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On Whittaker transform of generalized functions

N.K. AGRAWAL

RIASSUNTO: *Dopo aver esteso alle distribuzioni la trasformazione di Whittaker data in* (1.1), *viene stabilito un teorema di analiticità.*

ABSTRACT: *The Whittaker transform given by* (1.1) *has been extended to generalized functions. Different testing function spaces have been constructed and their properties have been studied. The Whittaker transform of generalized functions has been defined and the analyticity theorem has been proved.*

KEY WORDS: *Whittaker transform - Generalized Function - Analyticity theorem.*

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1 – Introduction

In the present paper we extend the Whittaker transform given by

$$F(x) = \int_0^\infty (xt)^{\lambda-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k,m}(xt) f(t)\,dt \tag{1.1}$$

to a class of generalized functions (1.1) is a generalization of the Laplace transform and reduces to Laplace transform for $\lambda = -m, k+m = \frac{1}{2}$; λ, k, m being complex numbers. MAINRA [1] has studied (1.1) in detail. For the sake of convenience, we quote below some of the results on Whittaker functions which we shall need off and on. From ERDELYI [2, pp.

264,258] we have

$$W_{k,m}(x) = e^{-\frac{1}{2}x} x^{m+\frac{1}{2}} \Psi(-k+m+\frac{1}{2}, 2m+1; x) \tag{1.2}$$

$$\begin{aligned} &\frac{d^q}{dx^q}\Big\{e^{-x}\Psi(-k+m+\frac{1}{2}, 2m+1; x)\Big\} = \\ &=(-1)^q e^{-x}\Psi(-k+m+\frac{1}{2}, 2m+1; x) \end{aligned} \tag{1.3}$$

From WHITTAKER and WATSON [3, pp.336,340]

$$W_{k,m}(tu) = (tu)^k e^{-\frac{1}{2}tu}\Big\{1 + 0(u^{-1})\Big\} \tag{1.4}$$

for $t > 0$, u large; and also

$$W_{k,m}(tu) = (tu)^{\pm m+\frac{1}{2}}(1 + 0(u)) \tag{1.5}$$

for $t > 0$ and u small.

The notation and terminology of the present work follows that of [4]. The symbol $D'(I)$ denotes the space of distributions defined over the testing function space $D(I)$, whereby $D(I)$ we mean the space of infinitely differentiable complex valued functions having compact support defined over the open set $I(0 < t < \infty)$. The topology of $D(I)$ is that which makes its dual the space $D'(I)$ of Schwartz distributions. $E(I)$ is the space of all infinitely differentiable functions on I. Its dual $E'(I)$ is the space distributions with compact support.

2 – The testing function spaces $W_{a,b}(I)$ and $\overline{W}_{a,b}(I)$

DEFINITION 2.1. *We define $W_{a,b}(I)$ to be the set of all those complex valued smooth functions on I for which the expression*

$$\gamma_n(\phi) \triangleq \gamma_{a,b,n}(\phi) \triangleq \sup_{0<t<\infty} \left| e^{bt} t^{a+n} \frac{d^n\phi(t)}{dt^n} \right| < \infty, \quad n = 0, 1, 2 \dots ;$$

where a, b are suitably fixed numbers. The topology on $W_{a,b}(I)$ is defined by means of the separating collection of seminorms $\{\gamma_n\}_n^\infty = 0$. Cauchy sequence and convergent sequence in $W_{a,b}(I)$ are defined as in [4, pp. 7-10] It can be readily seen that $W_{a,b}(I)$ is a locally convex, sequentially complete, Hausdorff topological vector space.

$W'_{a,b}(I)$ is dual of $W_{a,b}(I)$. $D(I)$ is a subspace of $W_{a,b}(I)$ and the topology of $D(I)$ is stronger than the topology induced on $D(I)$ by $W_{a.b}(I)$ and as such the restriction of any member of $W'_{a,b}(I)$ to $D(I)$ is in $D'(I)$.

DEFINITION 2.2. *We define $\overline{W}_{a,b}(I)$ to be the set of all those complex valued smooth functions $\phi(t)$ defined on I such that for each non-negative integer n, the expression*

$$P_n(\phi) \triangleq \operatorname*{Sup}_{0<t<\infty} \left| e^{bt} t^a \left(t \frac{d}{dt}\right)^n \phi(t) \right| < \infty,$$

for suitably fixed real numbers a and b.

The topology on $\overline{W}_{a,b}(I)$ is defined by means of the separating collection of seminorms $\{P_n\}_{n=0}^\infty$. The concepts of convergence and completeness in $\overline{W}_{a,b}(I)$ are defined in a way similar to that in $W_{a,b}(I)$. The space $\overline{W}_{a,b}(I)$ is also a locally convex, sequentially complete, Husdorff topological vector space.

