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Generating function and semi-orthogonal properties of a new class of polynomials

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RIASSUNTO: *Viene definita una nuova classe di polinomi per mezzo delle loro funzioni generatrici e viene stabilita per essi una proprietà di semi-ortogonalità (cfr. § 3).*

ABSTRACT: *We define a new class of polynomials by means of a generating function and establish its semi-orthogonal properties.*

KEY WORDS: *Generating function – Semi-orthogonality – B-polynomials – Bessel polynomials – Laguerre polynomials, Hermite polynomials – X-polynomials.*

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1 – Introduction

Recently [3], we have defined the B-polynomials:

$$(1.1)\ B_m(x) = \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m}\, {}_{r+q+1}F_{s+p}\left[\begin{matrix} c_r, 1-b_q-m, -m; \dfrac{\beta}{\alpha}x(-1)^{p-q-1} \\ \mathfrak{d}_s, 1-a_p-m \end{matrix}\right](\alpha)^m,$$

by means of the generating function:

$$(1.2) \qquad {}_pF_q\left[\begin{matrix}\mathfrak{a}_p;\ \alpha t\\ \mathfrak{b}_q\end{matrix}\right]{}_rF_s\left[\begin{matrix}\mathfrak{c}_r;\ \beta xt\\ \mathfrak{d}_s\end{matrix}\right]=$$

$$=\sum_{m=0}^{\infty}\frac{(\alpha t)^m}{m!}\frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m}{}_{r+q+1}F_{s+p}\left[\begin{matrix}\mathfrak{c}_r, 1-\mathfrak{b}_q-m, -m; \frac{\beta}{\alpha}x(-1)^{p-q-1}\\ \mathfrak{d}_s, 1-\mathfrak{a}_p-m\end{matrix}\right],$$

where the symbol $\mathfrak{a}_p$ represents the set of parameters $a_1, \ldots, a_p$, and $1-\mathfrak{a}_p-m$ the set of parameters $1-a_1-m, \ldots, 1-a_p-m$, and so on for $\mathfrak{b}_q, \mathfrak{c}_r, \mathfrak{d}_s$.

It is interesting to note that the B-polynomials lead to the generalization of some classical polynomials and yield some new polynomials.

We visualize at least three orthogonality properties of the B-polynomials for different weight functions on different intervals. However, we have not been successful to establish any of them. The proofs are complicated in view of the general nature of B-polynomials. In our attempt to establish orthogonality properties of B-polynomials, we have been able to find some new methods to estahlish ortogonality properties of orthogonal polynomials and define some new orthogonal polynomials.

In this paper, we obtain a generating function and two semi-orthogonal properties of the semi-orthogonal polynomials:

$$(1.3) \qquad X_n(x;a,y) = {}_2F_0\left[\begin{matrix}-n, a;\ -\frac{x}{y}\\ -\end{matrix}\right],$$

wich we shall call the X-polynomials.

The X-polynomials are related to the Bessel polynomials [7], the Laguerre polynomials and the Hermite polynomials by the following relations:

$$(1.4) \qquad X_n(x; n+a-1, b) = y_n(x : a, b)$$

$$(1.5) \qquad X_n(x; a, b) = y_n(x; 1+a-n, b)$$

$$(1.6) \qquad X_n(1; -a-n, x) = (-1)^n n! x^{-n} L_n^a(x)$$

(1.7) $$X_n(1;a,x) = (-1)^n n! x^{-n} L_n^{-a-n}(x)$$

(1.8) $$X_n(1;1/2-n,x^2) = (2x)^{-2n} H_{2n}(x)$$

(1.9) $$X_n(1;-1/2-n,x^2) = (2x)^{-2n-1} H_{2n+1}(x)\,.$$

Although X-polynomials are related to the above polynomials, but their basic properties would be different from those of the above polynomyals, in view of their unique generating function given in section 2.

In section 4, we obtain a known generating function and two known orthogonality properties as particular cases of our results.

