nabla A_i = -\frac{l}{2}\alpha_i \otimes U$$

(iv) $$\nabla v = \frac{l}{2}\psi \qquad \nabla \alpha_i = -\frac{l}{2}\alpha_i \otimes u$$

for $i \in \{1, \ldots, 2m\}$.

PROOF.

The proposition follows from (2.5), (4.1), (4.3) and using proposition 4.2 and the relations:

$$\nabla u = \frac{1}{l}\nabla\omega \quad , \quad \nabla_X V = (\nabla_X J)U + J(\nabla_X U). \qquad \square$$

Now, we deduce another result for a sasakian m-hyperbolic l.c.K. manifold V^{2n+2m}. Denote by L the Lie derivate on V^{2n+2m}.

PROPOSITION 4.5. *Let $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ be a sasakian m-hyperbolic l.c.K. manifold. Then, V is a Killing vector field for the metric g. Moreover, the following relations hold*

$$(4.7) \qquad [U,V]=0\,, \quad [V,A_i]=0\,, \quad [A_i,A_j]=0\,, \quad [U,A_i]=-\frac{l}{2}A_i$$

$$(4.8) \qquad L_U J=0\,, \quad L_V J=0\,, \quad L_{A_k}J=-\frac{l}{2}(v\otimes A_k - u\otimes A_{m+k})$$

$$(4.9) \qquad L_{A_{m+k}}J=-\frac{l}{2}(v\otimes A_{m+k} + u\otimes A_k)$$

$$(4.10) \qquad L_U v=0\,, \quad L_{A_i}v=0\,, \quad dv=\frac{l}{2}\psi\,,$$

for $i,j \in \{1,\dots,2m\}$ and $k \in \{1,\dots,m\}$.

PROOF. Using proposition 4.4 and since ∇ is a torsionless linear connection on V^{2n+2m} we obtain (4.7).

Let X, Y be vector fields on V^{2n+2m}. Then, we have that

$$2dv(X,Y) = (\nabla_X v)Y - (\nabla_Y v)X$$

and thus, from proposition 4.4, we deduce that

$$(4.11) \qquad dv(X,Y)=\frac{l}{2}\psi(X,Y).$$

On the other hand, by the classical formula of the Levi-Civita connection [13] we have that,

$$(L_V g)(X,Y) = 2g(\nabla_X V, Y) - 2dv(X,Y)$$

and therefore, using (4.11) and proposition 4.4, we obtain that V is a Killing vector field.

Now, from (2.5), (4.3), proposition 4.4 and from the fact that

$$(L_X J)(Y) = (\nabla_X J)(Y) - \nabla_{JY}X + J(\nabla_Y X)$$

for all $X, Y \in \mathfrak{X}(V^{2n+2m})$, we deduce (4.8) and (4.9).

Finally, using (4.11), (4.6), proposition 4.3 and the relations

$$L_U v = d(i_U v) + i_U(dv) \quad , \quad L_{A_j} v = d(i_{A_j} v) + i_{A_j}(dv)$$

with $1 \leq j \leq 2m$, we prove that $L_U v = L_{A_j} v = 0$, $1 \leq j \leq 2m$. □

Next, using proposition 4.5, we obtain an interesting result

COROLLARY 4.1. *A compact manifold cannot admit a sasakian m-hyperbolic l.c.K. structure with $m \geq 1$.*

PROOF. Let $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ be a compact sasakian m-hyperbolic l.c.K. manifold, with $m \geq 1$. Then, from proposition 4.3, we deduce that the $(2n+2m)$-form γ on V^{2n+2m} given by

$$\gamma = \alpha_1 \wedge \ldots \wedge \alpha_{2m} \wedge u \wedge v \wedge \psi^{n-1}$$

is a volume element.

On the other hand, using (3.9) and (4.10), we obtain that

$$\gamma = d\Big(\Big(\frac{1}{ml}\Big)\alpha_1 \wedge \ldots \wedge \alpha_{2m} \wedge v \wedge \psi^{n-1}\Big)$$

which, in view of Stokes' theorem, is a contradiction. □

REMARK. It is well known that the compact Hopf manifolds admit a l.c.K. structure with parallel Lee form (see [24] and [25]), i.e., the compact Hopf manifolds are compact sasakian 0-hyperbolic l.c.K. manifolds (other examples of compact sasakian 0-hyperbolic l.c.K. manifolds are obtained in [6]). Consequently, corollary 4.1 is not true for $m = 0$.

5 – The curvature tensor on a sasakian m-hyperbolic l.c.K. manifold

In this section, we shall study the Riemann curvature tensor of a sasakian m-hyperbolic l.c.K. manifold.

