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Space - time compatibility conditions for strains and velocities

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RIASSUNTO: *Si dimostra che le condizioni unificate di compatibilità per le deformazioni e le velocità derivano dall'invarianza di gauge del tensore di influenza in SO(3).*

ABSTRACT: *It is shown that unified compatibility conditions for strains and velocities follow from the SO(3) gauge-invariant strength tensor.*

KEY WORDS: *Continuum Mechanics - Compatibility conditions.*

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1 – Introduction

Within the framework of the model of simple mechanical deformable continuum, usually called the Cauchy continuum, the problem of compatibility for strain and velocity fields arises always when these fields are not computed as the gradients of vectors or another fields derivatives. The fundamental compatibility question, due to De Saint-Venant, asks in which conditions a given strain can be represented as a gradient of displacements. When the deformation gradients are small the Saint-Venant's compatibility conditions decide about compatibility or not. Yet long before as we have recognized the connection between Saint-Venant's conditions and the equivalent statement of Riemannian geometry, the problem was considered also within the framework of the finite deformation. In this case the problem is non-linear one: to find necessary and sufficient condition that for given six components of the Green strain tensor there exist a displacement vector. Let from many different solutions we record only these given by RIQUIER [6], MANVILLE [7], MARCOLONGO [8], CRUDELI [12], for an arbitrary three-dimensional deformation. TRUESDELL and TOUPIN [21] give an essential relation between the Saint-Venant and the flat geometry conditions by the statement that the Riemann-Christoffel curvature tensor, formed with the Green strain as the components of the metric tensor, vanish identically.

Nevertheless, from the point of view of a problem which we shall discuss here, the Saint-Venant's treatment of compatibility is not sufficient. The purpose of this paper is to consider a material continuum which undergo a motion. We mean here such problems as, for instance, a rolling of heavy, deformable ring on the plane or a free-free flexible beam undergoing large overall motion. During a such kind of motion there exists in the body, frequently a large domain that move only either as a rigid body or a nearly rigid one with, form the other hand, an explicit localized hardly deformed domain. Our aim is an appropriate formulation of the problem of a recovering of motion from a given strain and velocity that would contains also the case of rigid body motion.

Unfortunately, farther utilisation of Saint-Venant's and Cesaro's formulae for our non-linear problem seems to be impossible. Thus, a fundamental concept which may be a base for considerations is connected with the Cauchy fundamental theorem which says about decomposition of deformation at any point on a translation, a rigid rotation of principal axes of strain and a stretch along these axes. We have added here, important in our context, an supplementary Kirchhoff's theorem about decomposition of velocity [2]. In the hidden manner, we have to do with the rotation which is not only a time function (as in rigid body kinematics) but also the space co-ordinate function. Such a rotation acting on a global fixed reference frame produces the moving frame that is the non-inertial in time and also in space sense simultaneously. Speaking in the language of the group theory one can say that the such rotation is an element of a local (in space and time) Lie group SO(3). It would thus seems reasonable to expect that the integrability conditions for the local rotation field would deal with the non-classical form of the compatibility equations. The such approach, for the decomposition $\mathbf{F} = \mathbf{RU}$, was explored in a fully nonlinear version by SIGNORINI [18] and developed and completed by FERRARESE [20,22,24] who discovered the relation with the anholonomic geometry. Later on SHAMINA [28], quite independently, developed the same relations but written down in the convective Hencky co-ordinate system.

Adopting the same approach, PIETRASZKIEWICZ [33] generalized the linear Cesàro formulae into the finite deformation case. He divided all process of determination of the displacement on two etaps. First, a finite rotation in a point under consideration is calculated by using the ordered path integral connecting the considered point with an early fixed comparison point of a body. However, the calculation of stretch tensor form the Green-St. Venant strain presents a purely algebraic problem. In the second step, with the already known rotation and stretch tensors, the displacement vector can be obtained through quadratures in path integration procedure. The Cesáro-Pietraszkiewicz formula is the path-independent only for a simple connected body and becomes the path-dependent if there exists an isolated region where compatibility is not satisfied or in the case of a multi-connected domain.

In our paper we prefer the left decomposition $\mathbf{F} = \mathbf{VR}$. It is due to fact that kinematics of a rigid body is based on the relations similar to those which follow from **VR** decomposition. Such a similarity, applied to a one-dimensional body, is known as the Kirchhoff kinematic analogy [1]. In this analogy the arc length along the axis of an undeformed rod plays the role similar as time co-ordinate in a motion of heavy body about a fixed point [3]. Also Kirchhoff's decomposition of velocity $\mathbf{v} = \nu\mathbf{R}$ [2] agrees conceptually with rigid body mechanics. FERRARESE has indicated [20] on the possible interrelation between the left and the right decomposition especially within the context of the spin of a rigid body. But an another profit is due to fact that this approach can easily be prolonged to Cosserat continuum where a moving reper plays the leading role. In that sense the papers by FERRARESE [27,30], and FERRARESE ET AL [36], referring to anholonomic geometry, develop the Cosserat continuum precisely in the original line of reasoning.

In the first part of the paper we shall present the kinematical relation of the simple non-polar, Cauchy continuum in useful form where the spatial and temporal co-ordinates are treated in the equivalent manner. Therefore we shall prefer the Hencky convective co-ordinates, although, the final results are presented also in the general absolute notation. The main result concerns with an extension of Signorini-Shamina ansatz to the case of space-time compatibility equations. To this end, it has been used the mathematical formalism connected with the non-Abelian SO(3) group. It is characteristic that the space-time compatibility conditions here obtained coincide partially with those ones derived, via other postulate, by FERRARESE [30, eq. 11]. In the last parts, the Cesaro-Pietraszkiewicz formula has been extended in the spatial-temporal context as well as in the path-dependent one.

2 – The kinematical relations

Let us consider the four-dimensional Galilean space-time continuum which is endowed with the absolute time co-ordinate t. The additional structure deals with the geometry of three-dimensional layers of space-time for the slices of constant t. The Galilean approach requires that geometry of each "space slice" is completely flat, what means that each one is endowed with a three-dimensional Cartesian metric and the orthogonal co-ordinates, and also with the vanishing Galilean connection. We shall discuss a relation between two space slices which are determined by two fixed time $t = 0$ and $t = \tau$. Also we shall restrict ourselves to analyze only this domain of the sliced space which is occupied by our material body, and denote its by $\mathcal{B}$ and $\overline{\mathcal{B}}$, respectively.

