Rendiconti di Matematica, Serie VII
Volume 13 , Roma (1993), 87-94

Schunck classes of finite π-soluble groups with the D-property

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RIASSUNTO: *Si risolvono, per i gruppi π-risolubili, i problemi dell'esistenza dei sottogruppi di ricoprimento e del coniugio tra essi; sono studiate inoltre le classi di Schunck dei π-gruppi π-risolubili per i quali valga la proprietà D.*

ABSTRACT: *All groups considered are finite. We investigate in π-soluble groups the problems of existence and cnjugation of covering subgroups. We analyze the Schunck classes of π-soluble π-groups with the D-property.*

KEY WORDS: *Primitive - Homomorph - X-projector - X-covering.*

A.M.S. CLASSIFICATION: 20D10

1 – Introduction. Notation

All groups considered are finite. In any class of groups considered, we will suppose that if a group G belongs to the class, any group isomorphic to G belongs to the class too.

A group G is said to be primitive if it has a faithful representation as a primitive permutation group.

Primitive groups are characterized as those groups G which have a maximal subgroup M with $\mathrm{Core}_G(M) = 1$.

E denotes the class of all groups. S denotes the class of all soluble groups. W_π^S denotes the class of all π-soluble groups. P denotes the class of all primitive groups. We designate a maximal subgroup M of a group G with the notation $M < \cdot\, G$. We designate a minimal normal subgroup N of a group G with the notation $N \cdot \trianglelefteq G$.

Let X be a class of groups. Let us consider the following operators between classes of groups:

Let Q be such that

$$QX = (G \in E | G \cong H/N \text{ for some } H \in X \text{ and } N \trianglelefteq H).$$

A class X of groups is called Q-closed or homomorph if $QX \subseteq X$.

Let S be such that

$$SX = (G \in E | G \leq U \text{ with } U \in X).$$

A class X of groups is called S-closed if $SX \subseteq X$.

Let V be a non-empty homomorph, which we consider the universe. A class $X \subseteq V$ is called V-Schunck class if $X = PX$ where

$$PX = (G \in V | Q(G) \cap P \subseteq X).$$

Let X be a class of groups. Let G be a group and $H \leq G$. H is called X-maximal in G if:

i) $H \in X$, and

ii) If $H \leq L \leq G$ and $L \in X$, then $H = L$.

Let X be a class of groups. Let G be a group and $H \leq G$. H is called X-projector of G if:

$$HK/K \text{ is } X-\text{maximal in } G/K \text{ for every } K \trianglelefteq G.$$

The set of all X-projectors of G is denoted by $\mathrm{Proj}_X(G)$.

Let X be a class of groups. Let G be a group and $H \leq G$. H is called X-covering subgroup of G if:

$$H \in \mathrm{Proj}_X(U) \text{ when } H \leq U \leq G.$$

The set of all X-covering subgroups of G is denoted by $\mathrm{Cov}_X(G)$.

Let X be a class of groups. We will denote

$$\sigma(X) = \{p | p \text{ is a prime number and } p|\ |G| \text{ for some group } G \in X\}.$$

In this paper we resolve in π-soluble groups the problems of existence and conjugation of covering subgroups. Moreover, the results obtained in soluble groups by WOOD [6] are generalized to π-soluble groups and Schunck classes X with the D-property in π-soluble groups are studied, when the class X is a class of π-groups and therefore of soluble groups, in which case we prove that the D-property is fulfilled in π-soluble groups if and only if it is fulfilled in soluble groups. COVACI also studies these problems in [1] and [2], when X is a homomorph which fulfils:

$$G/O_{\pi'}(G) \in X \text{ implies } G \in X,$$

which this author calls π-homomorph. A π-homomorph which is a Schunck class is called π-Schunck class by this author.

We next investigate the existence and conjugation of covering subgroups in π-soluble groups.

PROPOSITION 1. a) *Let π be a set of prime numbers. If X is a class of π-soluble groups and $\sigma(X) \subseteq \pi$, then the following statements are equivalent:*

i) *X is a W_π^S-Schunck class.*

ii) *Every π-soluble group has X-covering subgroups.*

b) *Let π be a set of prime numbers. If X is a class of groups and $\sigma(X) \subseteq \pi$, then any two X-covering subgroups of a π-soluble group G are conjugated in G.*

PROOF. a) Let us see that i) implies ii). This is a consequence of FÖRSTER ([5], (4.2) Satz) and of ERICKSON ([4], Theorem 2), since the latter Theorem is valid when V is a non-empty S-closed homomorph and X is a V-Schunck class and $G \in V$, this generalization having an analogous proof to that of Erickson.