Let $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ be a $(2n+2m)$-dimensional sasakian m-hyperbolic l.c.K. manifold and let A_i be as in (4.2) and l, u, U, v and V as in (4.5). Then, if R is the Riemann curvature tensor of V^{2n+2m}, we have,

PROPOSITION 5.1. *On a sasakian m-hyperbolic l.c.K. manifold* V^{2n+2m}

$$R(X,Y)U = -\frac{l^2}{2}\sum_{i=1}^{2m}(\alpha_i \wedge u)(X,Y)A_i \tag{5.1}$$

$$R(X,U)Y = \left(\frac{l}{2}\right)^2 \sum_{i=1}^{2m}(\alpha_i(X)\alpha_i(Y)U - \alpha_i(X)u(Y)A_i) \tag{5.2}$$

$$R(X,Y)A_i = \frac{l^2}{2}\left\{\sum_{j=1}^{2m}(\alpha_i \wedge \alpha_j)(X,Y)A_j + (\alpha_i \wedge u)(X,Y)U\right\} \tag{5.3}$$

$$\begin{aligned} R(X,A_i)Y = -\left(\frac{l}{2}\right)^2 \Big\{ & u(X)\alpha_i(Y)U - u(X)u(Y)A_i + \\ & + \sum_{j=1}^{2m}(\alpha_j(X)\alpha_i(Y)A_j - \alpha_j(X)\alpha_j(Y)A_i)\Big\} \end{aligned} \tag{5.4}$$

where $i \in \{1, \dots, 2m\}$ *and* $X, Y \in \mathfrak{X}(V^{2n+2m})$.

PROOF. From proposition 4.4 we deduce that

$$\begin{aligned} R(X,Y)U &= \frac{l}{2}\sum_{i=1}^{2m}(2d\alpha_i(X,Y)A_i + \alpha_i(Y)\nabla_X A_i - \alpha_i(X)\nabla_Y A_i) = \\ &= l\sum_{i=1}^{2m} d\alpha_i(X,Y)A_i \end{aligned}$$

$$\begin{aligned} R(X,Y)A_i &= -\frac{l}{2}\{2d\alpha_i(X,Y)U + \alpha_i(Y)\nabla_X U - \alpha_i(X)\nabla_Y U\} \\ &= -\frac{l}{2}\{2d\alpha_i(X,Y)U - l\sum_{j=1}^{2m}(\alpha_i \wedge \alpha_j)(X,Y)A_j\} \end{aligned}$$

for all $X, Y \in \mathfrak{X}(V^{2n+2m})$.

Thus, using (3.9), we obtain (5.1) and (5.3).

(5.2) and (5.4) follow from (5.1) and (5.3) respectively and using the relation

$$g(R(X,Y)Z,W) = -g(R(Z,W)Y,X) \tag{5.5}$$

for all $X, Y, Z, W \in X(V^{2n+2m})$. □

Also, we have

PROPOSITION 5.2. *On a sasakian m-hyperbolic l.c.K. manifold* V^{2n+2m}

$$R(X,Y)V = \left(\frac{l}{2}\right)^2 \{-v(X)Y + v(Y)X + 2(v \wedge u)(X,Y)U + 2\sum_{i=1}^{2m}(v \wedge \alpha_i)(X,Y)A_i\} \tag{5.6}$$

$$R(X,V)Y = \left(\frac{l}{2}\right)^2 \{v(Y)X - u(X)v(Y)U + (u(X)u(Y) + \sum_{i=1}^{2m}\alpha_i(X)\alpha_i(Y) - g(X,Y))V - \sum_{i=1}^{2m}\alpha_i(X)v(Y)A_i\} \tag{5.7}$$

for all $X, Y \in \mathfrak{X}(V^{2n+2m})$.

PROOF. Using propositions 4.4 and 4.5 and since the 1-form u is closed we obtain that

$$\begin{aligned}
R(X,Y)V = {}& \\
= -\frac{l}{2}\{&(\nabla_X J)Y - (\nabla_Y J)X + l\psi(X,Y)U - l\sum_{j=1}^{2m}(v \wedge \alpha_j)(X,Y)A_j + \\
&+ u(X)(-\frac{l}{2}(JY + v(Y)U - u(Y)V + \sum_{i=1}^{m}(\alpha_i(Y)A_{m+i} - \alpha_{m+i}(Y)A_i))) + \\
&- u(Y)(-\frac{l}{2}(JX + v(X)U - u(X)V + \sum_{i=1}^{m}(\alpha_i(X)A_{m+i} - \alpha_{m+i}(X)A_i))) + \\
&+ \sum_{i=1}^{m}(2d\alpha_i(X,Y)A_{m+i} - 2d\alpha_{m+i}(X,Y)A_i - l\alpha_i(Y)\alpha_{m+i}(X)U + \\
&+ l\alpha_{m+i}(Y)\alpha_i(X)U)\}.
\end{aligned}$$

Thus, from (2.5), (3.9) and proposition 4.3, we deduce (5.6).

(5.7) follows from (5.5) and (5.6). □

Let x be a point of V^{2n+2m}. Denote by K_{XY} and by $\rho(X,X)$ the sectional curvature for the plane section in T_xM with orthonormal basis $\{X,Y\}$ and the Ricci curvature in the direction X respectively. Then, by using (5.1), (5.3) and (5.6), we obtain

COROLLARY 5.1. *On a sasakian m-hyperbolic l.c.K. manifold* V^{2n+2m}

$$K_{XU} = -\Big(\frac{l}{2}\Big)^2 \sum_{i=1}^{2m} (\alpha_i(X))^2,$$

$$K_{XA_i} = -\Big(\frac{l}{2}\Big)^2 \{(u(X))^2 + \sum_{j=1, j\neq i}^{2m} (\alpha_j(X))^2\}$$

$$K_{UA_i} = K_{A_iA_j} = -\Big(\frac{l}{2}\Big)^2$$

$$\rho(U,U) = \rho(A_i,A_i) = -2m\Big(\frac{l}{2}\Big)^2$$

for $i,j \in \{1,\dots,2m\}$.