LEMMA 2.1. *Let $W_{a,b}(I)$ and $\overline{W}_{a,b}(I)$ be testing function spaces as given in Definitions 2.1 and 2.2, a, b being some fixed real numbers; then*

(i) *$W_{a,b}(I) = \overline{W}_{a,b}(I)$ in store of elements; and*

(ii) *the topology T_1 generated by the sequence of seminorms $\{\gamma_n\}_{n=0}^\infty$ is the same as the topology T_2 generated by $\{P_n\}_{n=0}^\infty$.*

PROOF. The proof for (i) is obvious and that for (ii) can be obtained using the results in [4, p.18; [5], p.29]

LEMMA 2.2. *Let a, b be such that*
(i) $\mathrm{Re}(a+\lambda \pm m) \geq 0, \mathrm{Re}\, m \geq 0$;
(ii) $\mathrm{Re}\, s > b$, *s not living on the negative real axis; then $W(st) \in W_{a,b}(I)$, where*

$$W(st) = (st)^{\lambda-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)$$

PROOF. In view of properties of the Whittaker function $W_{k,m}(x)$ enlisted in section 1, the proof of the lemma is simple.

We have following corollaries:

COROLLARY 2.2-1. *Under the conditions of lemma 2.2,*

$$\frac{\partial^p W(st)}{\partial s^p} \in W_{a,b}(I).$$

COROLLARY 2.2-2. *Let r and p be non-negative integers and Ω any compact set of complex plane not containing any point of negative real axis; then*

$$\gamma_r\left\{\frac{\partial^p W(st)}{\partial s^p}\right\} \leq < \infty$$

uniformly for all s lying in Ω, B_Ω being a real constant depending on Ω.

3 – The generalized functions spaces $W_{a,b}(I)$

$W'_{a,b}(I)$ is dual of $W_{a,b}(I)$. Members of $W'_{a,b}(I)$ are generalized functions. Concepts of convergence and completeness in $W'_{a,b}(I)$ are defined as in [4, p.21]. As $W_{a,b}(I)$ is complete, $W'_{a,b}(I)$ is also complete. Certain elements of $W'_{a,b}(I)$ can be identified with conventional functions. For example, let $f(t)$ be a locally integrable function such that $e^{-bt}t^{-a}f(t))$, is absolutely integrable on I, then $f(t)$ generates a member of $W'_{a,b}(I)$ through the definition

$$\langle f, \Phi\rangle = \int_0^\infty f(t)\Phi(t)\, dt \quad , \quad \Phi \in W_{a,b}(I) \tag{3.1}$$

Indeed

$$|\langle f,\Phi\rangle| = \left|\int_0^\infty f(t)\Phi(t)\,dt\right| = \left|\int_0^\infty (e^{-bt}t^{-a}f(t))(e^{bt}t^{a}\Phi(t))\,dt \le\right.$$
$$\le \gamma_0(\Phi)\int_0^\infty |e^{-bt}t^{-a}f(t)|dt$$

which shows that (3.1) truly defines a functional f on $W_{a,b}(I)$. Linearity of this functional is obvious.

Also, if $\{\Phi_n\}_{n=1}^{\infty}$ converges in $W_{a,b}(I)$ to Zero, then $\gamma_0(\Phi_n) \longrightarrow 0$ so that $|\langle f,\Phi_n\rangle| \longrightarrow 0$. This shows that f is continuous too. Hence $f \in W'_{a,b}(I)$ and is a generalized function. Members of $W'_{a,b}(I)$ generated in this way from conventional functions of the stated type are called Regular generalized functions. A generalized function not generated by any such conventional function is known as singular generalized function. An example of a singular element of $W'_{a,b}(I)$ is given by $\delta_a (a>0)$, a translated δ-function defined by

$$\langle \delta a,\Phi\rangle = \Phi(a) \quad , \quad a>0 \quad , \quad \Phi \in W_{a,b}(I)\,.$$

4 – Properties of space $W_{a,b}(I)$ and its dual $W_{a,b}(I)$

PROPERTY 4.1. Since

$$|e^{b_1 t}t^{a+n}D^n\Phi(t)| < |e^{b_2 t}t^{a+n}D^n\Phi(t)|$$

for $b_1 < b_2$, we have

$$\gamma_{a,b_1,n}(\Phi) \le \gamma_{a,b_2,n}(\Phi) \tag{4.1}$$

Hence, whenever $b_1 < b_2$

(i) $W_{a,b_2} \subseteq W_{a,b_1}(I)$; and

(ii) the topology of $W_{a,b_2}(I)$ is stronger than the topology induced on it by $W_{a,b_1}(I)$ [4, Lemma 1.6.3] and thus the restriction of any $f \in W'_{a,b_1}(I)$ to $W_{a,b_2}(I)$ is in $W'_{a,b_2}(I)$ (speaking more loosely $f \in W'_{a,b_2}(I)$).