Let us see that ii) implies i). This is a consequence of FÖRSTER ([5], (4.2) Satz), since every X-covering subgroup is an X-projector.

b) Let us prove it by induction on $|G|$. Let $G \neq 1$ and let H_1 and H_2 be two X-covering subgroups of G. Two possibilities are considered:

1) There exists $M \cdot \trianglelefteq G$ such that $|H_1M| < |G|$ or $|H_2M| < |G|$; let us suppose that $|H_1M| < |G|$ since the other case is analogous. It holds that H_1M/M and H_2M/M are X-covering subgroups of G/M and

by induction are conjugated in G/M; therefore there exists some $x \in G$ such that $H_1M = H_2^X M$. Hence H_1 and H_2^X are X-covering subgroups of H_1M and by induction H_1 and H_2^X are conjugated in H_1M and therefore H_1 and H_2 are conjugated in G.

2) For every $M \cdot \trianglelefteq G$ it holds that $H_1M = G = H_2M$. Let us suppose that there exists a minimal normal subgroup M of G which is a π'-group; then as H_1 and H_2 are π-groups, they will be Hall π-subgroups of G and by the Hall-Čunihin Theorem they are conjugated in G. The other case is that every minimal normal subgroup M of G is a soluble π-group, in which case it is deduced that G is a π-group and therefore G is soluble. Then M is abelian; we can suppose that $G \notin X$. Then $H_1 \cap M = H_2 \cap M = 1$ for every $M \cdot \trianglelefteq G$. Hence $H_1 < \cdot G, H_2 < \cdot G$ and $\mathrm{Core}_G(H_1) = \mathrm{Core}_G(H_2) = 1$ and $G \in P$ and H_1 and H_2 are complements of the unique minimal normal subgroup and by the Theorem of Galois they are conjugated in G.

DEFINITION 2 ([2], §1). *Let π be a set of prime numbers. It is said that a W_π^S-Schunck class X is a D-class of π-soluble groups if it has the D-property, which means that for every π-soluble group G, every X-subgroup of G is contained in an X-covering subgroup of G.*

The case in which π is the set of all prime numbers has been considered by WOOD in [6]. The following definitions 3.a) and 3.b) are introduced in soluble groups by WOOD in [6] and in π-soluble groups by COVACI in [2]. We define the F-property which allows us to obtain a new characterization of the D-classes of π-soluble groups with $\sigma(X) \subseteq \pi$.

DEFINITION 3. *Let π be a set of prime numbers and X a W_π^S-Schunck class.*

a) *X has the A-property if for any π-soluble group G and for any subgroup H of G with $\mathrm{Core}_G(H) \neq 1$, every X-covering subgroup of H is contained in an X-covering subgroup of G.*

b) *X has the B-property if for any π-soluble group G and for any minimal normal subgroup N of G, the existence of an X-covering subgroup of G which avoids N implies that every X-maximal subgroup in G avoids N.*

c) *X has the F-property if for any π-soluble group G, the existence of an X-covering subgroup S of G with $S < \cdot G$ and* $\mathrm{Core}_G(S) = 1$ *implies that for every X-maximal subgroup in G there exists $M < \cdot G$ such that $H \leq M$ and* $\mathrm{Core}_G(H) = 1$.

From now on we will study the D-classes X of π-soluble groups which fulfil the condition $\sigma(X) \subseteq \pi$, since with these conditions we can assert that every π-soluble group has X-covering subgroups and that any two X-covering subgroups of a π-soluble group G are conjugated in G, which is fundamental to the following.

THEOREM 4. *Let π be a set of prime numbers. Let X be a class of soluble groups. Then the following statements are equivalent:*

i) *X is a D-class of soluble groups and $\sigma(X) \subseteq \pi$.*

ii) *X is a D-class of π-soluble groups and $\sigma(X) \subseteq \pi$.*

PROOF. ii) implies i). It is trivial.

i) implies ii). If X is a D-class of soluble groups it is obvious that X is a W_π^S-Schunck class. Therefore we must only see that if G is a π-soluble group, then the X-maximal subgroups in G are X-covering subgroups of G.

Let G be a π-soluble group and H be an X-maximal subgroup in G; then H is a π-group and therefore there exists a Hall π-subgroup L of G such that $H \leq L$; as L is soluble by the D-property in soluble groups it holds that H is an X-covering subgroup of L. Let S be an X-covering subgroup of G; then $S \leq L^g$ with $g \in G$ and thus $S^{g^{-1}} \leq L$ and L contains X-covering subgroups of G which are also X-covering subgroups of L and therefore H and $S^{g^{-1}}$ are conjugated in L and thus H is an X-covering subgroup of G.

THEOREM 5. *Let π be a set of prime numbers. If X is a W_π^S-Schunck class and $\sigma(X) \subseteq \pi$, then the following statements are equivalent:*

i) *X has the A-property.*

ii) *X has the D-property.*

iii) *X has the B-property.*

iv) *X has the F-property.*

v) *X fulfils that for every π-soluble group G, the existence of an X-covering subgroup S of G with $S < \cdot\, G$ and* $\mathrm{Core}_G(S) = 1$, *implies that for every X-maximal subgroup H in G it holds that $H < \cdot G$ and* $\mathrm{Core}_G(H) = 1$.