COROLLARY 5.2. *On a sasakian m-hyperbolic l.c.K. manifold* V^{2n+2m}

$$K_{XV} = \Big(\frac{l}{2}\Big)^2 \{1 - (u(X))^2 - \sum_{j=1}^{2m} (\alpha_i(X))^2\}$$

$$K_{A_iV} = K_{UV} = 0$$

$$\rho(V,V) = 2(n-1)(\frac{l}{2})^2$$

for $i \in \{1,\dots,2m\}$.

From proposition 5.1, we have

COROLLARY 5.3. *On a sasakian m-hyperbolic l.c.K. manifold* V^{2n+2m}

$$R(X,Y)Z = R(X',Y')Z' + \frac{l^2}{2}\{\sum_{i=1}^{m} (\alpha_i \wedge u)(X,Y)(\alpha_i(Z)U - u(Z)A_i) + \\ - \sum_{i,j=1}^{2m} \alpha_j(Z)(\alpha_i \wedge \alpha_j)(X,Y)A_i\}$$

for all $X, Y, Z \in \mathfrak{X}(V^{2n+2m})$, *where* X', Y' *and* Z' *are the orthogonal projections of* X, Y *and* Z *respectively onto the tangent planes of the leaves of the foliation* $\mathfrak{F}$ *given by* $u = 0$, $\alpha_i = 0$, *with* $1 \leq i \leq 2m$.

Let $\overline{R}$ be the curvature tensor of the Weyl connection $\overline{\nabla}$ given in (2.3). Then,

PROPOSITION 5.3. *On a sasakian m-hyperbolic l.c.K. manifold* V^{2n+2m}

$$\overline{R}(X,Y)Z = R(X',Y')Z' - \frac{l^2}{4}\{g(Y',Z')X' - g(X',Z')Y'\}, \tag{5.8}$$

for all $X, Y, Z \in \mathfrak{X}(V^{2n+2m})$, *where* X', Y' *and* Z' *are the orthogonal projections of* X, Y *and* Z *respectively onto the tangent planes of the leaves of the foliation* $\mathfrak{F}$ *given by* $u = 0$, $\alpha_i = 0$, *with* $1 \leq i \leq 2m$.

PROOF. Using proposition 4.4 and a well known relation (see [9], pg. 115) we deduce

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z + \frac{l^2}{4}\{\sum_{i=1}^{2m}(&\alpha_i(Y)\alpha_i(Z)X - \alpha_i(X)\alpha_i(Z)Y + \\ &+ g(Y,Z)\alpha_i(X)A_i - g(X,Z)\alpha_i(Y)A_i) + \\ &+ (u(X)g(Y,Z) - u(Y)g(X,Z))U + \\ &+ (u(Y)u(Z)X - u(X)u(Z)Y) - (g(Y,Z)X - g(X,Z)Y)\}\end{aligned}$$

for all X, Y, $Z \in \mathfrak{X}(V^{2n+2m})$, and thus the result follows from corollary 5.3. □

6 – The universal covering space of a sasakian m-hyperbolic l.c.K. manifold

In this section we shall study the universal covering space of a sasakian m-hyperbolic l.c.K. manifold.

Let $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ be a sasakian m-hyperbolic l.c.K. manifold and let A_i be $(1 \leq i \leq 2m)$ as in (4.2) and l, u, U, v, V as in (4.5). Denote by $c = -\frac{l}{2}$ and by $\mathfrak{F}$ the foliation given by $u = 0$, $\alpha_i = 0$,

$1 \leq i \leq 2m$. $\mathfrak{F}$ defines on V^{2n+2m} a foliation of dimension $(2n-1)$, which we call the *canonical foliation* of V^{2n+2m}. Using (4.7), proposition 4.4 and corollary 5.1, we deduce

PROPOSITION 6.1. *The canonical foliation $\mathfrak{F}$ of a sasakian m-hyperbolic l.c.K. manifold is totally geodesic with integrable normal bundle. Moreover, if $\mathfrak{F}^{\perp}$ is the foliation determined by the normal bundle of $\mathfrak{F}$, then $\mathfrak{F}^{\perp}$ also is totally geodesic and its leaves are of constant sectional curvature $-c^2$.*

Let $i : N \longrightarrow V^{2n+2m}$ be the inmersion of a generic leaf N of the canonical foliation $\mathfrak{F}$. We define an almost contact metric structure $(\varphi_N, \xi_N, \eta_N, g_N)$ on N by

$$(6.1)\quad \varphi_N X = JX + (i^*v)(X)U\,|_N,\ \xi_N = -V\,|_N,\ \eta_N = -(i^*v),\ g_N = i^*g$$

for all $X \in \mathfrak{X}(N)$. Then, we have

PROPOSITION 6.2. *The almost contact metric structure $(\varphi_N, \xi_N, \eta_N, g_N)$ on N is c-sasakian.*

