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Solution of equality-constrained quadratic programming problems by a projection iterative method

M. BENZI(*)

RIASSUNTO: *Viene sviluppato un procedimento iterativo per la soluzione del seguente problema di minimo vincolato:*

$$\tfrac{1}{2}x^T Ax - x^T s = \text{minimo}, \quad Ex = t,$$

dove A è una matrice $n \times n$ simmetrica definita positiva ed E è una matrice $m \times n$ di rango massimo, $m < n$. Il metodo è una generalizzazione dell'algoritmo di Kaczmarz per la soluzione di sistemi lineari, e fornisce allo stesso tempo approssimazioni sia per la soluzione x che per il corrispondente vettore λ dei moltiplicatori di Lagrange. Si dà una dimostrazione della convergenza dello schema iterativo e se ne mette in luce la relazione con il metodo SOR. Si suggerisce l'uso del metodo qui presentato nella soluzione di certi problemi di analisi di strutture elastiche.

ABSTRACT: *This paper is concerned with the development of an iterative scheme for the solution of the following constrained minimization problem:*

$$\text{minimize } \tfrac{1}{2}x^T Ax - x^T s, \quad \text{subject to} \quad Ex = t,$$

where A is a symmetric positive definite $n \times n$ matrix and E is a full-rank $m \times n$ matrix, $m < n$. The method is a generalization of Kaczmarz's projection scheme for solving linear systems, and it provides simultaneous approximations for both x and the corresponding Lagrange multiplier vector λ. Convergence is proved, and the relation between this scheme and the SOR method is clarified. The method appears to be potentially useful for the solution of problems of structural analysis.

KEY WORDS: *Constrained optimization - Lagrange multipliers - Iterative methods - Row-action methods - SOR method - Structural analysis.*

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1 – Statement of the problem

Consider the following constrained minimization problem:

$$\begin{aligned} &\text{minimize} && \tfrac{1}{2}\, x^T A x - x^T s \\ &\text{subject to} && Ex = t \end{aligned} \tag{1}$$

where A is a real symmetric positive definite (SPD) $n \times n$ matrix, E a real $m \times n$ matrix with full row rank ($m < n$), s a real n-vector, t a real m-vector, and x the unknown n-vector to be determined. The feasible set, defined by $Ex = t$, is an affine manifold of dimension $n - m$: it is closed and convex. The objective function $f(x) = \frac{1}{2}x^T Ax - x^T s$ is strictly convex.

From general results in mathematical programming, problem (1) admits one and only one solution. Geometrically, the solution is the A-orthogonal projection of A^{-1}s, the solution of the unconstrained problem, onto the feasible set (A-orthogonal means orthogonal with respect to the inner product $<x, y>= x^T Ay$).

The Lagrange optimality conditions yield the following system of linear equations (where λ is the vector of Lagrange multipliers):

$$Ax + E^T\lambda = s, \tag{2}$$

$$Ex = t. \tag{3}$$

Equations (2)-(3), often called *equilibrium equations*, can be written in the form

$$\begin{bmatrix} A & E^T \\ E & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}. \tag{4}$$

The $(n+m) \times (n+m)$ coefficient matrix in (4) is symmetric, nonsingular, indefinite. The unique solution of (4) will be denoted by (x^*, λ^*); notice that x^* and λ^* are related through the identities

$$x^* = A^{-1}(s - E^T\lambda^*), \tag{5}$$

$$\lambda^* = (EE^T)^{-1}E(s - Ax^*) \tag{6}$$

which are immediate consequences of (2)-(3).

There are many applications leading to the solution of equality-constrained quadratic programming problems of the type (1). At times, the problem presents itself already in the form of equilibrium equations (2)-(3).

Some of these applications (to fluid flow, electric networks and structures) have been elegantly described by Strang ([29], [30]). Other applications arise in statistics (data interpolation, surface fitting–see [28]) and in the image processing field ([14]).

Notice that any equality-constrained linear least squares problem:

$$\begin{aligned} &\text{minimize} && \| d - Gx \|_2 \\ &\text{subject to} && Ex = t \end{aligned}$$

(where G is a matrix with full column rank) is a special case of problem (1), with $A = G^T G$, $s = G^T d$.

Most often, equilibrium equations occur not by themselves, but as a step in a much more complex problem. For example, they are the result of a linearization process, or of a time discretization of an evolution problem; so, they must be solved a number of times. Clearly, it is important to develop efficient solution algorithms for (4). Given the variety of situations leading to problem (1) (or (4)), it is not surprising that the literature in this field is very rich. Moreover, research papers are scattered on a number of journals in Numerical Analysis, Linear Algebra, Optimization, Engineering, Fluid Dynamics, Image Processing, Statistics, etc. For the sake of brevity, we avoid going into any detail; a list of problems and relative bibliographic references is given in [2].

We conclude this section by pointing out that there are many important problems (in all of the above mentioned application areas) where the matrix A has a simple form, for example diagonal or block diagonal with blocks of small size. The last situation occurs, typically, in structural analysis. The point is that for such problems, it is easy to compute and to manipulate A^{-1} (and to compute the Cholesky factorization $A^{-1} = LL^T$, if necessary). This fact is important since, as we shall see, the matrix A^{-1} explicitly appears in many of the algorithms for the solution of the equilibrium equations.