Let introduce also, a preferred curvilinear, convected, co-ordinate system θ^i, $i = 1, 2, 3$, (the first Hencky) which, together with the base vectors $\mathbf{g}_i$ and $\bar{\mathbf{g}}_i$, defines the Hamilton operator $\nabla = \mathbf{g}^i \frac{\partial}{\partial \theta^i} = \mathbf{g}^i \partial_i$ and $\overline{\nabla} = \bar{\mathbf{g}}^i \frac{\partial}{\partial \theta^i} = \mathbf{g}^i \partial_i$ in both space slices, respectively. The position of an arbitrary point of the body under consideration is marked out by a vector $\mathbf{p}(\theta^i, t = 0)$ in the undeformed, and by $\bar{\mathbf{p}}(\theta^i, t = \tau)$ in the deformed configuration, respectively.

Let recall that in the framework of the Cauchy continuum the complete information about body kinematics contains the deformation gradient tensor $\mathbf{F}$ and the velocity vector $\mathbf{v}$ fields

$$\mathbf{F}(\theta^i, t) = \bar{\mathbf{p}} \otimes \nabla = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i, \quad \mathbf{v}(\theta^i, t) = \partial_t \bar{\mathbf{p}}_{|\theta^j = \mathrm{const}} \tag{1}$$

It is a fundamental assumption for our considerations that the deformation gradient $\mathbf{F}$ and the velocity vector $\mathbf{v}$ are decomposed into the product

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \quad \mathbf{v} = \boldsymbol{\nu}\mathbf{R} \tag{2}$$

where $\mathbf{V}$ is the positive defined spatial stretch and $\mathbf{R}$ the material rotation, however, $\boldsymbol{\nu}$ represents a velocity vector measured in a local, rotating under the action of $\mathbf{R}$, base. Here, the formula $(2)_2$ represents analogue of a space gradient decomposition but referred to a time "gradient". The such decomposition was introduced by KIRCHHOFF [2] in the context of the description of motion of a rotating body in fluid.

Now, it is straightforward to show that our starting formula has the form

$$\begin{aligned}(\partial_\alpha \mathbf{F})\mathbf{F}^{-1} =& (\partial_\alpha \mathbf{V})\mathbf{V}^{-1} + \mathbf{V}(\partial_\alpha \mathbf{R})\mathbf{R}^{-1}\mathbf{V}^{-1} = \\ =& (\partial_\alpha \mathbf{V})\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}_\alpha\mathbf{V}^{-1}, \quad \alpha = t, 1, 2, 3\end{aligned} \tag{3}$$

where we define

$$\boldsymbol{\Omega}_\alpha \equiv (\partial_\alpha \mathbf{R})\mathbf{R}^{-1} = -\boldsymbol{\Omega}_\alpha^T \tag{4}$$

and by $\partial_\alpha = (\partial_i, \partial_t)$, $i = 1, 2, 3$, denote the differentiation operators, for space and time, with respect to the holomic co-ordinates (θ^i, t). The time derivative will be, for convenience, denoted also by superposed dot and the results of time differentiation without any additional index, for instance $\boldsymbol{\Omega}_\alpha \equiv \boldsymbol{\Omega}$, when $\alpha = t$.

In particular, the above relation for $\alpha = t$ is well-known as the decomposition of the velocity gradient (TRUESDELL and TOUPIN [21, §76], DENIS [32])

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{L} = L_{ij}\bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}\mathbf{V}^{-1} \tag{5}$$

We anticipate that, in the similar way, formulae (5) can also be extended into a spatial relation, when $\alpha = i$, $i = 1, 2, 3$:

$$(\partial_i \mathbf{F})\mathbf{F}^{-1} = \mathbf{L}_i = (\partial_i \mathbf{V})\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}_i\mathbf{V}^{-1} \tag{6}$$

and then it has an interpretation of the gradient of geometrical velocity and may be called the space rate of the deformation gradient.

Adopting the fundamental decomposition of Cauchy and Stokes, we split the measures $\mathbf{L}_\alpha$ on the symmetric and the skew-symmetric parts

$$\begin{aligned} \alpha = t, &\quad \mathbf{L} = \mathbf{D} + \mathbf{W} \\ \alpha = i, &\quad \mathbf{L}_i = \mathbf{D}_i + \mathbf{W}_i \end{aligned} \tag{7}$$

where the tensors $\mathbf{D}_\alpha = \mathbf{D}_\alpha^T$ and $\mathbf{W}_\alpha = -\mathbf{W}_\alpha^T$ play the role of the rate of deformation and the vorticity tensors in space-time. Notice, that $\mathbf{D}$ and $\mathbf{W}$ are known in the literature as to be the rate and the vorticity tensors [21, §90].

However, the tensor $\boldsymbol{\Omega}_\alpha(\alpha = i, t)$ is called the material spin tensor with the spatial and temporal components. In particular, $\boldsymbol{\Omega}_i$ may be also called the tensor of change of curvature because it expresses the spatial rate of change of a body configuration.

3 – The first formula for $\boldsymbol{\Omega}_\alpha$

From the skew-symmetry property of $\boldsymbol{\Omega}_\alpha$ it follows that there exists an unique vector representation

$$\boldsymbol{\Omega}_\alpha = \boldsymbol{\omega}_\alpha \times \mathbf{1}\,. \tag{8}$$

Additionally, $\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^{-1} = \boldsymbol{\omega} \times \mathbf{1}$ is known to be a material spin [20].

From eqs (4) it follows that the spin tensors $\boldsymbol{\Omega}_\alpha(\theta^i, t)$ are the function of the rotation $\mathbf{R}(\theta^i, t)$ solo, and that a final form of $\boldsymbol{\Omega}_\alpha$ depends on three independent parameters of the rotation $\mathbf{R}$. For instance, when orthogonal rotation tensor $\mathbf{R}$ is represented by a unit vector $\mathbf{e}$ which is describing the axis of rotation and φ - the angle of clockwise rotation about $\mathbf{e}$, that is by the Finger representation [21, §38]

$$\mathbf{R} = \cos\varphi \mathbf{1} + \sin\varphi \mathbf{e} \times \mathbf{1} + (1 - \cos\varphi)\mathbf{e} \times \mathbf{e} \tag{9}$$

then, after simple calculations [20,22], from (4) and (9) we obtain

$$\begin{aligned} \boldsymbol{\Omega} &= \Big(\sin\varphi \dot{\mathbf{e}} - (1 - \cos\varphi)\dot{\mathbf{e}} \times \mathbf{e} + \dot{\varphi}\mathbf{e}\Big) \times \mathbf{1} \equiv \boldsymbol{\omega} \times \mathbf{1} \\ \boldsymbol{\Omega}_i &= \Big(\sin\varphi \partial_i \mathbf{e} - (1 - \cos\varphi)\partial_i \mathbf{e} \times \mathbf{e} + \partial\varphi \mathbf{e}\Big) \times \mathbf{1} \equiv \boldsymbol{\omega}_i \times \mathbf{1} \end{aligned} \tag{10}$$