PROPERTY 4.2. $D(I) \subset W_{a,b}(I) \subset E(I)$. Since $D(I)$ is dense in $E(I)$, it follows that $W_{a,b}(I)$ is dense in $E(I)$. It can be easily proved that if $\{\Phi_n\}$ converges to Φ in $D(I)$, then $\{\Phi_n\}$ converges to Φ in $W_{a,b}(I)$. For, supports of Φ_n and Φ are all contained in some closed interval $[c, d]$, $0 < c < d < \infty$, so that

$$\gamma_r(\Phi_n - \Phi) = \operatorname*{Sup}_{0<t<\infty} \left|e^{bt}t^{a+r}D^r(\Phi_n - \Phi)\right| =$$
$$= \operatorname*{Sup}_{c<t<d} \left|e^{bt}t^{a+r}D^r(\Phi_n - \Phi)\right| \le C \operatorname*{Sup}_{c<t<d} \left|D^r(\Phi_n - \Phi)\right|$$

Where $C = \operatorname*{Sup}_{c<t<d} |e^{bt}t^{a+r}|$. It follows that the restriction of any $f \in W'_{a,b}(I)$ to $D(I)$ is in $D'(I)$; hence $W'_{a,b}(I) \subseteq D'(I)$.

PROPERTY 4.3. For each $f \in W'_{a,b}(I)$, there corresponds a positive constant C and a non-negative integer n such that, for all $\Phi \in W_{a,b}(I)$

$$|\langle f, \Phi\rangle| \le C \max_{0<r\le n} \gamma_r(\Phi)$$

This follows from Theorem 1.8-1 of [4].

5 – The $W_{\lambda,k,m}$ - transform of generalized functions

We shall call f a $W_{\lambda,k,m}$ - transformable generalized function if it is a member of $W'_{a,b}(I)$ for some fixed real numbers a and b. From property 4.1 we see that if $f \in W'_{a,b}(I)$ then $f \in W'_{a,b}(I)$ for every $b' > b$. This implies that there exists a real number σ_f (possibly $\sigma_f = -\infty$) such that $f \in W'_{a,b}(I)$ for every $b > \sigma_f$ and $f \notin W'_{a,b}(I)$ for $b < \sigma_f$.

DEFINITION 5.1. *Let $f \in W'_{a,b}(I)$. Then $W_{\lambda,k,m}$ - transform of f denoted by $W_{\lambda,k,m}(f)$ is defined by the relation*

$$F(s) \triangleq (W_{\lambda,k,m}f)(s) = \langle f(t), W(st)\rangle \tag{5.1}$$

where $W(st) = (st)^{\lambda-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)$ and $s \in \Omega_f$. The region Ω_f is defined as follows:

$$\Omega_f = \left\{s | \operatorname{Re} s > \sigma_f,\ s \ne 0,\ -\pi < \arg s < \pi\right\} \tag{5.2}$$

If $\sigma_f < 0, \sigma_f$ is a cut half plane obtained by deleting all real non-positive values of s.

The Definition 5.1 has a sense as an application of $f \in W'_{a,b}(I)$ to $W(st) \in W_{a,b}(I)$ for fixed real numbers a and b with $\mathrm{Re}(s) > b$, $\mathrm{Re}(a + \lambda \pm m) > 0$. The number σ_f is called the abscissa of definition and Ω_f the region of definition for the generalized $W_{\lambda,k,m}$ - transform.

THEOREM 5.1 (Analyticity Theorem). *Let $F(s) = W_{\lambda,k,m}(f)$ for $s \in \Omega_f$. Then $F(s)$ is an analytic function of s and*

$$D_s^n F(s) = \langle f(t), \frac{\partial n}{\partial s^n} W(st)\rangle, \quad s \in \Omega_f \tag{5.3}$$

Where $W(st) = (st)^{\lambda-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)$ and $D_s \equiv \dfrac{\partial}{\partial s}$, $n = 1, 2, 3, \ldots$.

