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Some remarks about integral geometry in the complex space $\mathbb{C}^n$

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RIASSUNTO: *Si estendono a $\mathbb{C}^n$ alcuni risultati di geometria integrale, già dimostrati in $\mathbb{R}^n$: misure di sezioni complesse, formule di Kubota, misure di sottospazi lineari olomorfi che sezionano un corpo convesso e formule sulla densità di sottospazi lineari complessi.*

ABSTRACT: *We extend to $\mathbb{C}^n$ certain results of Integral Geometry on $\mathbb{R}^n$ such as complex cross sectional measures, Kubota's formula, the measure of the holomorphic linear subspaces that cut a convex body and formulas about densities of complex linear subspaces.*

KEY WORDS: *Integral geometry - Complex spaces - Complex cross sectional measure - Holomorphic subspace.*

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– Introduction

The Integral Geometry of linear subspaces L_r of $\mathbb{C}^n$ has been deeply studied; see for example [6]. We began this study in [3] giving some very simple properties about Integral Geometry of holomorphic linear subspaces L_r of $\mathbb{C}^n$. Using projections on holomorphic linear subspaces it seems to us natural to consider "complex cross sectional measures" of

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a real hypersurface imbedded in $\mathbb{C}^n$. Evidently, this measures differ from the "real cross sectional measures" because the integration is taken over the complex Grassmann manifold. In the books of L.A. SANTALO, [6], and Y.D. BURAGO - V.A. ZALGALLER, [1], one can find the definition and some properties about cross sectional measures.

In the § 1 we recall first of all some properties about densities of holomorphic linear subspaces and also we give some simple properties of the kinematic density in $\mathbb{C}^n$.

In the § 2 we define the "holomorphic cross sectional measures" and we obtain a generalization to the complex case of the Kubota's Formula. We finish this section giving a geometrical interpretation of this cross sectional measures as the measure of the holomorphic linear subspaces that cut a convex body. An open question is to give a geometric interpretation of this measures in function of the integrals of the principal curvatures of the hypersurface.

In § 3 we generalize to the complex case some formulas relating densities of complex linear subspaces that were obtained by S.S. CHERN, in the real case, [2].

Finally, in § 4 we give also a formula for the volume of a real compact manifold in function of the volume of their sections with holomorphic planes.

1 – Some general results

In this section we assume that all geometric objects are differentiable. Also to avoid the symbol of summation, we use always the Einstein convention over dumb indices.

Let $\mathbb{C}^n$ be the standard n-dimensional complex vector space with its usual topology. If J denotes its standard complex structure, we say that a basis $(e_1, \cdots, e_{2n})$ of the underlying $2n$-dimensional real vector space is a J-basis provided that $e_{i+n} = e_{i*} = Je_i$, for all $i, 1 \le i \le n$.

Since, the unitary group $U(n)$ acts on $\mathbb{C}^n$ in a natural way we can consider the unitary motions

$$x' = ax + b, \quad a \in U(n), \quad b \in \mathbb{C}^n .$$

Following the general method of [6], and considering the real representation of $U(n)$, the group of unitary motions can be represented by matrices of the form

$$g = \begin{pmatrix} a_1 & a_2 & b_1 \\ -a_2 & a_1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The structure equations with respect to a moving complex frame (P, e_A, e_{A^*}) are given by

$$\begin{pmatrix} d\omega_{AB} & d\omega_{AB^*} \\ -d\omega_{AB^*} & d\omega_{A^*B^*} \end{pmatrix} = -\begin{pmatrix} \omega_{AC} & \omega_{AC^*} \\ -\omega_{AC^*} & \omega_{A^*C^*} \end{pmatrix} \wedge \wedge \begin{pmatrix} \omega_{CB} & \omega_{CB^*} \\ -\omega_{CB^*} & \omega_{C^*B^*} \end{pmatrix} \tag{1.1}$$

and

$$\begin{pmatrix} d\omega_A \\ d\omega_{A^*} \end{pmatrix} = -\begin{pmatrix} \omega_{AC} & \omega_{AC^*} \\ -\omega_{AC^*} & \omega_{A^*C^*} \end{pmatrix} \wedge \begin{pmatrix} \omega_C \\ \omega_{C^*} \end{pmatrix} \tag{1.2}$$