From the other hand, if the rotation tensor $\mathbf{R}$ is unknown, then the material spin $\boldsymbol{\Omega}_\alpha$ can be calculated from formulae (5) and (10) in terms of the space-time rate of deformation $\mathbf{D}_\alpha$, the vorticity $\mathbf{W}_\alpha$ and without, what is some advantage, the derivatives of $\mathbf{V}$. To this end, by pre multiplying and post multiplying eq. (5) by $\mathbf{V}$ we obtain

$$\begin{aligned} &\partial_\alpha \mathbf{V} + \mathbf{V}\boldsymbol{\Omega}_\alpha = \mathbf{D}_\alpha \mathbf{V} + \mathbf{W}_\alpha \mathbf{V} \\ &\partial_\alpha \mathbf{V} - \boldsymbol{\Omega}_\alpha \mathbf{V} = \mathbf{V}\mathbf{D}_\alpha - \mathbf{V}\mathbf{W}_\alpha . \end{aligned} \tag{11}$$

Combining (11) we find

$$\mathbf{V}\boldsymbol{\Omega}_\alpha + \boldsymbol{\Omega}_\alpha \mathbf{V} \equiv \mathbf{A}_\alpha \tag{12}$$

the classical tensor algebraic equation put on the unknown tensors $\boldsymbol{\Omega}_\alpha$. Above, $\mathbf{V}$ is symmetric, and $\boldsymbol{\Omega}_\alpha$ and $\mathbf{A}_\alpha$ are skew-symmetric

$$\mathbf{A}_\alpha = \mathbf{W}_\alpha \mathbf{V} + \mathbf{V}\mathbf{W}_\alpha + \mathbf{D}_\alpha \mathbf{V} - \mathbf{V}\mathbf{D}_\alpha \tag{13}$$

If we use a vector representation for the skew-symmetric tensors $\mathbf{A}_\alpha = \mathbf{a}_\alpha \times \mathbf{1}$, then equations (12) lead to

$$[\operatorname{tr}\mathbf{V}\mathbf{1} - \mathbf{V}]\boldsymbol{\omega}_\alpha \times \mathbf{1} = \mathbf{a}_\alpha \times \mathbf{1} \tag{14}$$

The solution of (14) for $\alpha = t$ has been given by DENIS [22] to be

$$\boldsymbol{\omega}_\alpha = (\operatorname{tr}\mathbf{V}\mathbf{1} - \mathbf{V})^{-1}\mathbf{a}_\alpha \tag{15}$$

It can be represented in the more comfortable tensor notation as

$$\begin{aligned} \alpha = t \quad \boldsymbol{\Omega} =& \Delta^{-1}\Big((I_V^2\mathbf{1} - \mathbf{V}^2)\mathbf{A} + (\mathbf{V} - I_V\mathbf{1})\mathbf{A}\mathbf{V} + \\ &+ III_V(I_V\mathbf{V}^{-1} - \mathbf{1})\mathbf{A}\mathbf{V}^{-1}\Big) \\ \alpha = i \quad \boldsymbol{\Omega}_i =& \Delta^{-1}\Big((I_V^2\mathbf{1} - \mathbf{V}^2)\mathbf{A}_i + (\mathbf{V} - I_V\mathbf{1})\mathbf{A}_i\mathbf{V} + \\ &+ III_V(I_V\mathbf{V}^{-1} - \mathbf{1})\mathbf{A}_i\mathbf{V}^{-1}\Big) \end{aligned} \tag{16}$$

where I_V, II_V, III_V are the principal invariants of $\mathbf{V}$ and $2\Delta = I_V II_V - III_V$.

4 – The second formula for Ω_α

The formula (16) gives the needed relation which enables us to express Ω_α ether by rotation solo or by the components $\mathbf{L}_\alpha$ and $\mathbf{V}$ from the other hand. Now we shall show another realization of the Signorini-Shamina ansatz in which we will use the convective curvilinear co-ordinates and components of strain E_{ij} and velocity explicitly. A result will be equivalent to the formulae (16) but expressed in more convenient variables.

Let recall that in the convective co-ordinate description all information about a deformation contains the deformed actual base vectors $\bar{\mathbf{g}}_i$ via $\mathbf{F} = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i$, $\mathbf{F}^{-1} = \mathbf{g}_i \otimes \bar{\mathbf{g}}^i$ and part of them is used to definition of the deformed (actual) metric $\bar{g}_{ij} = \bar{\mathbf{g}}_i \cdot \bar{\mathbf{g}}_j$. The fundamental symmetric deformation measure E_{ij} can be related to undeformed and deformed basis in the Green-St. Venant and Almansi-Hamel strain tensors respectively

$$
(17) \qquad
\begin{aligned}
\mathbf{E} &= \frac{1}{2}(\mathbf{F}^T \bar{\mathbf{1}} \mathbf{F} - \mathbf{1}) = \frac{1}{2}(\bar{g}_{ij} - g_{ij}) g^i \otimes g^j = E_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \\
\bar{\mathbf{E}} &= \frac{1}{2}(\bar{\mathbf{1}} - \mathbf{F}^{-T} \mathbf{1} \mathbf{F}^{-1}) = \frac{1}{2}(\bar{g}_{ij} - g_{ij}) \bar{g}^i \otimes \bar{g}^j = E_{ij} \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j
\end{aligned}
$$

where the identity tensors $\mathbf{1} \equiv \bar{\mathbf{1}}$ are represented by

$$
\mathbf{1} = g_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \quad \bar{\mathbf{1}} = \bar{g}_{ij} \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j, \quad \bar{g}_{ij} = \bar{g}_i \cdot \bar{g}_j, \quad g_{ij} = g_i \cdot g_j
$$

Let $\Gamma^k_{ij}, \overline{\Gamma}^k_{ij}$ be the spatial Christoffel symbols of the second kind such that

$$
(18) \qquad \partial_i \bar{\mathbf{g}}_j = \overline{\Gamma}^m_{ij} \bar{\mathbf{g}}_m , \quad \partial_i \mathbf{g}^j = -\Gamma^j_{im} \mathbf{g}^m
$$

and let L^j_i (5) be the time symbols satisfying the relation

$$
(19) \qquad \partial_t \bar{\mathbf{g}}_i = L^j_i \bar{\mathbf{g}}_j \quad \partial_t \mathbf{g}_i = 0 ,
$$

From the point of view of the compatibility conditions it is important to note that formulae (18) and (19) involve the additional assumptions related to the geometry of deformed space. From the symmetry of Christoffel's symbols $\overline{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ji}$ follows that $\partial_i \partial_j \bar{\mathbf{p}} - \partial_j \partial_i \bar{\mathbf{p}} = \partial_{[i} \partial_{j]} \bar{\mathbf{p}} \equiv 0$.