2 – An equivalent formulation

It is important to observe that problem (1) is strictly related to the problem of finding the minimum l_2-norm solution of an underdetermined system of linear equations. This is easily seen with the help of a suitable change of variable (see [21], [22]). Since A is SPD, so is A^{-1}; hence we can consider the Cholesky factorization $A^{-1} = LL^T$ where L is lower triangular. Multiplying both sides of equation (2) on the left by L^T we get

$$L^{-1}x + L^T E^T \lambda = L^T s.$$

Rewriting (3) as

$$EL(L^{-1}x - L^T s) = t - EA^{-1}s$$

and introducing a new vector $z = L^{-1}x - L^T s$ (the change of variable), we obtain the system

$$z + (EL)^T \lambda = 0, \tag{7}$$

$$ELz = t - EA^{-1}s. \tag{8}$$

But (7)-(8) are the Lagrange optimality conditions for the following constrained minimization problem:

$$\begin{aligned} &\text{minimize} && \tfrac{1}{2} \parallel z \parallel_2^2 \\ &\text{subject to} && ELz = t - EA^{-1}s. \end{aligned} \tag{9}$$

In (9), $\| \cdot \|_2$ denotes the l_2-norm. Clearly, this problem has a unique solution z^*. Letting $x^* = Lz^* + A^{-1}s$ (the inverse change of variable), we recover the solution of problem (1). This relationship is quite useful, since many algorithms have been devised to compute the minimum norm solution of an underdetermined, consistent linear system. Each of those methods can be used, in principle, to solve the constrained minimization problem (1). As we shall see, sometimes it is not even necessary to perform the change of variable (and, in particular, the Cholesky factorization of A^{-1}) explicitly.

For some problems, however, A is diagonal or block diagonal with blocks of small size, so that inverting A and computing the Cholesky factorization of A^{-1} is an almost trivial task.

3 – Solving the equilibrium equations

A number of algorithms have been proposed for the solution of equality-constrained quadratic programming problems. The question is still open as to which solution method, for a given application and computing environment, should be preferred. In real-world applications, direct methods are usually chosen in structural analysis and electric networks computations, whereas iterative methods are used for the Stokes equations and in image processing (here, we regard conjugate-gradient type methods as iterative).

There are also methods which combine direct and iterative techniques. Recently, some attention has been paid to the use of graph-theoretical concepts, which is quite natural since the equilibrium matrix E can often be interpreted as the incidence matrix of some grid, or network, associated to the problem.

Also, the implementation of various algorithms on high-performance computers has received increasing attention in the last five years. See [2] and [3] for a few bibliographic references. Although our main interest is in iterative methods, in particular on those methods which provide successive approximations for both x^* and λ^* (simultaneously), it is worthwhile to say a few words about one of the major approaches used in solving equilibrium equations, which is essentially a block elimination on (4).

This technique is usually called the *displacement method*.

In the displacement method, λ^* (which in structural analysis represents the nodal displacements of the structure, up to the sign) is computed first; then x^* is computed from (5). Multiplying (2) by EA^{-1} on the left and using (3) to eliminate x, we get

$$EA^{-1}E^T\lambda = EA^{-1}s - t. \tag{10}$$

The coefficient matrix in (10) is SPD (since E has full rank), hence there exists one and only one solution λ^*. The linear system (10) can be solved in a variety of ways; a common procedure is to use Cholesky factorization.

This method is popular in structural analysis, where $s = 0$ and A^{-1} can be computed cheaply (if it is not already available from the formulation of the problem); clearly, obtaining the vector x^* from (5) is also inexpensive. However, there are some drawbacks to this method. Ex-

plicit computation of the matrix $EA^{-1}E^T$ is very much like forming the normal equations when solving least squares problems.

The usual disadvantages occur: loss of information due to round-off errors, but also loss of sparsity and structure which are usually present in both A and E. Also, computing $EA^{-1}E^T$ may not be feasible if the dimensions m and n are very large.

In the case of electric networks, it is sometimes possible to determine the entries of $EA^{-1}E^T$ without explicitly forming the matrix products. This holds because A^{-1} is a diagonal matrix and E an incidence matrix (see Strang [29]).

An alternative approach is the null space method (also known as force, or dual variable, method), where the vector x^*, which represents the internal forces in structural analysis, is computed first.

The vector λ^*, if necessary, is obtained from (6). The null space method does not require the computation of A^{-1}, but it does require the computation of a basis of $N(E)$, the null space of the equilibrium matrix E. For a recent paper on this approach, see [24].

Another major research area is the one devoted to iterative methods which generate two sequences $\{x^{(k)}\}$ and $\{\lambda^{(k)}\}$ for the approximation of x^* and λ^*. Early references on this approach are Hestenes [17], Powell [26] and Polyak [25]. The augmented Lagrangian approach, which belongs to this class of iterative methods, has been considered by many authors. A detailed treatment, and a substantial bibliography, are given by Fortin and Glowinski [10]; see also Glowinski [13]. Several algorithms and applications are considered in an interesting paper by Axelsson [1]. In these papers, the applications are to finite element computations, particularly in fluid mechanics and elasticity; Galligani et al. [11] consider an augmented Lagrangian approach, combined with a polynomial preconditioning for the conjugate gradient algorithm, which proves quite effective in solving the image restoration problem on the CRAY X-MP.

Other interesting contributions to the solution of equilibrium equations were given by Dyn and Ferguson [9] and by Plemmons [23].

We conclude this short overview by mentioning a relatively old paper by Bregman [6].

This author describes a general iterative method to find a common point of a family of convex, closed subsets of a (locally convex) topological vector space. He also shows how the method could be adapted to solve

constrained optimization problems. The paper is difficult to read, in part because of misprints and omissions in the English translation. In papers by Censor and Lent ([8] and [22]), these errors were corrected and the theory was further developed. These authors consider applications to the very large optimization problems arising in image reconstruction. The method of Bregman, together with some general results in the theory of iterative methods for linear systems, is the basis for the projection algorithm studied in the next sections.