PROOF. Let X, Y be vector fields on N and N_J, N_{φ_N} and L the Nijenhuis tensors of J and φ_N and the Lie derivate on V^{2n+2m} respectively. Then,

$$\begin{aligned} &N_{\varphi_N}(X,Y) + 2d\eta_N(X,Y)\xi_N = \\ &= N_J(X,Y) - v(Y)\{(L_U J)X + (L_U v)(X)U\} + \\ &+ v(X)\{(L_U J)Y + (L_U v)(Y)U\} + 2(dv(JX,Y) + dv(X,JY))U \end{aligned}$$

which, from (2.6), (4.8) and (4.10), implies that the structure $(\varphi_N, \xi_N, \eta_N)$ is normal, i.e., $N_{\varphi_N} + 2d\eta_N \otimes \xi_N = 0$.

On the other hand, if ϕ_N and Ω denote the fundamental 2-form of N and the Kähler 2-form of V^{2n+2m} respectively then, using (6.1), we obtain that

$$\phi_N = i^*\Omega = i^*\left(\psi + 2\sum_{i=1}^{m}(\alpha_i \wedge \alpha_{m+i}) + 2v \wedge u\right) = i^*\psi\,.$$

Thus, from (4.10), we deduce that

$$d\eta_N = c\phi_N.$$

Consequently, $(\varphi_N, \xi_N, \eta_N, g_N)$ is a c-sasakian structure on N. □

Now, consider the inmersion $j : M \longrightarrow V^{2n+2m}$ of a generic leaf M of the foliation $\mathfrak{F}^{\perp}$ on V^{2n+2m}. We define an almost contact metric structure $(\varphi_M, \xi_M, \eta_M, g_M)$ on M by

$$\begin{aligned} \varphi_M(Y) = JY + (j^*u)(Y)V \mid_M, \quad \xi_M = U \mid_M, \\ \eta_M = (j^*u), \quad g_M = j^*g, \end{aligned} \tag{6.2}$$

for all $Y \in \mathfrak{X}(M)$. Then, we have

PROPOSITION 6.3. *The almost contact metric structure* $(\varphi_M, \xi_M, \eta_M, g_M)$ *on* M *is c-kenmotsu.*

PROOF. Let X, Y be vector fields on M and N_{φ_M} the Nijenhuis tensor of φ_M. Then,

$$\begin{aligned} N_{\varphi_M}(X,Y) = N_J(X,Y) + u(Y)\{(L_V J)(X) - (L_V u)(X)V\}+ \\ - u(X)\{(L_V J)(Y) - (L_V u)(Y)V\} \end{aligned}$$

and thus, using (4.8), (2.6) and since $L_V u = 0$, we obtain that $N_{\varphi_M}(X,Y) = 0$.

On the other hand, it is clear that the 1-form η_M is closed. Moreover, if ϕ_M is the fundamental 2-form of M then, from (6.2), we deduce that $\phi_M = j^*\Omega$, which, using (2.6), implies that $d\phi_M = \phi_M \wedge j^*\omega$, i.e.,

$$d\phi_M = -2c\eta_M \wedge \phi_M.$$

This completes the proof. □

Let N be a leaf of the canonical foliation $\mathfrak{F}$ and $(\varphi_N, \xi_N, \eta_N, g_N)$ the induced c-sasakian structure on N.

Suppose that N is of constant φ_N-sectional curvature k. Then, from (6.1) and using a theorem of Ogiue [17] and the fact that the foliation $\mathfrak{F}$ is totally geodesic, we have that

$$
\begin{aligned}
&R(X,Y)Z = \\
&=\frac{1}{4}(k+3c^2)(g(Y,Z)X - g(X,Z)Y)+ \\
&+\frac{1}{4}(k-c^2)\{v(X)v(Z)Y - v(Y)v(Z)X + (g(X,Z)v(Y)+ \\
&-g(Y,Z)v(X))V + g(JY,Z)JX - g(JX,Z)JY+ \\
&+2g(X,JY)JZ + (v(X)g(JY,Z) - v(Y)g(JX,Z)+ \\
&+2v(Z)g(X,JY))U\}
\end{aligned}
\tag{6.3}
$$

for all $X,\ Y,\ Z\ \in \mathfrak{X}(N)$, where R is the Riemann curvature tensor of V^{2n+2m}.

Now, we give the following definition.

Definition 6.1. *A sasakian m-hyperbolic l.c.K. manifold is called* ***sasakian (k) m-hyperbolic l.c.K.*** ($k \in \mathbb{R}$) *if every leaf N of the canonical foliation $\mathfrak{F}$ is of constant φ_N-sectional curvature k, where $(\varphi_N, \xi_N, \eta_N, g_N)$ is the induced c-sasakian structure on N given by* (6.1).

If $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ is a sasakian(k) m-hyperbolic l.c.K. manifold then V^{2n+2m} is said to have a *sasakian(k) m-hyperbolic l.c.K. structure* $(J, g, \alpha_1, \dots, \alpha_{2m})$.