whence,

$$\begin{aligned} d\omega_{AB} &= -\omega_{AC} \wedge \omega_{CB} - \omega_{AC^*} \wedge \omega_{C^*B} \\ d\omega_{AB^*} &= -\omega_{AC} \wedge \omega_{CB^*} - \omega_{AC^*} \wedge \omega_{C^*B^*} \\ d\omega_A &= -\omega_{AC} \wedge \omega_C - \omega_{AC^*} \wedge \omega_{C^*} \\ d\omega_{A^*} &= -\omega_{A^*C} \wedge \omega_C - \omega_{A^*C^*} \wedge \omega_{C^*} \end{aligned} \tag{1.3}$$

where,

$$\begin{aligned} \omega_{AB} &= (de_A \cdot e_b) = -(e_A \cdot de_B) = \omega_{A^*B^*} \\ \omega_{AB^*} &= -(e_A \cdot de_{B^*}) = (de_A \cdot e_{B^*}) = -\omega_{A^*B} \\ \omega_a &= dx \cdot e_A : \omega_{A^*} = dx \cdot e_{A^*} : x \in \mathbb{C}^n. \end{aligned} \tag{1.4}$$

We remark that equations (1.2) could be equally expressed in terms of the moving frames of $\mathbb{C}^n$.

A calculation of the volume of the unitary group can be found in [4] along with the proof that its density is given by

$$dU(n) = \Big(\bigwedge_{1 \le A < B \le n} (\omega_{AB} \wedge \omega_{AB^*}) \Big) \bigwedge_C (\wedge \omega_{CC^*}) \tag{1.5}$$

and once the integration is accomplished the volume is given by

$$\int_{U(n)} dU(n) = \prod_{h=1}^{n} \frac{(2\pi)^h}{(h-1)!}\,. \tag{1.6}$$

Therefore, the density or volume element of the unitary group of motions K is given by

$$dk = (\bigwedge_{A<B} (\omega_{AB} \wedge \omega_{AB^*})) \wedge (\bigwedge_{C} \omega_{CC^*}) \wedge (\bigwedge_{D} (\omega_D \wedge \omega_{D^*}))\,. \tag{1.7}$$

In the following, since we work most of the time with holomorphic r-planes, we call such planes, r-planes.

PROPOSITION 1, [3]. *Let L_r be an holomorphic r-plane of $\mathbb{C}^n$. Then, the density of L_r is given by*

$$dL_r = (\bigwedge_{\alpha} (\omega_\alpha \wedge \omega_{\alpha^*})) \wedge (\bigwedge_{i,\alpha} (\omega_{i\alpha} \wedge \omega_{i\alpha^*})) \tag{1.8}$$

where $1 \leq i, j, \cdots, \leq r < \alpha, \beta, \cdots, \leq n$.

We will denote by $L_{r[d]}$ the r-planes in $\mathbb{C}^n$ that contain a d-plane.

PROPOSITION 2, [3]. *The density of $L_{r[d]}$ is given by*

$$dL_{r[d]} = dL^{n-d}_{r-d[0]} = (\bigwedge_{i,\alpha} (\omega_{i,\alpha} \wedge \omega_{i\alpha^*})) \tag{1.9}$$

$d+1 \leq i, j \cdots \leq r < \alpha, \beta, \cdots \leq n$. *In particular, the density of r-planes through the origin; i.e., the density for the complex Grassmann manifold is*

$$dL_{r[0]} = (\bigwedge_{i,\alpha} (\omega_{i\alpha} \wedge \omega_{i\alpha^*})) \tag{1.10}$$

$1 \leq i, j, \cdots \leq r < \alpha, \beta, \cdots \leq n$.

The volume of the complex Grassmann manifold $U(n)/U(r)U(n-r) = G_{r,n-r}$ is given by, [4]

$$v(G_{r,n-r}) = \frac{v(U(n))}{v(U(r))v(U(n-r))} \tag{1.11}$$

Since the complex projective space $\mathbb{CP}^n$ is the homogeneous manifold $U(n+1)/U(1)U(n)$, (i.e., the set of complex lines in $\mathbb{C}^n$ through 0), as a particular case of (1.10), we have

$$d\mathbb{CP}^n = \omega_{12} \wedge \omega_{13} \wedge \cdots \wedge \omega_{1n} \wedge \omega_{12^*} \wedge \omega_{13^*} \wedge \cdots \wedge \omega_{1n^*} \tag{1.12}$$

If we represent by $L_a^{(r)}$ the r-planes contained in L_r, we have the following.

