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Resurgence of a majorant series arising in the normalization of symplectic maps

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RIASSUNTO: *In questa nota studiamo l'equazione funzionale*

$$3g(z) = g\big(g(z)\big) + f(z)\,,$$

dove $f(z)$ è una funzione olomorfa tale che $f(0) = 0$, $f'(0) = 2$.

Nel caso $f(z) = z^2 + 2z$, l'unica soluzione formale tangente all'identità è una serie maggiorante per la trasformazione a forma normale delle mappe simplettiche $F : \mathbf{R}^{2n} \mapsto \mathbf{R}^{2n}$ con un punto fisso ellittico nell'origine (cfr. [1]).

Dimostriamo in questo lavoro che la soluzione formale è risorgente con le singolarità appartenenti a $\Omega = \mathbf{N}(\log 2 + 2\pi i\mathbf{Z})$. Da questo risultato si derivano delle informazioni precise sull'esistenza di una soluzione analitica e sul suo dominio massimale di esistenza.

ABSTRACT: *In this note we study the functional equation*

$$3g(z) = g(g(z)) + f(z)\,,$$

where $f(z)$ is a holomorphic function such that $f(0) = 0, f'(0) = 2$.

In the case $f(z) = z^2 + 2z$, the unique tangent to identity formal solution is a majorant series for the normalizing diffeomorphisms of symplectic maps $F : \mathbf{R}^{2n} \mapsto \mathbf{R}^{2n}$ with an elliptic fixed point at the origin (see [1]).

We prove here that the formal solution is resurgent with singularities on $\Omega = \mathbf{N}\,(\log 2 + 2\pi i\mathbf{Z})$. This gives us precise information on the existence of an analytic solution and its maximal existence domain.

KEY WORDS: *Functional equations – Asymptotic series – Borel summability – Resurgence – Normal forms.*

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1 – Introduction

Normal forms are an important tool for the investigation of the local properties of differential equations or diffeomorphisms and of Hamiltonian systems and symplectic maps. In constructing the estimates to the formal power series solutions of the theory of symplectic normal forms [1] the Cauchy method was used and the majorant series was first found by one of us [8] to satisfy a functional equation (2.1) which seems to retain some relevant aspects of the original conjugacy equation. Indeed it was proved that the coefficients of order n of its formal solution grow as $n!$ [9], that it has a solution analytic in a disc and it was numerically showed the existence of a self similar set of non integer dimension of singularities [3].

A mathematical investigation of this equation on its own seems therefore to be justified in order for instance to determine the maximal domain of holomorphy and the behaviour in a neighborhood of the origin, where the singularities form a parabolic cusp and have a density exponentially small with the length on any arc [9].

The appropriate mathematical framework to investigate the properties of this equation is the summability theory [7] and the resurgence theory [5].

Let $\hat{h}$ be an element of $\mathbf{C}[[\frac{1}{w}]]$

$$\hat{h}(w) = \sum_{n=0}^{\infty} \frac{a_n}{w^{n+1}} \ . \tag{1.1}$$

The *Borel transform* $\hat{h}_B = \mathcal{B}\hat{h}$ of $\hat{h}$ is the element of $\mathbf{C}[[t]]$ defined by

$$\hat{h}_B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \ . \tag{1.2}$$

If $\hat{h}_B$ has a non zero convergence radius it defines an element of $\mathbf{C}\{t\}$ and it will be denoted by h_B. Moreover, if h_B can be analytically continued to a neighborhood U of $\mathbf{R}_+$ and its growth at $+\infty$ is less than exponential, i.e. there exists two positive constants M and λ such that

$$|h_B(t)| \le M e^{\lambda |t|} \tag{1.3}$$

for all $t \in U$, we will say that the formal series $\hat{h}$ is *Borel–summable.* In this case the Laplace transform of h_B

$$(1.4) \qquad (\mathcal{L}h_B)(w) = \int_0^{+\infty} e^{-wt} h_B(t)dt$$

defines a function $h(w)$ analytic on $\{\mathrm{Re}\, w > \lambda\}$ which will be called the *Borel sum* of $\hat{h}$.