Let V^{2n+2m} be a sasakian m-hyperbolic l.c.K. manifold. Denote by $\overline{R}$ the curvature tensor of the Weyl connection $\overline{\nabla}$ on V^{2n+2m} given by (2.3).

From ( ) and using corollary 5.3 and proposition 5.3, we obtain

Corollary 6.1. *If $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ is a sasakian m-hyperbolic l.c.K. manifold then, the following conditions are equivalent:*

i) *$(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ is a sasakian(k) m-hyperbolic l.c.K. manifold.*

ii) *For all* $X, Y, Z \in \mathfrak{X}(V^{2n+2m})$

$$
\begin{aligned}
&R(X,Y)Z = \\
&= \frac{1}{4}(k+3c^2)(g(Y',Z')X' - g(X',Z')Y') + \\
&+ \frac{1}{4}(k-c^2)\{v(X)v(Z)Y' - v(Y)v(Z)X' + (g(X',Z')v(Y) + \\
&- g(Y',Z')v(X))V + g(JY',Z')JX' - g(JX',Z')JY' + \\
&+ 2g(X',JY')JZ' + (v(X)g(JY',Z') - v(Y)g(JX',Z') + \\
&+ 2v(Z)g(X',JY'))U\} + \frac{l^2}{2}\{\sum_{i=1}^{m}(\alpha_i \wedge u)(X,Y)(\alpha_i(Z)U + \\
&- u(Z)A_i) - \sum_{i,j=1}^{2m} \alpha_j(Z)(\alpha_i \wedge \alpha_j)(X,Y)A_i\}
\end{aligned} \tag{6.4}
$$

where X', Y' *and* Z' *are the orthogonal projections of* X, Y *and* Z *respectively onto the tangent planes of the leaves of the canonical foliation.*

iii) *For all* X, Y, $Z \in \mathfrak{X}(V^{2n+2m})$

$$
\begin{aligned}
&\overline{R}(X,Y)Z = \\
&= \frac{1}{4}(k-c^2)\{g(Y',Z')X' - g(X',Z')Y' + v(X)v(Z)Y' + \\
&- v(Y)v(Z)X' + (g(X',Z')v(Y) - g(Y',Z')v(X))V + \\
&+ g(JY',Z')JX' - g(JX',Z')JY' + 2g(X',JY')JZ' + \\
&+ (v(X)g(JY',Z') - v(Y)g(JX',Z') + 2v(Z)g(X',JY'))U\}
\end{aligned} \tag{6.5}
$$

where X', Y' *and* Z' *are the orthogonal projections of* X, Y *and* Z *respectively onto the tangent planes of the leaves of the canonical foliation.*

If $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ is a sasakian m-hyperbolic l.c.K. manifold then, every point $x \in V^{2n+2m}$ has an open neighbourhood U such that the structure $(J, e^{-\sigma}g)$ is Kähler on U and $\overline{R}$ is the curvature tensor of the local metric $e^{-\sigma}g$, where $\sigma : U \longrightarrow \mathbb{R}$ is a real differentiable function on U (see section 2). Moreover, using (6.5) and proposition 5.3, we deduce

COROLLARY 6.2. *Let V^{2n+2m} be a sasakian m-hyperbolic l.c.K. manifold. Then, the following conditions are equivalent:*

i) *V^{2n+2m} is a sasakian(c^2) m-hyperbolic l.c.K. manifold.*

ii) *The leaves of the canonical foliation are of constant sectional curvature c^2.*

iii) *The local metrics $e^{-\sigma}g$ are flat, i.e., $\overline{R} = 0$.*

Next, we introduce a definition which will be useful in the sequel.

Let N, k be a (2n-1)-dimensional manifold and a real number respectively and let $(H_c^{2m+1}, (ds^2)_c)$ be the $(2m+1)$-dimensional hyperbolic space, with $c < 0$.

DEFINITION 6.2. *A* ***distinguished sasakian m-hyperbolic(c) l.c.K.*** *(respectively* ***distinguished sasakian (k) m-hyperbolic(c) l.c.K.****) structure on $V^{2n+2m} = N \times H_c^{2m+1}$ is a sasakian m-hyperbolic l.c.K. (respectively sasakian(k) m-hyperbolic l.c.K.) structure $(J, g, \alpha_1, \dots$ $\dots, \alpha_{2m})$ on V^{2n+2m}, such that:*

i) *The metric g is of the form*

$$g = d\sigma^2 + (ds^2)_c$$

where $d\sigma^2$ is a Riemann metric on N and,

ii) *The Lee 1-form ω and the 1-forms α_i, $1 \leq i \leq 2m$, are given by*

$$\omega = -2\frac{dx_{2m+1}}{x_{2m+1}}, \quad \alpha_i = \frac{dx_i}{cx_{2m+1}}$$

where $(x_1, \dots, x_{2m+1})$ are the usual coordinates on H_c^{2m+1}.