4 – Description of the method

The proposed scheme is based on the following well-known results. Given a consistent system of linear equations $Ex = t$ and a starting vector $x^{(0)} \in R(E^T)$, the range of the transpose of E, one forms the sequence $\{x^{(k)}\}$ obtained by successively projecting the iterates $x^{(0)}, x^{(1)}, \ldots$ onto the hyperplanes corresponding to the equations in $Ex = t$, taken in a cyclic order. By a theorem proved by Tanabe ([31]), the sequence is convergent to the minimum norm solution of the system. For a square, nonsingular matrix the method was first considered by Kaczmarz ([19]; see also [12], [18]). The projections onto the hyperplanes are orthogonal with respect to the Euclidean norm.

If A is a symmetric positive definite $n \times n$ matrix, a norm is defined on $\mathbb{R}^n$ by setting $\| x \|_A = (x^T A x)^{\frac{1}{2}}$, for all $x \in \mathbb{R}^n$; the quantity $\| x \|_A$ will be called the A-norm of x. Two vectors $x, y \in \mathbb{R}^n$ are said to be A-orthogonal (or A-conjugate) if $x^T A y = 0$.

The A-orthogonal projection of a point $x \in \mathbb{R}^n$ onto an hyperplane $H \subset \mathbb{R}^n$ is the unique point $\hat{x} \in H$ such that $\| x - \hat{x} \|_A < \| x - y \|_A$ for all $y \in H$, $y \neq \hat{x}$.

In this section we show how to extend Kaczmarz's method and Tanabe's theorem to the solution of the equality-constrained minimization problem (1). By selecting in an appropriate fashion the initial vectors $x^{(0)}$ and $\lambda^{(0)}$, two sequences $\{x^{(k)}\}$ and $\{\lambda^{(k)}\}$ are generated, which converge to the solution (x^*, λ^*) of the equilibrium equations (4).

To get the iterate $x^{(k)}$, one takes the A-orthogonal projection of $x^{(k-1)}$ onto the kth hyperplane in the system $Ex = t$. Once $x^{(k)}$ is computed, the dual iterate $\lambda^{(k)}$ is easily obtained (as it turns out, at no additional cost) by requiring that $x^{(k)}$ and $\lambda^{(k)}$ solve the equation (2). Choosing $x^{(0)} = 0$,

the sequence $\{x^{(k)}\}$ converges to the minimum A-norm solution of $Ex = t$; choosing $x^{(0)} = A^{-1}s$, the sequence will converge to the solution x^* of $Ex = t$ which minimizes the quadratic function $f(x) = \frac{1}{2}x^TAx - x^Ts$, i.e. to the solution of problem (1). The dual variables $\{\lambda^{(k)}\}$ converge to the corresponding Lagrange multipliers vector λ^*. This convergence result is proved in the next section; since for $A = I$ and $s = 0$ we obtain the method of Kaczmarz, the theorem can be regarded as a generalization of Tanabe's theorem.

We introduce some notations, in order to formally describe the method. The feasible set is denoted by K:

$$K = \{x \in \mathbb{R}^n | Ex = t\}.$$

The rows of E are denoted by $r_i^T (1 \leq i \leq m)$ and the components of the m-vector t by t_i, so that the equation of the ith hyperplane in $Ex = t$ is $r_i^Tx = t_i$. Letting

$$K_i = \{x \in \mathbb{R}^n | r_i^Tx = t_i\}$$

we have $K = \bigcap_{i=1}^{m} K_i$. Given $x^{(0)}$, the projections are in the following cyclic order:

$$x^{(1)} \in K_1,\ x^{(2)} \in K_2, \ldots,\ x^{(m)} \in K_m,\ x^{(m+1)} \in K_1, \ldots$$

and so on. Thus, the iterate $x^{(k)}$ lies in the hyperplane K_i, where $i = k(\mathrm{mod}\ m) + 1$.

The standard basis vectors in $\mathbb{R}^m$ are denoted by $e_1, \ldots, e_m$; the nullspace and range of a matrix B by $N(B)$ and $R(B)$, respectively. Finally, we let

$$Z = \{x \in \mathbb{R}^n | \quad \text{there exists} \quad \lambda \in \mathbb{R}^m \quad \text{such that} \quad Ax + E^T\lambda = s\}.$$

Notice that Z is a closed subset of $\mathbb{R}^n$.

The proposed iterative scheme is the following:

– choose $x^{(0)} \in Z$, and $\lambda^{(0)} \in \mathbb{R}^m$ such that

$$Ax^{(0)} + E^T\lambda^{(0)} = s; \tag{11}$$

– for $k = 0, 1, 2, \ldots$ until convergence, let

$$\rho_k = \frac{t_i - r_i^T x^{(k)}}{r_i^T A^{-1} r_i} \tag{12}$$

$$x^{(k+1)} = x^{(k)} + \omega^{(k)} \rho_k A^{-1} r_i \tag{13}$$

$$\lambda^{(k+1)} = \lambda^{(k)} - \omega^{(k)} \rho_k e_i \tag{14}$$

$$(i = k(\mathrm{mod}\, m) + 1)$$

The factors $\omega^{(k)}$ in (13)-(14) form a sequence of relaxation parameters, introduced in order to accelerate the convergence. If $\omega^{(k)} = 1$ for all k, the iterate $x^{(k+1)}$ lies in the hyperplane K_i; if $0 < \omega^{(k)} < 1$, $x^{(k+1)}$ moves towards K_i without quite reaching it, and for $1 < \omega^{(k)} < 2$, $x^{(k+1)}$ moves past K_i, but it is closer to it than $x^{(k)}$ was. In practice, a single value $\omega^{(k)} = \omega$ is selected and kept fixed throughout the iteration process.