Also, the similar commutation conditions are related with the space-time differentiation [21, §150]

$$(20)\quad \partial_t(\partial_i\bar{\mathbf{p}}) = \partial_t\bar{\mathbf{g}}_i \equiv \partial_i(\partial_t\bar{\mathbf{p}}) = \partial_i\mathbf{v} = (\partial_i\bar{v}^j + \overline{\Gamma}^j_{im}\bar{v}^m)\bar{g}_j = \bar{v}^j_{|i}\bar{\mathbf{g}}_j = L^j_i\bar{\mathbf{g}}_j$$

These relations may also be interpreted as the vanishing of the torsion tensor $\bar{s}^k_{ij} = \overline{\Gamma}^k_{ij} - \overline{\Gamma}^k_{ji}$ for the deformed body configuration. In another words these conditions simplify the geometry of deformed space from non-Riemannian to Riemannian. As a result, the compatibility conditions should follows from the requirements identical with these ones that reduce the Riemannian to Euclidean space.

Now the relations (17) and (18) allow us to present the derivative $\partial_\alpha\mathbf{F}$ in eq. (5) in the following form

$$(21)\quad \begin{aligned} \partial_i\mathbf{F} &= \partial_i(\bar{\mathbf{g}}_k \otimes \mathbf{g}^k) = (\overline{\Gamma}^m_{ki} - \Gamma^m_{ki})\bar{\mathbf{g}}_m \otimes \mathbf{g}^k \\ \dot{\mathbf{F}} &= \partial_t(\bar{\mathbf{g}} \otimes \mathbf{g}^k) = L^m_k\bar{\mathbf{g}}_m \otimes \mathbf{g}^k \end{aligned}$$

Because

$$\begin{aligned} \overline{\Gamma}^k_{ij} &= \bar{g}^{kl}\overline{\Gamma}_{l,ij} = \frac{1}{2}\bar{g}^{kl}(\partial_j\bar{g}_{li} + \partial_i\bar{g}_{lj} - \partial_l\bar{g}_{ij}) = \\ &= \bar{g}^{kl}(\bar{g}_{lm}\Gamma^m_{ij} + \partial_jE_{il} + \partial_iE_{jl} - \partial_lE_{ij} - 2\Gamma^m_{ij}E_{lm}) \equiv \\ &\equiv \bar{g}^{kl}(\bar{g}_{lm}\Gamma_{ij} + E_{lij}) \end{aligned}$$

the components of the tensor $\partial_i\mathbf{F}$ can be simply represented as a function of E_{ij} and their derivatives e.i. so-called three index strain E_{lij} [28]

$$(22)\quad \overline{\Gamma}^k_{ij} - \Gamma^k_{ij} = \bar{g}^{kl}E_{lij}$$

The contravariant metric tensor $\bar{g}^{ij}$ is only the algebraic function of components E_{ij} and it is defined by [34]

$$\bar{g}^{il} = \frac{1}{2}g(\bar{g})^{-1}\,\epsilon^{ijk}\epsilon^{imn}\,(g_{jm} + 2E_{jm})(g_{kn} + 2E_{kn}), \bar{g} = |\bar{g}_{ij}|,\ g = |g_{ij}|$$

Now the left side of our starting relation $(\partial_\alpha\mathbf{F})\mathbf{F}^{-1} = (\partial_\alpha\mathbf{V})\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}_\alpha\mathbf{V}^{-1}$ can be calculated according to (18), (19) to be

$$(23)\quad \begin{aligned} \mathbf{L}_i &= (\partial_i\mathbf{F})\mathbf{F}^{-1} = \bar{g}^{kl}E_{lij}\bar{\mathbf{g}}_k \otimes \bar{\mathbf{g}}^j \\ \mathbf{L} &= (\partial_t\mathbf{F})\mathbf{F}^{-1} = L^m_k\bar{\mathbf{g}}_m \otimes \bar{\mathbf{g}}^k = \mathbf{v} \otimes \overline{\nabla} \end{aligned}$$

Additionally, if the velocity vector is represented by the cororational velocity $\nu(2)_2$ then the velocity gradient $\mathbf{L}$ is also the function of spin

$$\mathbf{L} = \partial_i(\nu\mathbf{R}) \otimes \bar{\mathbf{g}}^i = (\partial_i\nu + \nu\boldsymbol{\Omega}_i)\mathbf{R} \otimes \bar{\mathbf{g}}^i \tag{24}$$

Finally, the relation on $\boldsymbol{\Omega}_\alpha$ expressed in the convective co-ordinates in the terms of $E_{ij}, \bar{v}_i$ and their derivatives takes form

$$\boldsymbol{\Omega}_\alpha = \left[(V^{-1})^{ik} L_{\alpha km} V^{mj} - (V^{-1})^i_k V^{kj}_{|\alpha}\right] \bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j \tag{25}$$

5 – Four-dimensional Riemann-Christoffel tensor

The formula on $\boldsymbol{\Omega}_\alpha$ expressed in the unified manner by equations (16) or (25) will be used in the next section to introduce space-time compatibility derived from the integrability conditions. Now, it is arises the question; do these integrability conditions are equivalent, similar with the 3-dimensional case, to a condition of vanishing of a 4-dimensional Riemann-Christoffel curvature tensor. Taking the commutators of covariant derivative we are able to construct such curvature tensor only if we have an expression on the appropriate 4-dimensional Christoffel symbols, namely $\overline{\Gamma}^4_{ij}, \overline{\Gamma}^j_{4i}, \overline{\Gamma}^4_{i4}, \overline{\Gamma}^4_{44}$, with $\overline{\Gamma}^4_{\alpha\beta} = \overline{\Gamma}^4_{\alpha\beta}$ as the primary condition.

The solution of such formulated problem has been found by TH. LEHMANN [25] who have introduced the following ansatz $\overline{\Gamma}^i_{4j} = D^i_j, \overline{\Gamma}^4_{ij} = D_{ij}, \overline{\Gamma}^4_{44} = 0$ so that the rate of deformation $\mathbf{D} = \mathbf{D}^T$ fulfils the symmetry requirements. Then a time covariant derivative taken on a second order tensor $\boldsymbol{\tau} = \tau^{ij}\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j$ is expressed through

$$\tau^{ij}_{|4} = \dot{\tau}^{ij} + D^i_k \tau^{kj} + \tau^{ik} D^j_k \tag{26}$$

and the space-temporal components of the Riemann-Christoffel tensor do not vanish identically and are equal to [25]

$$R_{4ikl} = D_{ik|l} - D_{il|k} = W_{kl|i} \tag{27}$$

Of course, this last formula involves acommutativity of spatial and time derivatives. To identify the meaning of the time derivative (26) we remark only that (26) realize the objective Zaremba-Jaumann derivative, what can be simply proved by using its definition [21, §148] and the relations $\dot{\bar{\mathbf{g}}}_i = \mathbf{L}\bar{\mathbf{g}}_i = \bar{\mathbf{g}}_i\mathbf{L}^T, \mathbf{L} = \mathbf{D} + \mathbf{W}$

$$\begin{aligned}\frac{d_r}{dt} \equiv & \dot{\tau} - \mathbf{W}\tau + \tau\mathbf{W} = \dot{\tau}^{ij}\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j + \\ & +\tau^{ij}(\mathbf{L}\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j + \bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j\mathbf{L}^T) - \mathbf{W}\tau + \tau\mathbf{W} = \\ & =(\dot{\tau}^{ij} + D^i_k\tau^{kj} + \tau^{ik}D^j_k)\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j = \tau^{ij}_{|4}\bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_k\end{aligned} \tag{28}$$