PROOF. In the light of the Cor. 2.2 we see that the right hand of (5.3) is well defined. We prove the theorem for $n = 1$ and the theorem will follow for every $n = 2, 3, \ldots$ by method of induction. Let s be an arbitrary but fixed point in Ω_f. Let us choose real positive numbers b, r and r_1 such that

$$\sigma_f < b < b' = \mathrm{Re}\, s - r_1 < \mathrm{Re}\, s - r < \mathrm{Re}\, s\,.$$

Let C denotes the circle whose centre is at s and whose radius is r_1. Let us restrict r_1 (and thereby b and r) still further by requiring that C lies entirely within Ω_f (i.e., C does not intersect the non-positive real axis). Finally, let as be a non-zero complex increment such that $|\Delta s| < r$ and consider the expression

$$\frac{F(s + \Delta s) - F(s)}{\Delta s} - \langle f(t) D_s W(st)\rangle = \langle f(t), \Psi_{\Delta s}(t)\rangle \tag{5.4}$$

where $W(st) = (st)^{\lambda-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)$ and $\Psi_{\Delta s}(t) = \frac{W(\overline{s+\Delta s}t) - W(st)}{\Delta s} - D_s W(st)$.

From Cor. 2.2 we see that $D_s^n W(st)$, $n = 0, 1, 2, \ldots$; are members of $W_{a,b}(I)$, hence (5.3) and (5.4) are well defined. We shall now show that $\Psi_{\Delta s}(t) \longrightarrow 0$ in $W_{a,b}(I)$ as $\Delta s \longrightarrow 0$, which in consequence will prove that $\langle f(t), \Psi_{\Delta s}(t)\rangle \longrightarrow 0$ $\Delta s \longrightarrow 0$ (from continuity property of $f \in W'_{a,b}(I)$), whence proving the equality (5.3) for $n = 1$.

We can change the order of differentiation with respect to t to that with respect to s in $W(st)$. Applying Cauchy's integral formula to $\Psi_{\Delta s}(t)$ we get

$$\begin{aligned}\frac{d^n}{dt^n}\Psi_{\Delta s}(t) =& \frac{\partial^n}{\partial t^n}\left\{\frac{W(\overline{s+\Delta s}t) - W(st)}{\Delta s} - \frac{\partial W(st)}{\partial s}\right\} = \\ =& \frac{1}{\Delta s}\left[W_{n_t}(\overline{s+\Delta s}t) - W_{n_t}(st)\right] - \frac{\partial}{\partial s}W_{n_t}(st)\end{aligned}$$

The suffix n_t indicates n times partial derivative with respect to t.

Or,

$$\begin{aligned}\frac{d^n}{dt^n}\Psi_{\Delta s}(t) =& \frac{1}{\Delta s}\left[\frac{1}{2\pi i}\int_c \left\{\frac{1}{z-s-\Delta s} + \right.\right. \\ & \left.\left. - \frac{1}{z-s}\right\} W_{n_t}(zt)dz\right] - \frac{1}{2\pi i}\int_c \frac{W_{n_t}(zt)}{(z-s)^2}dz = \\ =& \frac{1}{2\pi i}\int_c W_{n_t}(zt)\left[\frac{1}{\Delta s}\left\{\frac{1}{z-s-\Delta s} - \frac{1}{z-s}\right\} - \frac{1}{(z-s)^2}\right]dz = \\ =& \frac{\Delta s}{2\pi i}\int_c \frac{W_{n_t}(zt)}{(z-s-\Delta s)(z-s)^2}dz.\end{aligned}$$

Now,

$$\left|e^{bt}t^{a+n}\frac{d^n}{dt}\Psi_{\Delta s}(t)\right| = \left|\frac{\Delta s}{2\pi i}\int_c \frac{e^{bt}t^{a+n}W_{n_t}(zt)}{(z-s-\Delta s)(z-s)^2}dz\right| \leq$$

$$\begin{aligned}\leq& \frac{|\Delta s|}{2\pi}\int_c \frac{|e^{b^t}t^{a+n}W_{n_t}(zt)|}{|(z-s-\Delta s)(z-s)^2|}|dz| \leq \\ \leq& \frac{|\Delta s|}{2\pi}\frac{2\pi r_1}{(r-r_1)r_1^2} = \frac{|\Delta s| M}{(r-r_1)r_1}\end{aligned} \tag{5.5}$$

From Cor. 2.3 $|e^{bt}t^{a+n}W_{n_t}(zt)|$ is bounded on any compact subset of Ω_f, let this bound be $\leq M$; hence (5.5) is justified. From (5.5) we see that

$$\operatorname*{Sup}_{0<t<\infty}\left|e^{bt}t^{a+n}\frac{d^n\Psi_{\Delta s}(t)}{dt^n}\right|\longrightarrow 0 \quad \text{as} \quad \Delta s\longrightarrow 0\,.$$

This proves the theorem.

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INDIRIZZO DELL'AUTORE:

N.K. Agrawal - Department of Mathematics - Marwari College - Ranchi - Address for correspondence - D-121 - Harmoo HSG Colony - Ranchi - 834012 (INDIA).