PROPOSITION 3, [3]. *With the same notations as above*

$$dL_a^{(r)} \wedge dL_r = dL_{r[a]} \wedge dL_a \tag{1.13}$$

and also

$$dL_{a[0]}^{(r)} \wedge dL_{r[0]} = dL_{r[a]} \wedge dL_{a[0]} \tag{1.14}$$

which are very useful expressions.

Formula (1.3) gives us dL_r in terms of a point x and $2r$ vectors of a J-basis contained in L_r. It is often useful to state the density dL_r in terms of the holomorphic space L_{n-r} orthogonal to L_r through a fixed point 0. We have that

$$d\sigma_{n-r} = \omega_{r+1} \wedge \cdots \wedge \omega_n \wedge \omega_{r+1^*} \wedge \cdots \wedge \omega_{n^*} \tag{1.15}$$

represents the volume element of $L_{n-r[0]}$. So, from (1.8) and (1.15) we have

$$dL_r = d\sigma_{n-r} \wedge dL_{n-r[0]} \, . \tag{1.16}$$

The kinematic density for the unitary group of motions was given by (1.7). We know also that $dx = \bigwedge_A (\omega_A \wedge \omega_{A^*})$ represent the volume element in $\mathbb{C}^n$ at the point x. So, if we represent by $dk_{[x]}$ the kinematic density of the unitary group we have $dk = dx \wedge dk_{[x]}$.

Let dk^r denote the kinematic density in L_r; that is,

$$dk^r = (\bigwedge_{i<j}(\omega_{ij} \wedge \omega_{ij^*})) \wedge (\bigwedge_{k} \omega_{kk^*}) \wedge (\bigwedge_{1}(\omega_1 \wedge \omega_{1^*})) \tag{1.17}$$

and let $dk_{[x]}^{n-r}$ denotes the kinematic density of the complex rotations about x in the $(n-r)$-plane. Then we have

PROPOSITION 4.

$$dk_{[x]}^{n-r} = (\bigwedge_{\alpha<\beta}(\omega_{\alpha\beta} \wedge \omega_{\alpha\beta^*})) \wedge (\bigwedge_{\gamma}(\omega_{\gamma\gamma^*})) \tag{1.18}$$

and using (1.7), (1.8), (1.17) we can prove

PROPOSITION 5.

$$dk = dL_r \wedge dk^r \wedge dk_{[x]}^{n-r}. \tag{1.19}$$

Using the same method, it is also possible to prove the following

PROPOSITION 6.

$$dk = dL_{r[0]} \wedge dk_r \wedge dk^{n-r} \tag{1.20}$$

2 – Complex cross sectional measures. Generalization of a Kubota's formula

Here we assume that the properties of convex bodies in $\mathbb{R}^n$ are well know, [6]. Let K be a convex set contained in $\mathbb{C}^n$ and let 0 be a fixed point. Consider all the holomorphic $(n-r)$-planes $L_{n-r[0]}$ through 0 and let K'_{n-r} be the orthogonal projection of K in $L_{n-r[0]}$. We want to define the mean value of the volume $V(K'_{n-r})$ of these projected sets.

DEFINITION. *The mean value of the projected volumes $V(K'_{n-r})$ is defined as*

$$(2.1)\quad W_r(K)=E(V(K'_{n-r}))=\frac{I_r(K)}{V(G_{r,n-r})}=\frac{V(U(n))}{V(U(r))xV(U(n-r))}I_r(K)$$

where

$$(2.2)\qquad I_r(K)=\int_{G_{r,n-r}} V(K'_{n-r})dL_{n-r[0]}\,.$$

REMARK The formula (2.2) defines $I_r(K)$ for $r=1,2,\cdots,n-1$. For completeness we define $I_O(K)=V(K)$.