5 – Convergence of the method

In this section we prove a convergence theorem for the projection algorithm (11)-(14). Our proof is direct and self-contained, and makes use of concepts and ideas which were used by Bregman in his 1967 paper and by Herman, Lent and Lutz in their convergence proof of Kaczmarz's method with relaxation [15]. A much shorter, but indirect proof will be given in Section 7, using the convergence of Kaczmarz's method as a starting point. The present proof, on the other hand, contains the convergence of Kaczmarz's method as a special case, and is interesting in itself. We start with a characterization of the solution of problem (1).

LEMMA. *Let $u \in K \cap Z$. Then u solves the constrained minimization problem* (1)*: minimize $f(x) = \frac{1}{2} x^T A x - x^T s$ subject to $Ex = t$.*

PROOF. Let $u \in K \cap Z$; then $f(u) \geq \min_{x \in K} f(x)$, and there exists $x \in K$ such that

$$f(u) - f(x) = a \geq 0.$$

To prove that $f(u) = \min_{x \in K} f(x)$, it is enough to show $a = 0$. Since $u \in Z$ and Z is closed, there exists a sequence $\{u^{(k)}\} \subset Z$ such that $\lim_{k \to \infty} u^{(k)} = u$.

Also, there is a "dual sequence" $\{z^{(k)}\}$ in $\mathbb{R}^m$ such that the equation

$$Au^{(k)} + E^T z^{(k)} = s$$

holds for all $k \in \mathbb{N}$.

Since $N(E)^\perp = R(E^T)$, for all $v \in N(E)$ we have $v^T(Au^{(k)} - s) = 0$. In particular, choosing $v = u - x$ (which is in $N(E)$), it is

$$(u - x)^T(Au^{(k)} - s) = 0 \quad \text{for all} \quad k \in \mathbb{N},$$

hence

$$(u - x)^T Au^{(k)} = (u - x)^T s \quad \text{for all} \quad k \in \mathbb{N}.$$

Thus

$$\begin{aligned} a = f(u) - f(x) &= \tfrac{1}{2} u^T Au - u^T s - \tfrac{1}{2} x^T Ax + x^T s = \\ &= \tfrac{1}{2} u^T Au - \tfrac{1}{2} x^T Ax - (u - x)^T s = \\ &= \tfrac{1}{2} u^T Au - \tfrac{1}{2} x^T Ax - (u - x)^T Au^{(k)} = \\ &= \tfrac{1}{2} (u - u^{(k)})^T A(u - u^{(k)}) - \tfrac{1}{2} (x - u^{(k)})^T A(x - u^{(k)}) = \\ &= \tfrac{1}{2} \| u - u^{(k)} \|_A^2 - \tfrac{1}{2} \| x - u^{(k)} \|_A^2, \quad \text{for all} \quad k \in \mathbb{N}. \end{aligned}$$

This clearly implies $0 \le a \le \frac{1}{2} \| u - u^{(k)} \|_A^2$, for all $k \in \mathbb{N}$. Since $u^{(k)} \to u$ as $k \to \infty$, this means that $a = 0$, as claimed. □

Since the solution x^* of problem (1) exists and is unique, the previous lemma implies that $K \cap Z = \{x^*\}$.

THEOREM 1. *Let $\{x^{(k)}\}$, $\{\lambda^{(k)}\}$ be the sequences generated by (12)-(14), with $x^{(0)} \in \mathbb{R}^n$ and $\lambda^{(0)} \in \mathbb{R}^m$ chosen so that $Ax^{(0)} + E^T\lambda^{(0)} = s$. If the sequence of real relaxation parameters $\{\omega^{(k)}\}$ satisfies*

$$0 < \liminf_{k \to \infty} \omega^{(k)} \le \limsup_{k \to \infty} \omega^{(k)} < 2$$

then $\lim_{k \to \infty} x^{(k)} = x^*$, $\lim_{k \to \infty} \lambda^{(k)} = \lambda^*$ where (x^*, λ^*) is the unique solution of the equilibrium equations (4).

PROOF. Let $x^{(k+1)} = x^{(k)} + \omega^{(k)} \frac{t_i - r_i^T x^{(k)}}{r_i^T A^{-1} r_i} A^{-1} r_i$, as in (12)-(13). Since $r_i^T A^{-1} r_i = \| A^{-1} r_i \|_A^2$, we have

$$\| x^{(k+1)} - x^{(k)} \|_A^2 = (\omega^{(k)})^2 \frac{(t_i - r_i^T x^{(k)})^2}{\| A^{-1} r_i \|_A^2}. \tag{15}$$

Let $z \in K$; then $r_i^T z = t_i$ for all $i = 1, 2, \dots, m$, and

$$\| x^{(k+1)} - z \|_A^2 = \| x^{(k+1)} - x^{(k)} + x^{(k)} - z \|_A^2 =$$

$$= \| x^{(k+1)} - x^{(k)} \|_A^2 + \| x^{(k)} - z \|_A^2 + 2(x^{(k+1)} - x^{(k)})^T A (x^{(k)} - z) =$$

$$= \| x^{(k+1)} - x^{(k)} \|_A^2 + \| x^{(k)} - z \|_A^2 + 2 \left[\omega^{(k)} \frac{t_i - r_i^T x^{(k)}}{\| A^{-1} r_i \|_A^2} A^{-1} r_i \right]^T A (x^{(k)} - z) =$$

$$= \| x^{(k+1)} - x^{(k)} \|_A^2 + \| x^{(k)} - z \|_A^2 + 2\omega^{(k)} \frac{t_i - r_i^T x^{(k)}}{\| A^{-1} r_i \|_A^2} r_i^T (x^{(k)} - z).$$