Note that if h_B and g_B are analytic germs at 0 the usual product of formal series becomes, under the action of the Borel transform, the convolution of analytic germs at 0

$$(1.5) \qquad (\hat{h}\hat{g})_B = h_B * g_B$$

where

$$(1.6) \qquad (h_B * g_B)(t) = \int_0^t h_B(u) g_B(t-u) du \; .$$

Moreover the derivative $\partial/\partial w$ is transformed by

$$(1.7) \qquad \left(\frac{\partial \hat{h}}{\partial w}\right)_B (t) = -t\, h_B(t) \; .$$

Let $\Omega \subset \mathbf{C}$ be a closed set consisting of isolated points (typically a lattice or a sub–lattice). According to ECALLE [5] (we recommend the reading of [4] for a nice elementary introduction) an element $\hat{h} \in \mathbf{C}[[\frac{1}{w}]]$ will define a resurgent function when its Borel transform is an element of $\mathbf{C}\{t\}$ and can be analytically continued (as a multivalued function) to $\mathbf{C} \setminus \Omega$ with a growth less than exponential at ∞. At each singular point $\omega \in \Omega$ the analytic continuation of h_B has locally the form

$$(1.8) \qquad \frac{\gamma_\omega}{2\pi i(t-\omega)} + h_\omega(t)\frac{\log(t-\omega)}{2\pi i} + k_\omega(t-\omega) \; ,$$

where h_ω and k_ω are elements of $\mathbf{C}\{t\}$. We denote by $\mathbf{A} = \mathbf{A}(\Omega)$ the set of resurgent functions with singularities in Ω.

For our needs, the main result contained in [5] is Ecalle's theorem

A *is a convolution sub-algebra of the algebra of germs holomorphic at* 0

We remark – but we will not use this fact – that Ecalle's resurgent functions are a special case of holomorphic microfunction, as noted by MALGRANGE [6].

The convolution algebra **A** is not unitary, since the Borel transform of 1 is Dirac's delta distribution. One can therefore extend the algebra **A** by considering the convolution algebra

$$\hat{\mathbf{A}} = \mathbf{C}\delta \oplus \mathbf{A} \tag{1.9}$$

with $\delta * h = h$. Thus one is led to consider the Borel transform as the map

$$\mathcal{B} : c + \sum_{n=0}^{\infty} \frac{a_n}{w^{n+1}} \mapsto c\delta + \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n . \tag{1.10}$$

Note also that one can consistently define for all $n \geq 0$

$$\mathcal{B}(w^n) = \delta^n(t) = \left(\frac{d}{dt}\right)^n \delta(t) . \tag{1.11}$$

2 – Statement of the results

We use throughout what follows the notations and the terminology of ECALLE [5]. In [9,3] it was proven that the functional equation

$$3g(z) = g(g(z)) + z^2 + 2z\ , \quad g(0) = 0,\ g'(0) = \frac{g''(0)}{2} = 1. \tag{2.1}$$

has a unique formal solution $\hat{g}(z) = \sum_{n=1}^{\infty} \hat{g}_n z^n$ which is divergent but of type Gevrey–1, see [7], i.e. there exists positive constants C and A such that

$$|\hat{g}_n| \leq CA^n n!\ . \tag{2.2}$$

Note that if one drops the initial conditions $g'(0) = \frac{g''(0)}{2} = 1$ in (2.1) then the functional equation admits a convergent solution $g(z) = 2z -$

$\frac{1}{3}z^2 + \ldots$. We now sketch the proof that the formal divergent solution $g(z) = z + z^2 + \ldots$ of (2.1) is actually resurgent.

PROPOSITION 1. *The unique formal solution of* (2.1) *is the asymptotic development of a resurgent function with singularities in* $\Omega := \mathbf{N}\,(\log 2 + 2\pi i\mathbf{Z})$.

SKETCH OF THE PROOF. Subtracting the identity to g and by the change of variables $z = 1/w$ we can write (2.1) in the form

$$2g(w) - g(w-1) = \frac{1}{w^2} + g(w - 1 + \phi(w)) - g(w-1)\,, \tag{2.3}$$

where, with a slight abuse of notation, we have denoted $g(w) = \frac{1}{w^2} + \frac{2}{w^3} + \ldots$ and

$$\phi(w) = 1 - \frac{w^2 g(w)}{1 + wg(w)} = \mathcal{O}\left(\frac{1}{w}\right)\,. \tag{2.4}$$

Note that

$$g(w - 1 + \phi(w)) - g(w-1) = \sum_{n=1}^{\infty} \frac{1}{n!}\left[\left(\frac{d}{dw}\right)^n g(w-1)\right](\phi(w))^n\,. \tag{2.5}$$