From (1.14) and proceeding as in ([6], p. 216) we have

$$(2.3)\qquad dL_{n-1[0]}\wedge dL^{(n-1)}_{r-1[0]}\wedge dL^{(n-1)}_{r[r-1]}=dL_{r[0]}\wedge dL^{(r)}_{r-1[0]}\wedge dL_{n-1[r]}\,.$$

Let us multiply now by $V(K'_{n-r})$ and perform the integration over all pairs $L_{r[0]}, L^{(r)}_{r-1[0]}$. We know also that K'_{n-r} is the holomorphic orthogonal projection onto $L_{n-r[0]}$ of the convex set K'_{n-1}, holomorphic orthogonal projection of K onto $L_{n-1[0]}$. Using (1.9), (1.11) and (2.3) we finally get

$$I_r(K)=\frac{V(U(1))V(U(r-1))}{V(U(r))}\int_{\mathbb{CP}_{n-1}} I_{r-1}(K'_{n-1})d\mathbb{CP}_{n-1}\,.$$

So, from (2.1) we have

$$(2.4)\qquad W_r(K)=\frac{V(G_{r,n-r})}{V(G_{1,r-1})}\int_{\mathbb{CP}_{n-1}} I_{r-1}(K'_{n-1})d\mathbb{CP}^{n-1}\,.$$

REMARK -i) (2.4) can be considered as a generalization of Kubota's Formula to the complex case.

ii) $E(V(K'_{n-r}))$ is defined as the holomorphic sectional measure of order r.

To give a geometric interpretation of the complex r-planes that cut a convex body, we have the following:

PROPOSITION 7. *The measure of all L_r that $L_r \cap K \neq \emptyset$ is given by*

$$
(2.5) \qquad \begin{aligned} m(L_r/L_r \cap K \neq \emptyset) &= \int_{L_r \cap K \neq \emptyset} dL_r = \int_{L_r \cap K \neq \emptyset} \sigma_{(n-r)} dL_{r[0]} = \\ &= \int_{L_r \cap K \neq \emptyset} \sigma_{(n-r)} dL_{n-r[0]} = I_r(K) . \end{aligned}
$$

The proof is immediate from (1.16) and (2.2).

Next we want to compute the integrals of the complex cross sectional measures $W_i^{(r)}(K \cap L_r)$ over all L_r that intersect K. To this end, working as in the real case, we have that

$$
dL^{(r)}_{a+1} \wedge dL_r = dL_{r[a+1]} \wedge dL_{a+1} .
$$

Now, considering the integral

$$
(2.6) \qquad I = \int_{L^{(r)}_{a+1} \cap K \neq \emptyset} dL^{(r)}_{a+1} \wedge dL_r
$$

keeping L_r fixed end integrating $dL^{(r)}_{a+1}$ we get

$$
(2.7) \qquad \begin{aligned} I &= \int I^{(r)}_{a+1}(K \cap L_r) dL_r = \int_{L_{a+1} \cap K \neq \emptyset} dL_{a+1} \int dL_{r[a+1]} = \\ &= \frac{V(U(n-a-1))}{V(U(r-a-1))V(U(n-r))} \int_{L_{a+1} \cap K \neq \emptyset} dL_{a+1} = \\ &= \frac{V(U(n-a-1))}{V(U(r-a-1))V(U(n))} I_{a+1}(K) . \end{aligned}
$$

So we can write

$$(2.8)\qquad \int I^{(r)}_{a+1}(K \cap L_r)dL_r = \frac{V(U(n-a-1))}{V(U(r-a-1))V(U(n-r))} I_{a+1}(K)\,.$$

3 – Some relations between densities of holomorphic linear subspaces

Let 0 be a fixed point (origin) and let $L_{q[0]}$ be an holomorphic fixed q-plane through 0. Let L_r be a moving holomorphic r-plane through 0 and assume $q+r-n>0$, so that $L_{q[0]} \cap L_{r[0]}$ is, general, an holomorphic $(r+q-n)$-plane through 0, which we represent by $L_{(r+q-n)[0]}$.

PROPOSITION 8. *With the above notation we have*

$$(3.1)\qquad dL_{r[0]} = \nabla^{r+q-n} dL_{r[(r+q-n)]} \wedge dL^{(q)}_{(r+q-n)[0]}$$

where ∇ *is a well determined determinant.*

PROOF. We construct two moving frames as follows

(I) a) $(e_1, \cdots, e_{r+q-n}, e_{1^*}, \cdots, e_{r+q-n^*})$ span $L_{q[0]} \cap L_{r[0]}$

b) $(e_{r+q-n+1}, \cdots, e_r, e_{r+p-n+1^*}, \cdots, e_{r^*})$ lie on $L_{r[0]}$

c) $(e_{r+1}, \cdots, e_n, e_{r+1^*}, \cdots, e_{n^*})$ are arbitrary unit vectors that complete orthonormal frame (I).