But

$$(t_i - r_i^T x^{(k)}) r_i^T (x^{(k)} - z) = -(t_i - r_i^T x^{(k)})^2$$

hence

$$2\omega^{(k)} \frac{t_i - r_i^T x^{(k)}}{\| A^{-1} r_i \|_A^2} r_i^T (x^{(k)} - z) = -2\omega^{(k)} \frac{(t_i - r_i^T x^{(k)})^2}{\| A^{-1} r_i \|_A^2} =$$

$$= -\frac{2}{\omega^{(k)}} \| x^{(k+1)} - x^{(k)} \|_A^2$$

(the last identity follows from (15)). Thus, we have

$$\| x^{(k+1)} - z \|_A^2 =$$

$$= \| x^{(k+1)} - x^{(k)} \|_A^2 + \| x^{(k)} - z \|_A^2 - \frac{2}{\omega^{(k)}} \| x^{(k+1)} - x^{(k)} \|_A^2 =$$

$$= \| x^{(k)} - z \|_A^2 + (1 - \frac{2}{\omega^{(k)}}) \| x^{(k+1)} - x^{(k)} \|_A^2$$

(observe that for $\omega^{(k)} = 1$ this is just the Pythagorean Theorem). Since

$$0 < \lim_{k \to \infty} \inf \omega^{(k)} \leq \lim_{k \to \infty} \sup \omega^{(k)} < 2 \tag{16}$$

we can assume that there exist numbers v, w such that

$$0 < v \leq \omega^{(k)} \leq w < 2 \quad \text{for all} \quad k \in \mathbb{N},$$

without loss of generality.

Indeed, from (16) it follows that all but finitely many members of the sequence $\{\omega^{(k)}\}$ are bounded away from 0 and 2; hence there exists $N \in \mathbb{N}$ such that all the $\omega^{(k)}$ for $k \geq N$ are bounded away from 0 and 2. Taking $x^{(N)}$ as the initial vector, we can say that $0 < v \leq \omega^{(k)} \leq w < 2$ holds for all k, for some v and w.

Thus, for all k, we have

$$\left(\frac{2}{\omega^{(k)}} - 1\right) \geq \left(\frac{2}{w} - 1\right) = \frac{2-w}{w} > 0,$$

hence $1 - \dfrac{2}{\omega^{(k)}} < 0$ and we see that

$$0 \leq \| x^{(k+1)} - z \|_A^2 \leq \| x^{(k)} - z \|_A^2 \quad \text{for all} \quad z \in K \quad \text{and all} \quad k \in \mathbb{N}.$$

As a consequence,

$$0 \leq \| x^{(k)} - z \|_A^2 \leq \| x^{(0)} - z \|_A^2 \quad \text{for all} \quad z \in K \quad \text{and all} \quad k \in \mathbb{N}.$$

Therefore, the sequence of real numbers $\{\| x^{(k)} - z \|_A\}$ is bounded and non increasing; thus, the $\lim_{k\to\infty} \| x^{(k)} - z \|_A$ exists, for each $z \in K$. Also, notice that $\lim_{k\to\infty} \| x^{(k+1)} - x^{(k)} \|_A = 0$ and that the sequence $\{x^{(k)}\}$ is bounded. From the last assertion, it follows that there is a subsequence $\{x^{(k_j)}\}$ of $\{x^{(k)}\}$ which is convergent: $\lim_{j\to\infty} x^{(k_j)} = \hat{x}$, $\hat{x}$ a limit point for $\{x^{(k)}\}$. We shall prove next that $\hat{x} \in K$ (i.e., $E\hat{x} = t$) and that $\hat{x}$ is independent of the particular subsequence we choose; in other words, $\hat{x}$ is the only limit point of $\{x^{(k)}\}$, and therefore it must be its limit: $\lim_{k\to\infty} x^{(k)} = \hat{x}$.

To prove that $\hat{x} \in K$, we show that $r_i^T \hat{x} = t_i$, for $1 \leq i \leq m$. Let $\epsilon > 0$ be arbitrary. Define (i is fixed):

$$\sigma = \min(1, w) / \| A^{-1} r_i \|_A .$$

Pick an element $x^{(t)}$ of the subsequence $\{x^{(k_j)}\}$ such that

$$\| \hat{x} - x^{(t)} \|_A < \frac{\sigma}{2m}\epsilon \tag{17}$$

(recall that m is the number of rows of the matrix E).

Since $\lim_{k\to\infty} \| x^{(k+1)} - x^{(k)} \|_A = 0$, we can choose $t \in \mathbb{N}$ so that

$$\| x^{(s+1)} - x^{(s)} \|_A < \frac{\sigma}{2m}\epsilon \quad \text{for all} \quad s \geq t$$

and (17) holds, too. Now, observe that there is an $s \in \mathbb{N}$ such that $t \leq s \leq t + m$ and $i = s(\mathrm{mod}\ m) + 1$. Using the triangle inequality (up to m times, if necessary) we can see that for this s it is:

$$\| \hat{x} - x^{(s)} \|_A < m\frac{\sigma}{2m}\epsilon \leq \frac{\epsilon}{2 \| A^{-1}r_i \|_A}.$$

Recalling the identity

$$\| x^{(k+1)} - x^{(k)} \|_A^2 = (\omega^{(k)})^2 \frac{(t_i - r_i^T x^{(k)})^2}{\| A^{-1}r_i \|_A^2} \qquad \text{(true for all } k \in \mathbb{N}\text{)},$$

we see that

$$(r_i^T x^{(s)} - t_i)^2 = \frac{\| A^{-1}r_i \|_A^2}{(\omega^{(s)})^2} \| x^{(s+1)} - x^{(s)} \|_A^2 < \frac{\| A^{-1}r_i \|_A^2}{(\omega^{(s)})^2} \frac{\sigma^2}{4m^2}\epsilon^2.$$