2 – The generating function

The following theorem holds:

THEOREM I. *The generating function of the polynomials $X_n(x;a,y)$, defined by* (1.3), *is given by the formula:*

(2.1) $$e^t(1-xt/y)^{-a} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X_n(x;a,y)\,.$$

PROOF. In (2.1), putting $\alpha = \beta = 1$, taking $p = q = s = 0$, $r = 1$, $c_1 = a$, and setting x/y for x, we have

(2.2) $${}_0F_0\left[\begin{matrix}-; & t\\ - & \end{matrix}\right] {}_1F_0\left[\begin{matrix}a; & \dfrac{xt}{y}\\ - & \end{matrix}\right] = \sum_{m=0}^{\infty} \frac{t^m}{m!}\, {}_2F_0\left[\begin{matrix}-m, a; - & \dfrac{x}{y}\\ - & \end{matrix}\right]$$

In (2.2), putting ${}_0F_0\left[\begin{matrix}-; & t\\ - & \end{matrix}\right] = e^t$, ${}_1F_0\left[\begin{matrix}a; & \dfrac{xt}{y}\\ - & \end{matrix}\right] = (1-xt/y)^{-a}$, and using (1.3), we obtain the generating function (2.1).

3 – The semi-orthogonal properties

In what follows for sake of brevity, we will write:

$$f(a,n)=\frac{n!\pi(a)_n}{\Gamma(1+n+a)\sin\pi(1+a)}$$

and

$$g(a,b)=\frac{(n+1)!\pi(a)_n}{\Gamma(2+n+a)\sin\pi(1+a)}.$$

The semi-orthogonal properties satisfied by the X-polynomials are the following:

THEOREM II. *For* $\operatorname{Re} a < -m$, *provided a is nonintegral, the X-polynomials are semi-orthogonal with weight function* $x^{a-n-1}\ e^{-z/x}$ *on* $(0,\infty)$ *according as*

(3.1) $$\int_0^\infty x^{a-n-1}e^{-z/x}X_m(x;a,z)X_n(x;a,z)dx=$$

(3.1a) $$=0,\quad \text{if}\quad m<n$$

(3.1b) $$=f(a,n)z^{a-n},\quad \text{if}\quad m=n$$

(3.1c) $$=g(a,n)z^{a-n},\quad \text{if}\quad m=n+1.$$

NOTE 1: On continuing as above, we can find the values of the integral (3.1) for $m=n+2,n+3,n+4,\ldots$.

THEOREM III. *For* $a=n-1/2,n-3/2,n-5/2,\ldots$, *provided a is nonintegral, the X-polynomials are semi-orthogonal with weight function* $x^{2(a-n-1/2)}\ e^{-z/x^2}$ *on* $(-\infty,\infty)$ *according as*

(3.2) $$\int_{-\infty}^\infty x^{2(a-n-1/2)}e^{-z/x^2}X_m(x^2;a,z)X_n(x^2;a,z)dx$$

(3.2a) $$=0,\quad \text{if}\quad m<n$$

(3.2b) $$=f(a,n)z^{a-n},\quad \text{if}\quad m=n$$

(3.2c) $$=g(a,n)z^{a-n},\quad \text{if}\quad m=n+1$$

NOTE 2: On continuing as above, the values of the integral (3.2) can be obtained for $m = n+2, n+3, n+4, \dots$.

PROOF. To prove (3.1), let us consider

$$X_m(x;a,z)X_n(x;a,z) = {}_2F_0\left[\begin{matrix}-m, a; - & \frac{x}{z}\\ - & \end{matrix}\right] {}_2F_0\left[\begin{matrix}-n, a; - & \frac{x}{z}\\ - & \end{matrix}\right]$$

so that

$$\begin{aligned}&\int_0^\infty x^{a-n-1}e^{-z/x}X_m(x;a,z)X_n(x;a,z)dx\\ (3.3)\quad &= \sum_{r=0}^{m}\frac{(-m)_r(a)_r}{r!(-z)^r}\sum_{u=0}^{n}\frac{(-n)_u(a)_u}{u!(-z)^u}\int_0^\infty x^{a-n-1+r+u}e^{-z/x}dx\end{aligned}$$

On evaluating the last integral by using the definition of gamma function, viz.

$$(3.4)\qquad \int_0^\infty x^{-n-2}e^{-z/x}dx = \Gamma(n+1)/z^{n+1}, \mathrm{Re}\, n > -1$$

and using [4,p.3,(4)], the right hand side of (3.3) reduces to the form

$$(3.5)\qquad z^{a-n}\sum_{r=0}^{m}\frac{(-m)_r(a)_r}{(-1)^r r!}\Gamma(n-a-r)\, {}_2F_1\left[\begin{matrix}-n, & a;1\\ 1-n+a+r\end{matrix}\right]$$

On applying Vandermonde's theorem:

$$(3.6)\qquad {}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n}, n = 0,1,2,\dots$$

to (3.5) and using the relation $(1-n+r)_n = (-1)^n(-r)_n$, we have

$$(3.7)\qquad z^{a-n}\sum_{r=0}^{m}\frac{(-m)_r(-r)_n(a)_r\Gamma(n-a-r)}{r!(1-n+a+r)_n}(-1)^{n-r}$$

If $r < n$, the numerator of (3.7) vanishes, and since r runs from 0 to m, it follows that (3.7) also vanishes, when $m < n$. Now, it is clear that for $m < n$ all terms of (3.7) vanish, which proves (2.1a).