(II) a) $(e_1, \cdots, e_{r+q-n}, e_{1^*}, \cdots, e_{r+q-n^*})$ span $L_{q[0]} \cap L_{r[0]}$

b) $(b_{r+q-n+1}, \cdots, b_r, b_{r+q-n+1^*}, b_{r^*})$ are constant unit vectors in the holomorphic $(n-q)$-plane $L_{(n-q)[0]}$ orthogonal to $L_{q[0]}$

c) $(b_{r+1}, \cdots, b_n, b_{r+1^*}, \cdots, b_{n^*})$ are contained in $L_{q[0]}$ and such that together with $(e_1, \cdots, e_{r+q-n}, e_{1^*}, \cdots, e_{r+q-n^*})$ form an orthonormal frame in $L_{q[0]}$.

We know, (1.8), that

$$(3.2)\qquad dL_{r[0]} = \bigwedge_{i,\alpha}((e_\alpha \cdot de_i) \wedge (e_\alpha \cdot de_{i^*})) \bigwedge_{i,\alpha}((e_\alpha \cdot de_i) \wedge (e_\alpha \cdot de_{i^*}))$$

$1 \leq i, j, \cdots \leq r+q-n < u, v, \cdots \leq r < \alpha, \beta, \cdots \leq n$. Also, we have

$$dL_{r[(r+q-n)]} = \bigwedge_{u,\alpha}((e_\alpha \cdot de_u) \wedge (e_\alpha \cdot de_{u^*})) \tag{3.3}$$

and

$$dL^{(q)}_{r+q-n[0]} = \wedge((b_\beta \cdot de_i) \wedge (b_{\beta^*} \cdot de_i)) . \tag{3.4}$$

Since

$$e_\alpha = u^\beta_\alpha b_\beta + u^{\beta^*}_\alpha b_{\beta^*} + v^v_\alpha b_v + v^{v^*}_\alpha b_{v^*}$$

and

$$e_{\alpha^*} = -u^{\beta^*}_\alpha b_\beta + u^\beta_\alpha b_{\beta^*} - u^{v^*}_\alpha b_v + v^v_\alpha b_{v^*} \tag{3.5}$$

and considering that b_β and b_{β^*} are constant vectors, we have

$$(e_\alpha \cdot de_u) = u^\beta_\alpha (b_\beta \cdot de_u) + u^{\beta^*}_\alpha (b_{\beta^*} \cdot de_u)$$

and

$$(e_{\alpha^*} \cdot de_u) = -u^{\beta^*}_\alpha (b_\beta \cdot de_u) + u^\beta_\alpha (b_{\beta^*} \cdot de_u) . \tag{3.6}$$

Now, from (3.2) - (3.6) we get (3.1) where ∇ is the determinant

$$\begin{bmatrix} u^\beta_\alpha & u^{\beta^*}_\alpha \\ -u^{\beta^*}_\alpha & u^\beta_\alpha \end{bmatrix} .$$

If we integrate (3.1) over all $L_{r[0]}$, using the value of the volume of the complex Grassmann manifold and since ∇ depends only on $L_{r[(r+q-n)]}$, we obtain

$$\int \nabla^{r+q-n} dL_{r[r+q-n]} = \frac{v(U(n))v(U(r+q-n))}{v(U(q))v(U(r))} \tag{3.7}$$

which can be written as

$$\int_{G_{n-r,n-q}} \nabla^{r+q-n} dL^{(2n-r-q)}_{n-q[0]} = \frac{v(U(n))v(U(r+q-n))}{v(U(q))v(U(r))} . \tag{3.8}$$

To simplify notation we will write $r+q-n=N$, $2n-r-q=\nu$, $n-q=\rho$. So,

$$\int_{G_{\rho,\nu-\rho}} \nabla^N dL^{(\nu)}_{\rho[0]} = \frac{v(U(\nu+N))v(U(N))}{v(U(\nu+N-\rho))v(U(N+\rho))}\,. \tag{3.9}$$