But $\sigma < \dfrac{w}{\| A^{-1}r_i \|_A}$, and $w \geq \omega^{(s)}$; hence

$$(r_i^T x^{(s)} - t_i)^2 < \frac{\epsilon^2}{4}, \quad \text{or} \quad |r_i^T x^{(s)} - t_i| < \frac{\epsilon}{2}.$$

Furthermore,

$$|r_i^T \hat{x} - t_i| = |r_i^T(\hat{x} - x^{(s)} + x^{(s)}) - t_i| = |r_i^T x^{(s)} - t_i + r_i^T(\hat{x} - x^{(s)})|$$

$$\leq |r_i^T x^{(s)} - t_i| + |r_i^T(\hat{x} - x^{(s)})| < \frac{\epsilon}{2} + |r_i^T(\hat{x} - x^{(s)})|.$$

But

$$r_i^T(\hat{x} - x^{(s)}) = (A^{-1}r_i)^T A(\hat{x} - x^{(s)})$$

and the Cauchy-Schwarz inequality yields

$$|r_i^T(\hat{x} - x^{(s)})| \leq \| A^{-1}r_i \|_A \| \hat{x} - x^{(s)} \|_A .$$

Since

$$\| \hat{x} - x^{(s)} \|_A < \frac{1}{\| A^{-1}r_i \|_A} \frac{\epsilon}{2}$$

we get

$$|r_i^T(\hat{x} - x^{(s)})| < \frac{\epsilon}{2}$$

and thus

$$|r_i^T\hat{x} - t_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $i = 1, 2, \ldots, m$. Since ϵ is arbitary, we can conclude that $\hat{x}$ solves $Ex = t$; that is, $\hat{x} \in K$.

To prove uniqueness, suppose that there is another $x' \in \mathbb{R}^n$ which is a limit point for $\{x^{(k)}\}$. Let $\{x^{(k_l)}\}$ be a subsequence of $\{x^{(k)}\}$ such that $\lim_{l\to\infty} x^{(k_l)} = x'$.

Using the same argument as above, we conclude that $x' \in K$. But we have proved that the $\lim_{k\to\infty} \| x^{(k)} - z \|_A$ exists, for all $z \in K$. So, the limits $\lim_{k\to\infty} \| x^{(k)} - \hat{x} \|_A$ and $\lim_{k\to\infty} \| x^{(k)} - x' \|_A$ exist. Let

$$\alpha = \lim_{k\to\infty} (\| x^{(k)} - \hat{x} \|_A^2 - \| x^{(k)} - x' \|_A^2).$$

Using the subsequence $\{x^{(k_j)}\}$ we get

$$\alpha = - \| \hat{x} - x' \|_A^2,$$

whereas using $\{x^{(k_l)}\}$ we get

$$\alpha = \| x' - \hat{x} \|_A^2$$

therefore $\alpha = 0$ and $x' = \hat{x}$.

This proves that $\hat{x}$ is the only limit point of $\{x^{(k)}\}$, hence that $\lim_{k\to\infty} x^{(k)} = \hat{x} \in K$.

The next step in the proof is to show that $\hat{x} = x^*$, the unique solution of problem (1). We already know that $\hat{x} \in K$; because of the Lemma, we only need to show that $\hat{x} \in Z$. But this is true, because $x^{(k)} \in Z$ for all $k \in \mathbb{N}$, and Z is closed. To see that $x^{(k)} \in Z$ for all $k \in \mathbb{N}$, recall that $x^{(0)}$ is chosen in Z (and $\lambda^{(0)}$ is such that the identity $Ax^{(0)} + E^T\lambda^{(0)} = s$ holds). But $x^{(k+1)}$ is in Z if $x^{(k)}$ is; indeed, $x^{(k)} \in Z$ if and only if there is $\lambda^{(k)} \in \mathbb{R}^m$ such that $Ax^{(k)} + E^T\lambda^{(k)} = s$; since $x^{(k+1)} = x^{(k)} + \omega^{(k)}\rho_k A^{-1}r_i$, it is

$$Ax^{(k+1)} = Ax^{(k)} + \omega^{(k)}\rho_k r_i = -E^T\lambda^{(k)} + s + \omega^{(k)}\rho_k r_i.$$

Since $\lambda^{(k+1)} = \lambda^{(k)} - \omega^{(k)}\rho_k e_i$, we get

$$Ax^{(k+1)} = -E^T\lambda^{(k+1)} + s$$

so that $x^{(k+1)} \in Z$. This implies that the sequence $\{x^{(k)}\}$ is entirely contained in the closed subset Z, provided that $x^{(0)} \in Z$. Therefore $\lim_{k\to\infty} x^{(k)} = \hat{x} \in Z$, as we claimed, and $\hat{x} = x^*$, the unique solution of problem (1).