Applying (28) we can calculate directly the commutator of the spatial and Zaremba-Jaumann derivative as follows

$$\partial_i\Big(\frac{d_r}{dt}\tau\Big) - \frac{d_r}{dt}(\partial_i\tau) = \partial_{[t}\partial_{i]}\tau + \tau(\partial_i\mathbf{W}) - (\partial_i\mathbf{W})\tau \tag{29}$$

what is equivalent to Lehmann's formula [25]

$$\tau^{kl}_{|4i} - \tau^{kl}_{|i4} = R_{m.i4}^{\ \ k}\tau^{ml} + R_{m.i4}^{\ \ l}\tau^{mk} \tag{30}$$

only if we assume that

$$\partial_{[t}\partial_{i]}\tau = 0\,. \tag{31}$$

As a result, we note that the presence of the non-Euclidean part of curvature tensor does not violate the commutation between the material time and spatial derivatives (31). This fact will be utilized in the next section.

6 – The compatibility for non-abelian SO(3)

Now we write the formula (6) using the algebra of the orthogonal, proper, rotation group SO(3)

$$\Omega_\alpha = (\partial_\alpha\mathcal{R})\mathcal{R}^{-1} = \omega^a_\alpha\mathcal{T}_a, \quad a = 1, 2, 3, \quad \alpha = t, i \tag{32}$$

where the skew-symmetric matrices $\mathcal{T}_a$ represent the adjoint element of Lie algebra of so(3) [35],

$$\mathcal{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{T}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{T}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{33}$$

which obey the following commutation rule

$$[\mathcal{T}_a, \mathcal{T}_b] = \mathcal{T}_a\mathcal{T}_b - \mathcal{T}_a\mathcal{T}_b = \epsilon_{abc}\, \mathcal{T}_c . \tag{34}$$

Now any rotation $\mathcal{R} \in \mathrm{SO}(3)$ is described as an exponential function of Lie group parameters

$$\mathcal{R}(\theta^i, t) = \exp(\varphi_a(\theta^i, t)\mathcal{T}_a) \tag{35}$$

where φ_a and also ω^a_α in eq. (32) are the components of the rotation vector and material spin vectors (10) referred to some fixed orthogonal base $\mathbf{i}_a$, e.i. $\boldsymbol{\varphi} = \varphi\mathbf{e} = \varphi_a\mathbf{i}_a$, $\boldsymbol{\omega}_\alpha = \omega^a_\alpha\mathbf{i}_a$. Using of the Lie algebra so(3) with the constant matrix element $\mathcal{T}_a$ (33) we qualitatively change our calculations and a meaning of equations referred now not to the base vectors but to the base matrices. For instance, the commutation relations (34) play the role identical with the vector multiplication $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\, \mathbf{g}^k$. Nevertheless another, very important, difference lies in the fact that $\mathcal{T}_a$ is constant whereas $\mathbf{g} = \mathbf{g}(\theta^i)$ is a co-ordinate-dependent. Only for the Cartesian co-ordinates when $\theta^i = \theta_i$, $\mathbf{g}_i = \delta^a_i\mathbf{i}_a$, the form of "gauge equations" and "tensor equations" is the same. In this section we shall introduce compatibility equations expressed in Cartesian co-ordinates what are commonly used in classical gauge field theory.

In our opinion, the curvilinear co-ordinate system needs either to introduce a co-ordinate-dependent Lie algebra $\mathcal{T}_a(\theta_i)$ or to use an additional field describing geometry of undeformed configuration that in field theory is known as the "background field". By using this second way we shell introduce, in the chapter 9, compatibility conditions important for arbitrary curvilinear co-ordinates. The such generally written down compatibility conditions are especially important in the case of low-dimensional continua that, in the most cases, must be parameterized by curvilinear co-ordinates.

Let notice that $\boldsymbol{\Omega}_\alpha$ plays here the role of a pure gauge potential which is known in gauge continuum with the local non-Abelian group SO(3) [19]. Since, the four-vector $\boldsymbol{\Omega}_\alpha$ with three values in Lie algebra so(3) is expressed, from one side, via the parameters of SO(3) solo (see eqs. 10), and from the second side via the symmetric strains and rates measures (eqs 16 or 25), the compatibility conditions for rotations can also be simultaneously used to be the conditions for strains and rates. The fact that Ω_α is represented by eq. (10) as well as by eq. (16) is the fundamental in a formulation of the non-classical compatibility conditions.

From the gauge field theory point of view, the compatibility conditions for Ω_α are nothing else but the conditions for vanishing the field strength tensor $\mathcal{F}_{\alpha\beta}$ [37]. In general, for an arbitrary gauge group the problem is formulated inversely - for a physically available strength field so that it fulfills $\mathcal{F}_{\alpha\beta} = 0$, in some domain, we are looking for a such form of the vector of potential which directly leads to the vanishing of $\mathcal{F}_{\alpha\beta}$. For the groups such as SU(2), SU(3), U(1) and SO(3) the solutions have a form similar to $\Omega_\alpha \equiv (\partial_\alpha \mathcal{R})\mathcal{R}^{-1}$ which are frequently called "the pure gauge potentials" [19].