We consider now $L_{q[0]}$ fixed and moving $L_r, r+q>n$. Let x be a point in the intersection $L_r \cap L_{q[0]}$ and consider the above moving frames (I) and (II) above with reference to the point x. In order to apply (3.1) we notice that

$$dL_{n-r[0]} = dL_{r[0]} = dL_{r[x]}$$

and

$$d\sigma_{(n-r)} = (dx\cdot e_{r+1}) \wedge \cdots \wedge (dx\cdot e_n) \wedge (dx\cdot e_{r+1^*}) \wedge \cdots \wedge (dx\cdot e_{n^*})\,.$$

Hence we have

$$dL_r = dL_{r[x]} \wedge (dx\cdot e_{r+\alpha}) \wedge (dx\cdot e_{r+\alpha^*})\,. \tag{3.10}$$

Since the vectors b_β, b_{β^*} are perpendicular to $L_r \cap L_{q[0]}$ and $x \epsilon L_r \cap L_{q[0]}$, we have $dx\cdot b_\beta = dx\cdot b_{\beta^*} = 0$ and thus (3.5) gives

$$\begin{aligned}(dx\cdot e_\alpha) &= u_\alpha^\beta (dx\cdot b_\beta) + u_\alpha^{\beta^*}(dx\cdot b_{\beta^*})\\ (dx\cdot e_{\alpha^*}) &= -u_\alpha^{\beta^*}(dx\cdot b_\beta) + u_\alpha^\beta (dx\cdot b_{\beta^*})\,.\end{aligned} \tag{3.11}$$

So, it follows that

$$\bigwedge_\alpha ((dx\cdot e_\alpha) \wedge (dx\cdot e_{\alpha^*})) = \nabla \bigwedge_\beta ((dx\cdot b_\beta) \wedge (dx\cdot b_{\beta^*}))\,. \tag{3.12}$$

Since

$$dL^{(q)}_{r+q-n} = dL^{(q)}_{r+q-n[0]} \bigwedge_\beta ((dx\cdot b_\beta) \wedge (dx\cdot b_{\beta^*}))$$

using (3.1), (3.10) and (3.12) we get

$$dL_r = \nabla^{r+q-n+1} dL_{r[r+q-n]} \wedge dL^{(q)}_{r+q-n}\,. \tag{3.13}$$

If $F(L_r)$ is an integrable function that depends only on $L^{(q)}_{r+q-n} = L_r \cap L_{q[0]}$. We get

$$\int F(L_r)dL_r = \int \nabla^{r+q-n+1} dL_{r[r+q-n]} \int F(L^{(q)}_{r+q-n})dL^{(q)}_{r+q-n}\,. \tag{3.14}$$

Applying now (3.8)

$$\int \nabla^{r+q-n+1} dL_{r[r+q-n]} = \frac{v(U(n+1))v(U(r+q-n+1))}{v(U(q))v(U(r))} \tag{3.15}$$

and finally

$$\int F(L_r)dL_r = \frac{v(U(n+1))v(U(r+q-n+1))}{v(U(q))v(U(r))}$$

$$\int F(L^{(q)}_{r+q-n})dL^{(q)}_{r+q-n}\,. \tag{3.16}$$

4 – Linear holomorphic subspaces that intersect a compact real manifold of odd dimension

Let (x, e_A) be a moving orthonormal frame and consider the holomorphic r-plane L_r determined by $x, e_1, \cdots, e_r, e_{1^*}, \cdots, e_{r^*})$. The density for L_r is given by

$$dL_r = \bigwedge_{\alpha}(\omega_\alpha \wedge \omega_{\alpha^*}) \bigwedge_{i,\beta}(\omega_{i\beta} \wedge \omega_{i\beta^*})\,. \tag{4.1}$$