To conclude the proof, we only need to show that the sequence $\{\lambda^{(k)}\}$ converges to λ^*, the dual variable which corresponds to x^*, so that (x^*, λ^*) is the solution to the equilibrium equations (4). Since $x^{(k)}$ and $\lambda^{(k)}$ satisfy the equation (2) for all k, they also satisfy

$$\lambda^{(k)} = (EE^T)^{-1}E(s - Ax^{(k)}) \quad \text{for all} \quad k \in \mathbb{N}.$$

Taking the limit as $k \to \infty$, we get

$$\lim_{k\to\infty} \lambda^{(k)} = (EE^T)^{-1}E(s - Ax^*).$$

From (6), we obtain $\lim_{k\to\infty} \lambda^{(k)} = \lambda^*$. This concludes our proof. □

6 – Implementation of the method

Some remarks on the implementation of the projection algorithm are in order. A natural choice for the starting vectors would be $x^{(0)} =$

$A^{-1}s$, $\lambda^{(0)} = 0$. Notice that $A^{-1}s$ is the solution of the unconstrained problem:

$$\text{minimize} \quad f(x) = \tfrac{1}{2}x^TAx - x^Ts.$$

As already observed, for $A = I$ and $s = 0$ the method reduces to Kaczmarz's algorithm for the minimum norm solution of the system $Ex = t$; the only difference is that usually, in that problem, no attention is paid to the corresponding Lagrange multipliers vector. Also observe that, in algorithm (12)-(14), the computation of $\lambda^{(k)}$ is trivial once ρ_k has been determined.

It is clear that the applicability of the method is limited, in practice, to the case where A is of simple form: for example, diagonal or block diagonal with blocks of moderate size. Indeed, not only A must be easily invertible, but it is also necessary that A^{-1} be of simple form.

In the fundamental problem of linear elastic analysis the element flexibility matrix A is block diagonal, with blocks ranging in size from 1×1 to 6×6; furthermore, the element-level force-deformation matrix A^{-1} is often available without computing the inverse of A (see Robinson [27]). Thus for this kind of problems the computation of the vectors $A^{-1}r_i$ and the dot products $r_i^TA^{-1}r_i$, which accounts for much of the work involved in (12)-(14), is easily accomplished. It is important to remark that the equilibrium matrix E is often sparse and structured; a careful exploitation of this structure is crucial in order to keep the computational costs acceptable.

In practice, the m equations in $Ex = t$ are scaled by dividing the ith equation by the quantity $(r_i^TA^{-1}r_i)^{\frac{1}{2}}$. Denoting by

$$\hat{r}_i^Tx = \hat{t}_i \qquad (1 \le i \le m)$$

the ith equation in the new linear system, the generic step (13) becomes

$$x^{(k+1)} = x^{(k)} + \omega(\hat{t}_i - \hat{r}_i^Tx^{(k)})A^{-1}\hat{r}_i,$$

while (14) becomes

$$\lambda^{(k+1)} = \lambda^{(k)} - \omega\frac{\hat{t}_i - \hat{r}_i^Tx^{(k)}}{(r_i^TA^{-1}r_i)^{\frac{1}{2}}}e_i.$$

Finally, we emphasize that the projection algorithm can be implemented as a row-action method, just like Kaczmarz's method ([7]). That is, at each iterative step only one row of the equilibrium matrix E is required to perform the computation. This feature of the method can be very important if the number m of rows is very large, as it is sometimes the case.

7 – Relation to the SOR method

It is a fact that projection methods often suffer from slow convergence. For some applications, this is not necessarily a problem (e.g., in image processing - see [16]). Nevertheless, it is important to attempt to improve the convergence by means of acceleration techniques. The introduction of a relaxation parameter is, perhaps, the simplest way to do so. Numerical experiments show that the rate of convergence of algorithm (12)-(14) can be substantially improved by overrelaxation (i.e., selecting for $\omega^{(k)} = \omega$ a suitable value in the interval (1,2)). This fact is not surprising if one considers the strict relationship existing between the projection method (12)-(14) and the SOR method, which we are going to demonstrate. In turn, this relationship suggests the use of other acceleration techniques.

In Section 2, we gave an equivalent formulation of problem (1), in terms of minimum norm solution of an underdetermined, consistent linear system (problem (9)).

According to Tanabe's theorem, mentioned in Section 4, we can use Kaczmarz's method to solve problem (9). The resulting sequence $\{z^{(k)}\}$ will converge to the unique solution z^* of (9), provided that $z^{(0)}$ is in $R((EL)^T)$. Notice that a relaxation parameter $\omega \in (0,2)$ can be used in order to accelerate the convergence (for simplicity, we consider the case $\omega^{(k)} = \omega$ for all k). The key observation is the following: if we let

$$x^{(k)} \equiv Lz^{(k)} + A^{-1}s \quad \text{for all} \quad k \in \mathbb{N},$$

we obtain precisely the sequence defined by (12)-(13). To see this, observe that the ith row of EL is $r_i^T L$, and the ith component of $t - EA^{-1}s$ is $t_i - r_i^T A^{-1}s$. Thus, the sequence $\{z^{(k)}\}$ generated by Kaczmarz's algorithm applied to (9) is

$$z^{(k+1)} = z^{(k)} + \omega \frac{(t_i - r_i^T A^{-1} s - r_i^T L z^{(k)})}{\| L^T r_i \|_2^2} L^T r_i =$$

$$= z^{(k)} + \omega \frac{(t_i - r_i^T (A^{-1} s + L z^{(k)}))}{\| L^T r_i \|_2^2} L^T r_i.$$

Premultiplying by L and adding $A^{-1}s$ in both sides, we get

$$Lz^{(k+1)} + A^{-1}s = Lz^{(k)} + A^{-1}s + \frac{(t_i - r_i^T (Lz^{(k)} + A^{-1}s))}{r_i^T A^{-1} r_i} A^{-1} r_i$$

(we also used the fact that $\| L^T r_i \|_2^2 = (L^T r_i)^T L^T r_i = r_i^T L L^T r_i$, and $LL^T = A^{-1}$). Replacing, for all $k = 0, 1, 2, \ldots$, $Lz^{(k)} + A^{-1}s$ by $x^{(k)}$ (the change of variable), we obtain the sequence $\{x^{(k)}\}$ as defined by (12)-(13). As k tends to infinity, $x^{(k)}$ tends to x^*, the solution of problem (1); indeed, from $\lim_{k \to \infty} z^{(k)} = z^*$ it follows that

$$\lim_{k \to \infty} x^{(k)} = \lim_{k \to \infty} (Lz^{(k)} + A^{-1}s) = Lz^* + A^{-1}s = x^*.$$

As a consequence, the sequence $\{\lambda^{(k)}\}$ defined by (14) converges to λ^*. Hence we have obtained an alternative proof of Theorem 1. Notice that the change of variable, and in particular the Choleski factorization of A^{-1}, need not be performed explicitly.