Starting from the definition $\mathcal{F}_{\alpha,\beta}$ [19]

$$\mathcal{F}_{\alpha\beta} = \partial_\beta \Omega_\alpha - \partial_\alpha \Omega_\beta + [\Omega_\alpha, \Omega_\beta] \tag{36}$$

after substituting the formula on the pure gauge potential (32) to (36) and taking into account (34) we obtain

$$\mathcal{F}_{\alpha\beta} = \mathcal{R}^{-1}(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)\mathcal{R} \equiv 0 \tag{37}$$

for a nowhere non-singular $\mathcal{R}(\theta^i, t)$. It means that the conditions $\mathcal{F}_{\alpha\beta} = 0$ are equivalent to the integrability conditions $\partial_\alpha\partial_\beta\mathcal{R} = \partial_\beta\partial_\alpha\mathcal{R}$ put on a rotation matrix. Since three-parameter group of local rotation SO(3) is homomorphic to the three-parameter unimodular unitary group SU(2) it is possible to write the tensor $\mathcal{F}_{\alpha\beta}$ and vector Ω_α exactly in the Yang-Mills form [19]:

$$\Omega_\alpha = \omega_\alpha^a \mathcal{T}_a \equiv \boldsymbol{\omega}_\alpha \mathcal{T}, \quad \mathcal{F}_{\alpha\beta} = F_{\alpha\beta}^a \mathcal{F}_a \equiv \mathbf{f}_{\alpha\beta}\mathcal{T} \tag{38}$$

where now the objects with the indexes (a) spain a three-dimensional iso-space with the orthogonal base z_1, z_2, z_3, defining the following iso-space vector multiplication $\boldsymbol{\omega}_\alpha \times \boldsymbol{\omega}_\beta = \omega_\alpha^a \omega_\beta^b \varepsilon_{abc} z_c$. Additionally, we can identify base z_a with the introduced above base $\mathbf{i}_a$. Thus it follows the useful form of iso-matrices ($\mathcal{T}$), iso-vectors ($\boldsymbol{\omega}_\alpha$) and iso-tensors ($\mathbf{f}_{\alpha\beta}$)

$$\mathcal{T} = \mathcal{T}_1 z_1 + \mathcal{T}_2 z_2 + \mathcal{T}_3 z_3, \qquad \boldsymbol{\omega}_\alpha = \omega_\alpha^1 z_1 + \omega_\alpha^2 z_2 + \omega_\alpha^3 z_3$$
$$\mathbf{f}_{\alpha\beta} = \partial_\beta \boldsymbol{\omega}_\alpha - \partial_\alpha \boldsymbol{\omega}_\beta + \boldsymbol{\omega}_\alpha \times \boldsymbol{\omega}_\beta$$

Similar to the non-Abelian electrodynamics [19], the skew-symmetric tensor $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$ possesses only six independent components each with three values in Lie algebra so(3). It is useful to split $\mathbf{f}_{\alpha\beta}$ into the space-time, "electric" vector $\mathbf{E}_i = \mathbf{f}_{ti}$ and the purely spatial part, "magnetic" vector $\mathbf{B}_i = \frac{1}{2} \in_{ijk} \mathbf{f}^{jk}$

(39) $$\begin{aligned} \mathbf{E}_i &= \dot{\omega}_i - \partial_i \omega - \omega \times \omega_i = 0 \\ \mathbf{B}^i &= \frac{1}{2} \in^{ijk} (\partial_j \omega_k - \partial_k \omega_j + \omega_k \times \omega_j) = 0 \,. \end{aligned}$$

A yet more compact form we obtain when $\mathbf{E}_i$ and $\mathbf{B}_i$ are expressed through "double" vectors in the space and the iso-space;

$$\underline{\mathbf{E}} \equiv (E_i^a) z_a \bar{\mathbf{g}}^i, \quad \underline{\mathbf{B}} \equiv (B_i^a) z_a \bar{\mathbf{g}}^i$$

Then the compatibility conditions (39) with $\underline{\omega} \equiv \omega_i \bar{\mathbf{g}}^i$, $\operatorname{grad}(\cdot) \equiv \overline{\nabla} \otimes (\cdot)$, and $\operatorname{curl}(\cdot) \equiv \overline{\nabla} \times (\cdot)$ read

(40) $$\begin{aligned} \underline{\mathbf{E}} &= \dot{\underline{\omega}} - \operatorname{grad} \underline{\omega} - \underline{\omega} \times \underline{\omega} \\ \underline{\mathbf{B}} &= \operatorname{curl} \underline{\omega} + \frac{1}{2} (\underline{\omega} \times) \times \underline{\omega} \end{aligned}$$

Together with (16) or (25) the above equations are the space-time $\underline{\mathbf{E}} = 0$ (9 equations) and space $\underline{\mathbf{B}} = 0$ (6 equations, from $\underline{\mathbf{B}} = \underline{\mathbf{B}}^T$) conditions for compatibility of strains and velocities.

7 – Discussion

The equations similar to $(39)_2$ were introduced first by SIGNORINI [18, eq. 75] exactly in the following notation

(41) $$\frac{\partial \mathcal{D}}{\partial y_{r+1}} - \frac{\partial \mathcal{D}_{r+1}}{\partial y_r} = \mathcal{D}_r \wedge \mathcal{D}_{r+1} \,, \quad r = 1, 2, 3$$

Replacing his iso-spin multiplication symbol $\wedge$ by our $\times$ and taking into account properties of the Ricci tensor $\epsilon = -\mathbf{1} \times \mathbf{1} = \in^{ijk} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k$ we rewrite (41) in the more familiar form [23]

(42) $$\frac{1}{2} \in^{ijk} \left(\partial_j \mathcal{D}_k - \partial_k \mathcal{D}_j - \mathcal{D}_k \times \mathcal{D}_j \right) = 0$$

Because the vectors $\mathcal{D}_i$ in (42) are referred to curvilinear base $\mathbf{g}^i$ in opposition to ω_i from (32) referred to the constant matrices base $\mathcal{T}_a$ the Signorini's equations (42) are more general then the "Cartesian" $(39)_2$ ones.

A basic difference between our $(39)_2$ and Signorini's (42) follows from use different representation of polar decomposition: the right $\mathbf{F} = \mathbf{RU}$ in SIGNORINI's paper (see [15] eq. 78, $\frac{dP}{dP_*} = \alpha_\rho \alpha_\delta$) and left $\mathbf{F} = \mathbf{VR}$ in the present one. This different starting point implies also another definition of the material spin: $\mathcal{D}_i \times \mathbf{1} = \mathbf{R}^{-1}\partial_i\mathbf{R}$ in Signorini paper's (exactly $\mathcal{D}_r = \vee[\alpha_\rho^{-1}\frac{\partial \alpha_\rho}{\partial y_r}]$, [15] eq. 93) and $\omega_i \times \mathbf{1} = (\partial_i\mathbf{R})\mathbf{R}^{-1}$ in the present one. If in analogy to $\mathbf{\Omega}_i$ we define $\mathbf{\Delta}_i = \mathbf{R}^{-1}\partial_i\mathbf{R}$, it is easy to see form comparison of

$$\partial_i\mathbf{R} = \mathbf{\Omega}_i\mathbf{R} = \mathbf{R}\mathbf{\Delta}_i \tag{43}$$

that $\mathbf{\Omega}_i$ and $\mathbf{\Delta}_i$ plays the role of the left and right tensors of change of curvature, respectively. They are connected between themselves by a relation $\mathbf{\Omega}_i = \mathbf{R}\mathbf{\Delta}_i\mathbf{R}^T$ for tensors and by $\omega_i = \mathbf{R}\mathcal{D}_i$ for their axial vectors.