The exterior product $\bigwedge_\alpha(\omega_\alpha \wedge \omega_{\alpha^*}) = \bigwedge_\alpha(dx{\cdot}e_\alpha) \wedge (dx{\cdot}e_{\alpha^*})$ equals the volume element $d\sigma_{(n-r)}$ of the holomorphic $(n-r)$-plane $L_{(n-r)}$ orthogonal to L_r at x. The exterior product $\wedge\,(\omega_{i\beta} \wedge \omega_{i\beta^*}) = \wedge\,[(e_\beta{\cdot}de_i) \wedge (e_{\beta^*}{\cdot}de_i)]$ is the density $dL_{r[x]}$ of the holomorphic r-planes about x. We have

$$dL_r = d\sigma_{(n-r)} \wedge dL_{r[x]}\,. \tag{4.2}$$

Let M^q be a compact differentiable real manifold of odd dimension q imbedded in $\mathbb{C}^n$, let us assume that it is piecewice smooth. Assume also that $2r + q - 2n \geq 0$ and consider the set of holomorphic r-planes that contain a fixed point x of M^q. The interaction $L_r \cap M^q$ is in general a manifold of real dimension $2r + q - 2n$. Take the frame (x, e_A) such that $x \in L_r \cap M^q$ and $e_1, \cdots, e_t, e_{1^*}, \cdots, e_{t^*}, e_{2t+1}, \cdots, e_{2r+q-2n}$ are orthonormal tangent vectors to $L_r \cap M^q$. Let $b_{2r+1}, \cdots, b_{2n}$ be a set of orthonormal vectors such that the tangent space to M^q at x is spanned by $e_a, e_{a^*}, e_f, e_k, (1 \leq a \leq t < f \leq 2r + q - 2n < k \leq q)$. Since we consider only holomorphic r-planes that intersect M^q, we may assume that the point x in (4.2) is such that

$$dx = \sum_{a=1}^{t}(\wedge_a e_a + \wedge_a^* e_{a^*}) + \sum_{f=2t+1}^{2r+q-2n} \mu_f e_f + \sum_{k=2r+q-2n}^{2n} \tau_k b_k \tag{4.3}$$

where $\wedge_a, \wedge_{a^*}, \mu_f$ and τ_k are differential 1-forms. So, we have

$$\omega_\alpha = \sum_{k=2r+q-2n}^{2n} \tau_k (b_k \cdot e_\alpha); \omega_{\alpha^*} = \sum_{k=2r+q-2n}^{2n} \tau_k (b_k \cdot e_{\alpha^*}) \tag{4.4}$$

and thus

$$d\sigma_{(n-r)} = \bigwedge_{\alpha=1}^{n-r} (\omega_\alpha \wedge \omega_{\alpha^*}) = \nabla (\bigwedge_k \tau_k) \tag{4.5}$$

where $\nabla = \det\ (b_k \cdot e_1), 1 = r+1, \cdots, n, r+1^*, \cdots, n^*$.

If $d\sigma_{2r+q-2n}$ denotes the volume element of $L_{2r} \cap M^q$ and $d\sigma_q$ the volume element of M^q at x and since $\wedge \tau_k$ is the $2(n-r)$-dimensional volume element of M^q orthogonal to $L_{2r} \cap m^q$, we have

$$(\wedge \tau_k) \wedge d\sigma_{2r+q-2n} = d\sigma_q \tag{4.6}$$

and from (4.2), (4.5) and (4.6) we obtain

$$d\sigma_{2r+q-2n} \wedge dL_r = \nabla\, d\sigma_q \wedge dL_r \tag{4.7}$$

Note that:

a) ∇ depends of the position of L_r with respect to the tangent q-plane of M^q at x, but it is independent of x.

b) $2r+q-2n \neq 0$

Let $\sigma(M^q)$ denote the q-dimensional volume of M^q. Integrating both sides of (4.7) over all holomorphic r-planes that intersect M^q, we get

$$\int_{L_r \cap M \neq \emptyset} \sigma_{2r+q-2n}(M^q \cap L_r) dL_r = c\sigma_q(M^q) \tag{4.8}$$

where $c = \int \nabla dL_{r[x]}$ is a constant that we want to calculate. To this end we shall calculate directly the left hand side of (4.8) for the q-dimensional unit sphere U_q in $\mathbb{C}^n$. Since q is odd, let us write $q = 2p+1$. Let $L_m^{(p+1)}, m \leq p$, denote the holomorphic m-planes of the holomorphic $(p+1)$-plane that contains U_q. Consider first the integral $\int \sigma_{2m-1}(U_q \cap L_m^{(p+1)}) dL_m^{(p+1)}$ over all $L_m^{(p+1)}$ that intersect U_q.