Björk and Elfving ([4]) pointed out the close relationship between Kaczmarz's method and the method of Gauss-Seidel (or, more generally, between Kaczmarz's method with relaxation and the SOR and SSOR iterations). They showed, as a consequence, that it is possible to use Chebyshev and conjugate gradient acceleration to improve the convergence of Kaczmarz's method. These remarks carry over in a natural way to the projection algorithm considered in this chapter. Following Björk and Elfving, we can establish a link with the SOR method. Let $\{v^{(k)}\}$ be the sequence obtained by the SOR method applied to the solution of the linear system

$$EA^{-1}E^T v = t - EA^{-1}s \tag{18}$$

starting with an arbitrary $v^{(0)} \in \mathbb{R}^m$. Since the coefficient matrix in (18) is SPD, the solution v^* is unique; notice that $v^* = -\lambda^*$ (see (10)).

From the results of Björk and Elfving, it follows that the sequence $\{z^{(k)}\}$ defined by

$$z^{(k)} = (EL)^T v^{(k)}, \quad k \in \mathbb{N},$$

is precisely the sequence obtained when Kaczmarz's method is applied to problem (9). In turn, the sequence

$$x^{(k)} = Lz^{(k)} + A^{-1}s$$

coincides with the sequence generated by (12)-(13), as we already observed. Also, the condition

$$Ax^{(0)} + E^T\lambda^{(0)} = s \tag{19}$$

on the initial vectors $x^{(0)}, \lambda^{(0)}$ is automatically fulfilled; indeed, from $z^{(0)} = L^T E^T v^{(0)}$ and $x^{(0)} = Lz^{(0)} + A^{-1}s$ we get $x^{(0)} = A^{-1}(E^T v^{(0)} + s)$, so that condition (20) holds with $\lambda^{(0)} = -v^{(0)}$, and $v^{(0)} \in \mathbb{R}^m$ arbitrary.

It is interesting to observe that the classical method of Uzawa ([25], [13]) can be obtained by applying a Cimmino-like iterative scheme to the solution of problem (9). This is in turn equivalent to an implicit Richardson method applied to (18) (see [2]).

Thus, we have a new interpretation of the projection algorithm for the solution of the equilibrium equations. Algorithm (12)-(14) results from implicitly applying the SOR method to (18), with no explicit computation of the coefficient matrix $EA^{-1}E^T$. Indeed, instead of the $\frac{m(m+1)}{2}$ entries of $EA^{-1}E^T$, only its m diagonal entries, given by $r_i^T A^{-1} r_i$, must be computed. Such an implementation of the SOR method is quite efficient, since explicit formation of $EA^{-1}E^T$ would lead to a loss of sparsity and structure (usually present in A and E), not to mention the loss of information due to round-off errors; see Section 3.

Furthermore, as a result of this implementation, we obtain both an approximating sequence for the primal variable x^* and an approximating sequence for the dual variable λ^*, at no extra cost.

Another advantage is that the SOR method, in its standard implementation, is not well-suited for vector computers; this is not the case for the implementation as a projection algorithm, since formulas (12)-(13) are rich in vector operations. Furthermore, the method can easily be cast in a "block" form, well-suited for implementation on multivector

computers such as the CRAY Y-MP and the Alliant FX/8. The idea is to partition the equilibrium matrix E into blocks of rows and project the successive iterates onto the linear manifolds defined by the equations corresponding to the various blocks. This results, of course, in an implicit block SOR scheme for (18). Similar ideas have been used by other authors in order to introduce some parallelism in row-action schemes for linear systems (see [5], [20]).

Finally, we observe that the projection method is symmetrizable: to do so, it is enough to alternate a cycle of successive projections onto the hyperplanes $K_1, K_2, \ldots, K_m$ with a cycle of projections onto the same hyperplanes taken in the reverse order. The resulting algorithm corresponds to an implicit SSOR scheme for the solution of (18) (see Björk and Elfving [4]). This allows for the use of Chebyshev and conjugate gradient acceleration techniques, in the same spirit as in the paper by Björk and Elfving. However, this makes the resulting algorithm substantially more complicated, and we do not wish to further investigate this question, which is beyond the scope of the present paper.

8 – Conclusions and acknowledgements

In this paper we have studied a projection algorithm for the solution of certain quadratic programming problems subject to linear equality constraints, and we have shown how this procedure is related to the classical SOR method. The algorithm is easy to implement, and appears to be attractive for those problems where the matrix A^{-1} is either available or easily computed, as it is the case in finite element computations for structural analysis problems.

In a forthcoming paper ([3]) it will be shown how to apply the projection idea when the matrix A^{-1} is not availabe (this is the case in many fluid flow problems), and the results of numerical experiments will be discussed. Some of these experiments ([2]) indicate that the projection algorithm may be a viable alternative to the widely used method of Uzawa, especially when no estimate for the convergence factor of Uzawa's iteration is available.

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INDIRIZZO DELL'AUTORE:

Michele Benzi - Department of Mathematics - North Carolina State University - Raleigh, NC 27695-8205 - USA.