When $m = n$, using [4,p.3,(6)] and the standard result:

$$(3.8) \qquad (-r)_n = \begin{cases} \dfrac{(-1)^n r!}{(r-n)!}, & \text{if } 0 \le n \le r \\ 0, & \text{if } n > r \end{cases}$$

we have

$$(3.9) \qquad \int_0^\infty x^{a-n-1} e^{-z/x} \{X_n(x; a, z)\}^2 \, dx = f(a, n) z^{a-n},$$

which proves (3.1b).

In (3.7), putting $m = n + 1$, using [4,p.3,(6)] and (3.8) and adding the resulting two terms $(m = n, n + 1)$, we obtain

$$(3.10) \qquad \int_0^\infty x^{a-n-1} e^{-z/x} X_{n+1}(x; a, z) X_n(x; a, z) dx = g(a, n) z^{a-n},$$

wich proves (3.1c).

To prove (3.2), we employ the same method of proof as for (3.1), except instead of (3.4), we use the following formula:

$$\int_0^\infty x^{-2n} e^{-z/x^2} dx = \frac{\Gamma(n - 1/2)}{z^{n-1/2}}, \quad n = 1, 2, 3, \ldots$$

4 – Fourier series expansion involving the X-polynomials

Based on the relation (3.1a) and (3.1b), we can generate a theory concerning the expansion fo arbitrary polynomials, or functions in genral, in a finite-series expansion of the X-polynomials. Specially if $F(x)$ is a suitable function defined for all x, we consider for expansion of the general from

$$(4.1) \qquad F(x) = \sum_{m=0}^{n} C_m x^{-m} X_m(x; a, z), \quad 0 < x < \infty$$

where the Fourier coefficients C_m are given by

$$C_m = \frac{z^{m-a}}{f(a,m)} \int_0^\infty F(x) x^{a-m-1} e^{-z/x} X_m(x; a, z) dx \,. \tag{4.2}$$

NOTE 3: Based on the relations (3.2a) and (3.2b), we can easily establish another finite series expansion similar to (4.1).

5 – Particular cases

(a) *Generating functions*

On applying (1.5) to (2.1), we obtain the following generating function (in somewhat lesser generality) for the Bessel polynomials:

$$e^t (1 - xt/z)^{-a} = \sum_{n=0}^{\infty} \frac{t^n}{n!} y_n(x; 1 + a - n, z) \,. \tag{5.1}$$

The above generating function in a slightly different notation was obtained by Al-Salam [1] by using two generating functions due to Feldheim [6] and the relation between the Bessel polynomails and the Jacobi polynomials.

(b) *Orthogonality properties*

(i) In (3.9), putting $z = 1$, setting $n + a - 1$ for a in X_n and in the index of x, we have

$$\begin{aligned} &\int_0^\infty x^{a-2} e^{-1/x} \{X_n(x, n + a - 1, 1)\}^2 dx \\ &= \frac{n! \pi (n + a - 1)_n}{\Gamma(a + 2n) \sin \pi (n + a)}, \operatorname{Re} a < 1 - m - n \,. \end{aligned} \tag{5.2}$$

Now, using (1.4) and simplifying, we obtain the orthogonality property for the Bessel polynomials given by Exton [5, p. 215, (14)].

(ii) In (3.9), putting $z = 1$, replacing x by $1/x$, setting $n + a - 1$ for a in X_n and in the index of x, and using the relation $X_n(1/x; a, 1) =$

$X_n(1; a, x)$, we obtain

$$(5.3)\qquad \int_0^\infty x^{-a}e^{-x}\{X_n(1; n+a-1, x)\}^2\,dx = \frac{n!\pi(n+a-1)_n}{\Gamma(a+2n)\sin\pi(n+a)}, \operatorname{Re} a < 1-m-n\,.$$

Now, using (1.4) and simplifying, we get the orthogonality property obtained by the author [(2, p. 78, (2.1)]:

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