If we insert (2.5) in the r.h.s. of (2.3) and we apply the Borel transform we find

$$(2 - e^t)g_B(t) = t + \sum_{n=1}^{\infty} \frac{1}{n!}\left[(-t)^n e^t g_B(t)\right] * \phi_B^{*n}\,, \tag{2.6}$$

where ϕ_B^{*n} denotes the convolution of ϕ_B with itself n times. We can now compute explicitly ϕ_B^{*n} by means of the definition (2.4) of ϕ, and applying the result to (2.6) we find

$$\begin{aligned}(2 - e^t)g_B(t) = {}& t + \sum_{n=1}^{\infty} \frac{1}{n!}\left[(-t)^n e^t g_B(t)\right] * \delta^{(n)}(t) * \sum_{h=0}^{n}\binom{n}{h}(-1)^h \delta^{(h)}(t) *\\ & * g_B^{*(n-h)}(t) * \left(g_B^{*h}(t) + \sum_{p=0}^{h-1}(-1)^{h-p}\binom{h}{p} t^{2(h-p)-1} * g_B^{*p}(t)\right) *\\ & * \left[\delta(t) + \sum_{k=1}^{n}\binom{n}{k}\sum_{s=k}^{\infty}(-1)^s\binom{s-1}{k-1}\delta^{(s)}(t) * g_B^{*s}(t)\right]\end{aligned} \tag{2.7}$$

where we have adopted the convention $g_B^{*0} = \delta$.

We want to stress that all the terms in the r.h.s of (2.7) are at least of order g_B^{*2}. Therefore one may solve the equation by applying standard perturbative methods.

Note, moreover, that the first term of the perturbative solution, i.e.

$$g_{B,0}(t) = \frac{t}{2-e^t}\,, \tag{2.8}$$

is the Borel transform of the solution of the linear functional equation

$$2g_0(w) - g_0(w-1) = \frac{1}{w^2}\,, \tag{2.9}$$

which has been studied in [2] since its formal solution provides a majorant series for the first integrals of symplectic maps in a neighborhood of an elliptic fixed point. The perturbation expansion of g_B will contain only convolutions of (2.8) with itself, with powers of t and with derivatives of $\delta(t)$. Then one can see that g_B will have singularities only in the set $\mathbf{N}(\log 2 + 2\pi i \mathbf{Z})$, since the convolution of two singularities gives rise to a new one if and only if they lie on the same line through the origin [4]. Moreover g_B will have less than exponential growth at ∞ in every direction that does not pass through the singularities.

3 – Remarks and generalizations

In what follows Δ_ω denotes the Ecalle's derivative (alien derivative; we refer to [5], [4] and [6] for its definition and properties).

We now consider the (slightly) more general problem given by the functional equation

$$3g(z) = g(g(z)) + f(z)\,, \quad g(0) = 0,\ g'(0) = 1. \tag{3.1}$$

where $f(z)$ is a germ of holomorphic function in a neighborhood of $z = 0$ verifying the conditions $f(0) = 0, f'(0) = 2$. (3.1) has therefore a unique formal solution $\hat{g}(z) = \sum_{n=1}^{\infty} \hat{g}_n z^n$, the coefficients of which can be recursively determined from those of $f(z) = \sum_{n=1}^{\infty} f_n z^n$: $\hat{g}_1 = 1$, $\hat{g}_2 = f_2$ and for all $n \geq 3$

$$\hat{g}_n = f_n + \sum_{s=2}^{n-1} \hat{g}_s \sum_{k_1+\ldots+k_s=n\, k_i \geq 1} \hat{g}_{k_1} \cdots \hat{g}_{k_s} \,. \tag{3.2}$$

The following proposition holds:

PROPOSITION 2. *Let g_B denote the Borel transform of the solution of* (3.1). *If $\Delta_\omega g_B = 0$ for all $\omega = \log f_2 + 2\pi i k$, where $k \in \mathbf{Z}$, then $\Delta_\omega g_B = 0$ for all $\omega \in \mathrm{N}(\log f_2 + 2\pi i \mathbf{Z})$.*

The proof of proposition 2 is simply obtained by mimicking the proof of proposition 1. Moreover one has the following (obvious) corollary:

COROLLARY. (3.1) *has a holomorphic solution if and only if the Borel transform f_B of f has zeros in* $(\log f_2 + 2\pi i \mathbf{Z})$.

REMARKS.

1. The analysis of the growth of the Borel transform g_B is related (by Laplace transform) to the maximal domain D of definition of the analytic solution g of (3.1) (or (2.1)). For the equation (2.1) some preliminary results are reported in [9], where numerical evidence for the existence of a fractal natural boundary is given.
2. The equations studied in this note are an example of a much more general class of functional equations

$$a_0(z) + a_1(z)g(z) + a_2(z)g \circ g(z) + \ldots + a_n(z)g^{\circ n}(z) = 0 \,, \tag{3.3}$$

   where $\circ$ denotes composition, and for all $j = 0, \ldots, n$ one has that $a_j \in \mathrm{C}\{z\}$. This class evidently contains both equations (2.1) and (3.1), and the iteration problem solved in [5], which corresponds to the case $a_1(z) = \ldots = a_{n-1}(z) \equiv 0$, $a_n(z) \equiv 1$, $a_0'(0) =$ root of unity. To our knowledge this problem, in the general form (3.3), has not yet been investigated.

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