We have,

PROPOSITION 6.4. *If $(J, g, \alpha_1, \dots, \alpha_{2m})$ is a distinguished sasakian m-hyperbolic(c) l.c.K. structure on $V^{2n+2m} = N \times H_c^{2m+1}$, then the manifold N carries an induced c-sasakian structure $(\varphi_N, \xi_N, \eta_N, g_N)$ and the almost hermitian structure (J, g) on V^{2n+2m} is given by (3.2). Moreover, if $(J, g, \alpha_1, \dots, \alpha_{2m})$ is a distinguished sasakian(k) m-hyperbolic(c) l.c.K. structure on V^{2n+2m}, then N is of constant φ_N-sectional curvature k.*

PROOF. From definition 6.2, we obtain that

$$g = d\sigma^2 + (ds^2)_c \quad , \quad U = (cx_{2m+1})\frac{\partial}{\partial x_{2m+1}} \quad , \quad A_i = (cx_{2m+1})\frac{\partial}{\partial x_i}$$

for all $i \in \{1, \dots, 2m\}$, where $(x_1, \dots, x_{2m+1})$ are the usual coordinates on the hyperbolic space H_c^{2m+1}.

By using (4.6) and first and second relation of (4.7) and (4.10) we deduce that $\xi_N = -JU = -V$ and $\eta_N = u \circ J = -v$ define a vector field and a 1-form respectively on N.

Let X be a vector field on N. Then, $X = \overline{X} + v(X)V$ with $v(\overline{X}) = 0$. Define $\varphi_N X = J\overline{X}$.

From (4.9) and first and third relation of (4.8) we have that φ_N defines a $(1,1)$-tensor field on N.

Now, it is easy to check that $(\varphi_N, \xi_N, \eta_N, g_N = d\sigma^2)$ is an almost contact metric structure on N.

On the other hand, from definition 6.2, we deduce that the leaves of the canonical foliation of V^{2n+2m} are $N \times \{(x_1^0, \dots, x_{2m+1}^0)\}$, with $(x_1^0, \dots, x_{2m+1}^0) \in H_c^{2m+1}$. Thus, by proposition 6.2, we get a c-sasakian structure on each $N \times \{(x_1^0, \dots, x_{2m+1}^0)\}$, $(x_1^0, \dots, x_{2m+1}^0) \in H_c^{2m+1}$. In fact, if $(x_1^0, \dots, x_{2m+1}^0) \in H_c^{2m+1}$ then, it is not difficult to check that the application $i_{(x_1^0, \dots, x_{2m+1}^0)}$ of $N \times \{(x_1^0, \dots, x_{2m+1}^0)\}$ into N given by $i_{(x_1^0, \dots, x_{2m+1}^0)}(x, x_1^0, \dots, x_{2m+1}^0) = x$ is an almost contact isometry.

This, in view of proposition 6.2 and definition 6.1, completes the proof. □

REMARK. Let $(N, \varphi_N, \xi_N, \eta_N, g_N)$ be a c-sasakian manifold. Then, using corollary 3.1, we obtain that the product manifold $N \times H_c^{2m+1}$ carries an induced distinguished sasakian m-hyperbolic(c) l.c.K. structure $(J, g, \alpha_1, \dots, \alpha_{2m})$. Moreover, it is clear that if N is of constant φ_N-sectional curvature k then $(J, g, \alpha_1, \dots, \alpha_{2m})$ is a distinguished sasakian(k) m-hyperbolic(c) l.c.K. structure on $N \times H_c^{2m+1}$. Therefore, the converse of proposition 6.4 is also true.

Using the above remark and corollary 6.2 we obtain

COROLLARY 6.3. *On the sasakian m-hyperbolic l.c.K. manifold $S_{c^2}^{2n-1} \times H_c^{2m+1}$ the local conformal Kähler metrics are flat.*

Next, we shall describe the universal covering space of a sasakian m-hyperbolic l.c.K. manifold.

THEOREM 6.1. *The universal covering space of a $(2n+2m)$-dimensional complete sasakian m-hyperbolic l.c.K. manifold V^{2n+2m} with Lee form ω is a product space $\overline{V}^{2n+2m} = N \times H_c^{2m+1}$, where N is the universal covering space of an arbitrary leaf of the canonical foliation of V^{2n+2m}, $c = -\|\omega\|/2$ and H_c^{2m+1} is the $(2m+1)$-dimensional hyperbolic space. The lift of the sasakian m-hyperbolic l.c.K. structure to $\overline{V}^{2n+2m}$ gives a distinguished sasakian m-hyperbolic(c) l.c.K. structure on $\overline{V}^{2n+2m}$. Moreover, if the structure of V^{2n+2m} is a sasakian(k) m-hyperbolic l.c.K. structure, then, considering the induced c-sasakian structure on N, we have:*

i) *If $k > -3c^2$, then N is almost contact isometric to $S^{2n-1}(c,k)$;*
ii) *If $k = -3c^2$, then N is almost contact isometric to $\mathbb{R}^{2n-1}(c)$;*
iii) *If $k < -3c^2$, then N is almost contact isometric to $(\mathbb{R}\times CD^{n-1})(c,k)$.*

PROOF. Let $(V^{2n+2m}, J, g, \alpha_1, \dots, \alpha_{2m})$ be a $(2n+2m)$-dimensional complete sasakian m-hyperbolic l.c.K. manifold and u the unit Lee form of V^{2n+2m}.