In order to clarify the differences between (42) and $(39)_2$ let us start, similar to (37), from the integrability conditions $\partial_{[i}\partial_{j]}\mathbf{R} = 0$ where by using of differential formulae (43) we obtain a left and right form of compatibility to be expressed primary by $\mathbf{\Omega}_i$ and $\mathbf{\Delta}_i$

$$\begin{aligned} &\partial_i\mathbf{\Omega}_j - \partial_j\mathbf{\Omega}_i + \mathbf{\Omega}_j\mathbf{\Omega}_i - \mathbf{\Omega}_i\mathbf{\Omega}_j = 0 \\ &\partial_i\mathbf{\Delta}_j - \partial_j\mathbf{\Delta}_i - \mathbf{\Delta}_j\mathbf{\Delta}_i + \mathbf{\Delta}_i\mathbf{\Delta}_j = 0 \end{aligned} \tag{44}$$

Next, by using the additional relation, appropriate for skew-symmetric tensors

$$(\omega_j \times \mathbf{1})(\omega_i \times \mathbf{1}) - (\omega_i \times \mathbf{1})(\omega_j \times \mathbf{1}) = (\omega_j \times \omega_i) \times \mathbf{1}$$

we easy obtain from $(44)_1$ our equation $(39)_2$ (their an exact tensor form) and from $(44)_2$ the Signorini-Ferrarese equations (42), respectively.

In 1974 MADAME SHAMINA [28] obtained, starting from integrability conditions $\partial_{[i}\partial_{j]}\mathbf{\Phi} = 0$ imposed on the finite rotation vector $\mathbf{\Phi} = \sin\varphi\mathbf{e}$, the compatibility conditions exactly in the right form $(44)_2$. The curvature vectors $\mathcal{D}_i$ (κ_i in [28]) where explicitly calculated with use of an additionally introduced stretched convective base. SHAMINA's approach enables us to determine the vectors $\mathcal{D}_i$ or ω_i to be the curvature measures being a difference between a deformed (say $\bar{\omega}_i$) and undeformed $\overset{\circ}{\omega}_i$ Darboux's vectors. Of course, a simple subtraction $\bar{\omega}_i - \overset{\circ}{\omega}_i$ like a simple one $(\bar{\mathbf{1}} - \mathbf{1})$ in the process of determination of strain, has no sense. It was indicated by Madame Shamina that correct definition of curvature measures must be

$$\mathcal{D}_i = \mathbf{R}^{-1}\bar{\omega}_i - \overset{\circ}{\omega}_i \tag{45}$$

for the right vector of change of curvature and analogically, what we can define

$$\omega_i = \bar{\omega}_i - \mathbf{R}\overset{\circ}{\omega}_i \tag{46}$$

for the left vector. These correct definitions are especially important when geometry of undeformed continuum is Riemannian and the influence of $\overset{\circ}{\omega}_i$ vectors are important. Additionally, what is important for correct justification of non classical compatibility conditions (42), Shamina have shown ([28], eq. 3.22) that fulfilment of (42) leads in consequence to the simultaneous vanishing of all components of the Riemann-Christoffel curvature tensor.

Notice also that eqs $(39)_2$ have the form of compatibility of Cosserat's strain measures [4, §36, eq A]. From the pure manifold geometry point of view it is nothing surprising. There are of the second part of the so-called Maurer-Cartan equations of structure. Since the first part of the Maurer-Cartan equations in the Cauchy continuum is fulfilled identically and, from the other hand, the Riemann-Christoffel tensor $\overline{R}_{ijkl}$ is connected with $\mathcal{F}_{kl}$ via $\overline{R}_{ijkl} = U_i^a U_j^b \mathcal{F}_{abkl}$, $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$, then the equations $\underline{\mathbf{B}} = 0$ are equivalent to the classical $\overline{R}_{ijkl} = 0$ ones.

Another difficulties are due to presence in our equations (39) new space-temporal conditions of compatibility. As we have mentioned, not every given (for instance, from the appropriate measurements) fields of strains and velocities can be expressed as the first derivatives of a vector. Even if a field of strain satisfy the space compatibility conditions, a velocity field can not be in the good agreement with a motion created by the strain rates. The best example of so given incompatible velocity is the field of velocity of a linear defect moving in a deformable body and measured with respect of a fixed reference frame. Thus the conditions $(39)_1$ say that if the field of velocity gradient and the rates of strain are mutually compatible then motion of continuum is primary described by one vector function.

From the other hand, remembering about analogy with the Cosserat continuum [26, 27, 30] one can expect of the equations of type $(39)_1$ in the Cosserat field theory too. Indeed, such a type equations was introduced first by FERRARESE [30,§2].

8 – Application

An application of our space-temporal equations $(39)_1$, but in the fully linear version, one can find in the theory of continuously distributed dislocations and disclinations. This theory, described in exhaustive paper by KOSSECKA and DEWIT [31], is completely based on a displacement vector as primary unknowns. Defects take part as the additional, generally not compatible fields such as the defect density α_{kl}, θ_{kl} and the defect current J_{kl}, S_{kl} which reflect the plastic property of crystalline medium and constitute the dense distribution of moving line defects. Let us first note that linear expression on the rotation vector is now $\varphi_i = \frac{1}{2}\varepsilon_{ijk}\partial_k u_j$ and next that linear expressions on the curvature vectors $\boldsymbol{\omega}_i = \omega_{im}\mathbf{i}_m$, $\boldsymbol{\omega} = \omega_m \mathbf{i}_m$ (25) are respectively [31]

$$\omega_{im} = \frac{1}{2}\varepsilon_{mkj}\partial_i\partial_k u_j, \quad \omega_m = \frac{1}{2}\varepsilon_{mjk}\partial_k \dot{u}_j \tag{47}$$

then after denoting additionally $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, $v_i = \dot{u}_i$ we obtain

$$\begin{array}{lll} \text{from} & \partial_{[i}\partial_{j]}\bar{\mathbf{p}} = 0 \Longrightarrow & \varepsilon_{imk}(\partial_m e_{kj} + \varepsilon_{kjn}\omega_{mn}) = 0 \\ \text{from} & (31)_2 \Longrightarrow & \varepsilon_{imk}\partial_m\omega_{kj} = 0 \\ \text{from} & \partial_{[t}\partial_{j]}\bar{\mathbf{p}} = 0 \Longrightarrow & \partial_j v_i - \dot{e}_{ij} - \varepsilon_{ijk}\omega_k = 0 \\ \text{from} & (31)_1 \Longrightarrow & \partial_k\omega_j - \dot{\omega}_{kj} = 0 \end{array} \tag{48}$$

These are the compatibility conditions exactly the same as has been introduced by KOSSECKA and DEWIT [31, §4.1]. The above form of the compatibility equations seems to be helpful in the formulation of more complex models of plastic flow. Especially when the so-called plastic spin and plastic bend-twist have to essential contribution to a evolution of plastic state.