If O is the center of U_q and ρ denotes the distance from O to $L_m^{(p+1)}$, then $U_q \cap L_m^{(p+1)}$ is a sphere of dimension $2m-1$ and radius $(1-\rho^2)^{1/2}$ whence $\sigma_{2m-1}(U_q \cap L_m^{(p+1)}) = (1-\rho^2)^{2m-1/2} O_{2m-1}$. On the other hand, since the volume element of the holomorphic $(p+1-m)$-plane perpendicular to $L_m^{(p+1)}$ through O at its intersection point with $L_m^{(p+1)}$ is equal to $\rho^{2(p-m)+1} du_{2(p-m)+1} \wedge d\rho$, (with $du_{(\cdot)}$ the density on the sphere), we have $dL_m^{(p+1)} = \rho^{2(p-m)+1} du_{2(p-m)+1} \wedge d\rho \wedge dL^{(p+1)}_{(p-m+1)[0]}$. Taking into account that, ([6], p. 245)

$$\int_0^1 \rho^{2(p-m)+1}(1-\rho^2)^{2m-1/2} d\rho = \frac{O_{2p+2}}{O_{2p-2m+1} O_{2m}} \tag{4.9}$$

and using (1.11) applied to the complex Grassmann manifold $G^{\mathbb{C}}_{p+1-m,m}$, we have

$$\int \sigma_{2m-1}(U_{2p+1} \cap L_m^{(p+1)}) dL_m^{(p+1)} = \frac{O_{2p+2} O_{2m-1} v(U(p+1))}{O_{2m} v(U(m)) v(U(p-m+1))}. \tag{4.10}$$

Now we return to the case of a general holomorphic m-plane L_m of $\mathbb{C}^n$. We apply formula (3.16) to the case in which F is the volume of

$U_{2p+1} \cap L_m$. So, we have

$$\int \sigma_{2r+q-2n}(U_{2p+1} \cap L_r)dL_r = \frac{v(U(n+1))v(U(r+p-n+2))}{v(U(p+1))v(U(r))}$$

$$\int \sigma_{2(p+r-n+1)-1}(U_{2p+1} \cap L^{(p+1)}_{(r+p+1-n)})dL^{(p+1)}_{(r+p+1-n)} =$$

(4.11)

$$= \frac{v(U(n+1))v(U(r+p-n+2))O_{2p+2}O_{2(p+r-n)+1}v(U(p+1))}{v(U(r))O_{2(r+p+1-n)}v(U(r+p-n+1))v(u(n-r-2))} =$$

$$= cO_{2p+1} .$$

Substituting in (4.8) the value of c obtained in (4.11), we have the final result

$$\int \sigma_{2r+q-2n}(M_q \cap L_r)dL_r =$$

$$(4.12) \quad = \frac{v(U(n+1))v(U(r+o-n+2))v(U(p+1))O_{2p+2}O_{2(p+r-n+1)}}{v(U(r)v(U(r+p-n+1))V(U(n-r-2))O_{2(r+p-n+1)}} \cdot$$

$$\cdot \sigma_{2p+1}(M^q) .$$

REFERENCES

[1] Y.D. Burago - V.A. Zalgaller: *Geometric inequalities*, Grund. der Math. Wiss. 285. Springer-Verlag, (1988).

[2] S.S. Chern: *On the kinematic formula in integral geometry*, J. Math. and Mech. 16, (1966), 101-118.

[3] A.M. Naveira - F. Gargia: *Some results of integral geometry for densities of linear subspaces of* $\mathbb{C}^n$, Rend. Mat., 12 (1992), 921-935.

[4] L.A. Santalo: *Integral geometry in hermitian spaces*, Amer. J. of Math.74, (1952), 423-434.

[5] L.A. Santalo: *Sur la mesure des espaces linéaires qui coupent un corps convexe et problèmes qui s'y rattachent. Colloque sur la questions de réalité en géométrie*, Liège, (1955), 177-190.

[6] L.A. Santalo: *Integral geometry and geometric probability*, Addision - Wesley. Reading. Mass., (1976).

[7] T. SHIFRIN: *The kinematic formula in complex integral geometry*, Trans. A.M.S. 264, (1981), 255-293.

[8] F.W.WARNER: *Foundations of differentiable manifold and Lie groups*, Scott Foresmann, (1971).

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