Denote by $\overline{g}$ the induced metric on $\overline{V}^{2n+2m}$. Then, using proposition 6.1 and theorem A of [4], we deduce that $(\overline{V}^{2n+2m}, \overline{g})$ is the Riemannian product $N \times H_c^{2m+1}$, where N is the universal covering space of an arbitrary leaf of the canonical foliation $\mathfrak{F}$ and $c = -\frac{\|\omega\|}{2}$. Moreover, if $\mathfrak{F}^\perp$ is the foliation determined by the normal bundle of $\mathfrak{F}$ then, the lift of the foliations $\mathfrak{F}$ and $\mathfrak{F}^\perp$ to $\overline{V}^{2n+2m}$ are the foliations with leaves of the form $N \times \{x\}$ $(x \in H_c^{2m+1})$ and $\{n\} \times H_c^{2m+1}$ $(n \in N)$ respectively.

Now, let $\overline{\alpha}_i$ and $\overline{u}$ be the lift of α_i $(1 \le i \le 2m)$ and u respectively to $\overline{V}^{2n+2m}$. Then, it is clear, from (3.9) and from the fact that $\overline{u}$ is a closed 1-form, that $\{\overline{u}, \overline{\alpha}_1, \dots, \overline{\alpha}_{2m}\}$ is a global basis of 1-forms on H_c^{2m+1}. The dual basis of vector fields on H_c^{2m+1} is given by $\{\overline{U}, \overline{A}_1, \dots, \overline{A}_{2m}\}$, being $\overline{U}$ and $\overline{A}_i$ $(1 \le i \le 2m)$ the lift of U and A_i $(1 \le i \le 2m)$ respectively to $\overline{V}^{2n+2m}$. Thus, using the following lemma 6.1, we obtain that

$$\overline{U} = (cx_{2m+1})\frac{\partial}{\partial x_{2m+1}}\ , \quad \overline{A}_i = (cx_{2m+1})\frac{\partial}{\partial x_i}$$

for $i \in \{1, \dots, 2m\}$, where $(x_1, \dots, x_{2m+1})$ are the usual coordinates on H_c^{2m+1}. Consequently,

$$\overline{u} = \frac{dx_{2m+1}}{cx_{2m+1}} \quad , \quad \overline{\alpha}_i = \frac{dx_i}{cx_{2m+1}}$$

for $i \in \{1, \dots, 2m\}$, which implies that the lift of the sasakian m-hyperbolic l.c.K. structure $(J, g, \alpha_1, \dots, \alpha_{2m})$ to $\overline{V}^{2n+2m}$ is a distinguished sasakian m-hyperbolic(c) l.c.K. structure on $\overline{V}^{2n+2m}$.

If $(J, g, \alpha_1, \dots, \alpha_{2m})$ is a sasakian(k) m-hyperbolic l.c.K. structure on V^{2n+2m}, then the lift of this sasakian(k) m-hyperbolic l.c.K. structure to $\overline{V}^{2n+2m}$ gives a distinguished sasakian(k) m-hyperbolic(c) l.c.K. structure on $\overline{V}^{2n+2m}$ and therefore, since N is a simply connected complete manifold, the rest of theorem follows using proposition 6.4 and proposition 2.2. □

LEMMA 6.1. *Let M be a $(2m+1)$-dimensional complete, simply connected, Riemannian manifold of constant negative curvature $-c^2$ $(c \neq 0)$ and U, A_i vector fields on M such that $\{U, A_1, \dots, A_{2m}\}$ form an orthonormal basis for M and $[U, A_i] = cA_i$, $[A_i, A_j] = 0$ for $i, j \in \{1, \dots, 2m\}$. Then, there is an isometry F of M to the $(2m+1)$-dimensional hyperbolic space H_c^{2m+1}, satisfying*

$$F_*U = (cx_{2m+1})\frac{\partial}{\partial x_{2m+1}} \quad , \quad F_*A_i = (cx_{2m+1})\frac{\partial}{\partial x_i},$$

for $i \in \{1, \dots, 2m\}$, where $(x_1, \dots, x_{2m+1})$ are the usual coordinates on H_c^{2m+1}.

PROOF. Let x be a point of M. We consider the linear isometry L of T_xM onto $T_{(0,\dots,0,1)}(H_c^{2m+1})$ given by

$$L(U_x) = c(\frac{\partial}{\partial x_{2m+1}}) \mid_{(0,\dots,0,1)} \quad , \quad L((A_i)_x) = c(\frac{\partial}{\partial x_i}) \mid_{(0,\dots,0,1)}$$

for $i \in \{1, \dots, 2m\}$. Then, there is an isometry F of M onto H_c^{2m+1} such that the differential of F at x is L (see, for instance, [13]) and thus, using the relations $[U, A_i] = cA_i$, $[A_i, A_j] = 0$ $(1 \leq i, j \leq 2m)$ we prove that

$$F_*U = (cx_{2m+1})\frac{\partial}{\partial x_{2m+1}} \quad , \quad F_*A_i = (cx_{2m+1})\frac{\partial}{\partial x_i},$$