PROOF. i) implies ii). It is obvious that if i) is fulfilled, then X is a S-Schunck class which fulfils the A-property in soluble groups and therefore X is a D-class of soluble groups according to WOOD [6, Theorem 3], hence according to Theorem 4 X is a D-class of π-soluble groups.

ii) implies iii). This is due to the fact that if X has the D-property, then for every π-soluble group G, every X-maximal subgroup in G is an X-covering subgroup of G and by the conjugation of the X-covering subgroups the conclusion is obtained.

iii) implies iv). X is a S-Schunck class which fulfils the B-property in soluble groups since it fulfils it in π-soluble groups. Hence, according to WOOD [6, Theorem 3], X is a D-class of soluble groups. According to Theorem 4 X is a D-class of π-soluble groups. Finally, it is obvious that if X has the D-property, then X has the F-property.

iv) implies v). If we suppose that iv) is true and we want to obtain v) as a conclusion, it is sufficient to prove that X has the D-property and we obtain the conclusion by the conjugation of the X-covering subgroups.

Let G be a π-soluble group; we must see that the X-maximal subgroups in G are X-covering subgroups of G; for this it will be seen that if H is an X-maximal subgroup in G and S is an X-covering subgroup of G, then H and S are conjugated in G. We prove it by induction on $|G|$. Let $G \neq 1$ and H be an X-maximal subgroup in G and S be an X-covering subgroup of G. If $G \in X$ it is obvious. Let us suppose that $G \notin X$ and let $N \cdot \trianglelefteq G$. Applying the induction to G/N we deduce the existence of an element $g \in G$ such that $H \leq S^g N$.

There are two cases:

a) $S^g N < G$. The conclusion is evident in this case by the inductive hypothesis.

b) $S^g N = G$, and therefore $SN = G$.

If $\mathrm{Core}_G(S) \neq 1$ we consider a minimal normal subgroup N_1 of G such that $N_1 \leq \mathrm{Core}_G(S)$ and it will hold that, on applying the induction to G/N_1, there exists an element $g_1 \in G$ such that $H \leq S^{g_1}$ and therefore $H = S^{g_1}$.

If $\text{Core}_G(S) = 1$, there are two cases, one when N is a soluble π-group and another when N is a π'-group.

Let us suppose that N is a π'-group; in this case, since $G = SN$ and $S \in X$, then S is a Hall π-subgroup of G. As $H \in X$, then H is a π-group and therefore $H \leq S^{g_2}$ for some $g_2 \in G$ according to the Hall-Čunihin Theorem; since $S^{g_2} \in X$ we deduce that $H = S^{g_2}$.

Let us suppose now that N is a soluble π-group; as $G = SN$, then G is a π-group and therefore G is soluble.

Let us see now that $S < \cdot G$. Obviously $S < G$ and if $S \leq S^* < G$, then $S^* = S(S^* \cap N) = S$.

Hence $S < \cdot G$ with $\text{Core}_G(S) = 1$ and as X fulfils the F-property there exists $M < \cdot G$ such that $H \leq M$ and $\text{Core}_G(M) = 1$. Then, since G is a soluble primitive group, it holds that by the Theorem of Galois there exists an element $g_3 \in G$ such that $M = S^{g_3}$. Hence $H \leq S^{g_3}$ and as $S^{g_3} \in X$, it holds that $H = S^{g_3}$ which was what we were looking for.

v) implies i). It is obvious that v) implies iv). In the previous reasoning we have seen that iv) implies ii) and it is evident that ii) implies i).

Acknowledgements

This paper is part of the author's Doctoral Thesis at the University of Valencia (Spain). The author expresses his sincerest thanks to this supervisor Professor M.J. Iranzo and also to Professor F. Pérez Monasor for their valuable guidance.

REFERENCES

[1] R. Covaci: *Projectors in Finite π-Solvable Groups*, Studia Univ. Babeş-Bolyai, Math. 1(1977), 3-5.

[2] R. Covaci: *π-Schunck Classes with the D-Property*, Studia Univ. Babeş-Bolyai, Math. 2(1977), 3-8.

[3] K. Doerk - T.O. Hawkes: *Finite Soluble Groups*, (Preprint).

[4] R.P. Erickson: *Projectors of finite groups*, Comm. Algebra 10(1982), 1919-1938.

[5] P. Förster: *Projektive Klassen endlicher Gruppen. I. Schunck- und Gaschütz-klassen*, Math. Z. 186(1984), 149-178.

[6] G.J. WOOD: *A Lattice of Homomorphs*, Math. Z. 130(1973), 31-37.

[7] M.O. VALERO OLTRA: *Operadores en clases de grupos finitos y familias de subgrupos asociadas*, Tesis Doctoral, Universidad de Valencia (1989).

Lavoro pervenuto alla redazione il 14 febbraio 1992
ed accettato per la pubblicazione il 1 ottobre 1992

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