for $i \in \{1, \dots, 2m\}$. □

Finally, from theorem 6.1, we deduce

COROLLARY 6.4. : *Let V^{2n+2m} be a complete sasakian(k) m-hyperbolic l.c.K. manifold, $\overline{V}^{2n+2m}$ the universal covering space of V^{2n+2m} and $c = -\|\omega\|/2$, where ω is the Lee 1-form of V^{2n+2m}.*

i) *If $k > -3c^2$, then $\overline{V}^{2n+2m}$ is almost complex isometric to $S^{2n-1}(c,k) \times H_c^{2m+1}$,*

ii) *If $k = -3c^2$, then $\overline{V}^{2n+2m}$ is almost complex isometric to $\mathbb{R}^{2n-1}(c) \times H_c^{2m+1}$ and,*

iii) *If $k < -3c^2$, then $\overline{V}^{2n+2m}$ is almost complex isometric to $(\mathbb{R} \times CD^{n-1})(c,k) \times H_c^{2m+1}$.*

In particular, if V^{2n+2m} is a complete sasakian(c^2) m-hyperbolic l.c.K. manifold then $\overline{V}^{2n+2m}$ is almost complex isometric to $S_{c^2}^{2n-1} \times H_c^{2m+1}$.

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REFERENCES

[1] L.C. DE ANDRES - L.A. CORDERO - M. FERNANDEZ - J.J. MENCIA: *Examples of four-dimensional compact locally conformal Kähler solvmanifolds*, Geometriae Dedicata **29** (1989), 227-232.

[2] D.E. BLAIR: *The theory of quasi-sasakian structures*, J. Diff. Geom. **1** (1967), 331-345.

[3] D.E. BLAIR: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., **509**, Springer (1976).

[4] R.A. BLUMENTHAL - J.J. HEBDA: *De Rham decomposition theorems for foliated manifolds*, Ann. Inst. Fourier, Grenoble, **33** (1983), 183-198.

[5] E. CALABI - B. ECKMANN: *A class of compact, complex manifolds which are not algebraic*, Ann. of Math. **58** (1953), 494-500.

[6] L.A. CORDERO - M. FERNANDEZ - M. DE LEON: *Compact locally conformal Kähler nilmanifolds*, Geometriae Dedicata **21** (1986), 187-192.

[7] D. CHINEA - J.C. GONZALEZ: *A classification of almost contact metric manifolds*, Ann. Mat. Pura ed appl. (IV) **156** (1990), 15-36.

[8] B.Y. CHEN - P. PICCINNI: *The canonical foliations of a locally conformal Kähler manifold*, Ann. di Mat. Pura ed appl. **141** (4) (1985), 289-305.

[9] S.I. GOLDBERG: *Curvature and homology*, Academic Press, New York, 1962.

[10] A. GRAY - L.M. HERVELLA: *The sixteen classes of almost hermitian manifolds and their linear invariants*, Ann. Mat. Pura ed appl. **123** (1980), 35-58.

[11] D. JANSSENS - L. VANHECKE: *Almost contact structures and curvature tensors*, Kodai Math. J. **4** (1981), 1-27.

[12] K. KENMOTSU: *A class of almost contact Riemannian manifolds*, Tôhoku Math. Journ. **24** (1972), 93-103.

[13] S. KOBAYASHI - K. NOMIZU: *Foundations of Differential Geometry*, Vol. I and II. Intersc. Publ., New York (1969).

[14] P. LIBERMANN: *Sur les structures presque complexes et autres structures infinitesimales regulieres*, Bull. Soc. Mat. France **83** (1955), 195-224.

[15] J.C. MARRERO: *The local structure of trans-sasakian manifolds*, Ann. di Mat. Pura ed appl. (IV) **162** (1992), 77-86.

[16] K. MATSUMOTO: *On locally conformal Kähler space forms*, Internat. J. Math. Sci. **8** (1) (1985), 69-74.

[17] K. OGIUE: *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, Kodai Math. Sem. Rep. **16** (1964), 223-232.

[18] M. OKUMURA: *On infinitesimal conformal and proyective transformations of normal contact spaces*, Tôhoku Math. J. **14** (1962), 398-412.

[19] G. TALLINI: *Metriche locali dotate di una connessione globale su una varieta differentiabile*, Period. Math. **46** (1968), 340-358.

[20] S. TANNO: *The topology of contact Riemannian manifolds*, Illinois J. Math. **12** (1968),700-717.

[21] S. TANNO: *Sasakian manifolds with constant φ-holomorphic sectional curvature*, Tôhoku Math. J. **21** (1969), 501-507.

[22] F. TRICERRI: *Some examples of locally conformal Kähler manifolds*, Rend. Sem. Mat. Univ. Politec. Torino **40** (1) (1982), 81-92.

[23] I. VAISMAN: *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976), 338-351.

[24] I. VAISMAN: *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Mat. VI 12 (1979), 263-284.

[25] I. VAISMAN: *Generalized Hopf manifolds*, Geometriae Dedicata 13 (1982), 231-255.

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