9 – The gauge covariant form

Let us return to the problem of writing down our "gauge like" equation (39) not in the Cartesian but in the arbitrary curvilinear co-ordinates. We shall use the method introduced by the author [35, 37] in which all geometrical quantities associated with curvilinear co-ordinates are reinterpreted within the framework of Lie algebra as the "background compensating fields". For instance, for the orthogonal curvilinear co-ordinates we introduce anholonomic base $\mathbf{i}_a$ such that $\mathbf{i}_a = \delta^i_a |\mathbf{g}_i|^{-1}\mathbf{g}_i$. Then, for an ordered triad $\mathring{\mathbb{R}} = \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ with a dimension compatible with the algebra of so(3) (33), the derivative formula for the base is represented by

$$\partial_i \mathring{\mathbb{R}} = \mathring{\Omega}_i \mathring{\mathbb{R}} \quad \text{instead of} \quad \partial_i \mathbf{g}_j = \Gamma^k_{ij} \mathbf{g}_k \tag{49}$$

The components of the objects $\mathring{\Omega}_i = \mathring{\Omega}^{\mathring{a}}_i \mathcal{T}_a = -\mathring{\Omega}^T_i$, due to name suggested by TRUESDELL [21, §61], are called the wryness coefficients and in an arbitrary curvilinear co-ordinates are represented by the generalised Frenet coefficients. Next, from (49) follows the covariant derivative with respect to the undeformed triad

$$\mathring{\nabla}_i \mathring{\mathbb{R}} = 0 \quad \text{with} \quad \mathring{\nabla}_1 = \partial_i - \mathring{\Omega}_i \tag{50}$$

An analogical form, for the rotated triad $\mathbb{R} = \mathcal{R}\mathring{\mathbb{R}}$ can be obtain as follows

$$\mathring{\nabla}_i \mathbb{R} = \left[(\partial_i \mathcal{R})\mathcal{R} + \mathcal{R}\mathring{\Omega}_i \mathcal{R} - \mathring{\Omega}_i \right] \mathbb{R} \equiv \Omega_i \mathbb{R} \tag{51}$$

what means, that a covariant derivative with respect to the rotated triad is defined to be

$$\nabla_i \mathbb{R} = 0, \quad \text{with} \quad \nabla_i = \mathring{\nabla}_i - \Omega_i . \tag{52}$$

Now, instead of (36), let us define $\mathcal{F}_{\alpha\beta}$ to be a result of the following operation

$$\mathcal{F}_{\alpha\beta} = \left[\nabla_\alpha, \nabla_\beta\right] = \nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha \tag{53}$$

and taking into account the form of ∇_i (52) and $\nabla_t \equiv \partial_t$ we obtain

$$\mathcal{F}_{\alpha\beta} = \mathring{\mathcal{F}}_{\alpha\beta} + \left(\mathring{\nabla}_\beta \omega_\alpha - \mathring{\nabla}_\alpha \omega_\beta + [\omega_\alpha, \omega_\beta]\right) \tag{54}$$

If the curvilinear co-ordinates are introduced in Euclidean space then always the "background strength tensor"

$$\mathring{\mathcal{F}}_{\alpha\beta} = \partial_\beta \mathring{\omega}_\alpha - \partial_\alpha \mathring{\omega}_\beta + \left[\mathring{\omega}_\alpha, \mathring{\omega}_\beta\right] \equiv 0 \tag{55}$$

identically vanishing everywhere. Dividing (54) on the same "magnetic" $\mathbf{B}^i$ and "electric" $\mathbf{E}_i$ parts we read

$$\begin{aligned} \mathbf{E}_i &= \dot{\omega}_i - \mathring{\nabla}_i \omega - \omega \times \omega_i = 0 \\ \mathbf{B}^i &= \frac{1}{2}\varepsilon^{ijk}\left(\mathring{\nabla}_j \omega_k - \mathring{\nabla}_k \omega_j + \omega_k \times \omega_j\right) = 0 \end{aligned} \tag{56}$$

with $\mathring{\nabla}_j \equiv \mathring{\nabla}_j^a z_a$.

10 – Extension of the Cesaro-Pietraszkiewicz formula

Let all geometric quantities for the reference body configuration be known. Also assume to have, from the appropriate measurements, strains and velocities functions. In the reference configuration choose arbitrarily a comparison point $p_0 = p_0(\theta_0^1, \theta_0^2, \theta_0^3)$ where naught values of co-ordinates are taken for convenience. An arbitrary point $p(\theta^1, \theta^2, \theta^3)$ can be analyzed from p_0 along a curve C described by the set of equations $\theta^i = \theta^i(\sigma)$, where σ is a parameter along C with $\sigma = 0$ at p_0 and $\sigma = \bar{\sigma}$ at p. During a motion of body the point p_0 move into $\bar{p}_0, p$ into $\bar{p}$ etc. Our problem is to find position of point $\bar{p}$ calculated with accuracy of a rigid-body finite translation $\mathbf{u}_0 = \bar{\mathbf{p}}_0 - \mathbf{p}_0$ and a rigid-body finite rotation $\mathbf{R}_0$ of the point $\bar{p}_0$.

The naught values of $\mathbf{u}_0$ and $\mathbf{R}_0$, in general time-dependent, play the role of the described boundary conditions or in the case of free motion they can be identified to be motion of the gravity center of a body. Having continuum theory in mind we consider the cases when a body occupied a regular, simple connected region of three-dimensional Euclidean slice space for every time in the period under consideration.

The rotation $\mathbf{R}$ can now be determined as a solution of linear first-order differential equation (6) written down along the curve C connecting p_0 and $\bar{p}$ in a form

$$\partial_i \mathbf{R} = \mathbf{\Omega}_i \mathbf{R} \tag{57}$$

or in the term of the rotation vector $\boldsymbol{\varphi} = \varphi \mathbf{e}$

$$\partial_i \boldsymbol{\varphi} = \left(2 \operatorname{tg}\frac{\omega}{2}\right)^{-1} \left(\varphi \boldsymbol{\omega}_i - \partial_i \varphi (\boldsymbol{\varphi} - \mathbf{e})\right) + \frac{1}{2} \boldsymbol{\varphi} \times \boldsymbol{\omega}_i \tag{58}$$

Since $\mathbf{\Omega}_j(\theta^i, t)$ is known, the finite rotation of the point p at time t may be expressed as a path-ordered exponential [33]

$$\mathbf{R}(\mathcal{P}) = \mathcal{P} \exp \int_0^{\bar{\sigma}} \left[\mathbf{\Omega}_i(\sigma) d\theta^i(\sigma)\right] d\sigma \tag{59}$$

where the $\mathcal{P}$ indicates that in expanding the exponential integral, all matrices are to be ordered

$$\mathcal{P} \exp \int_0^{\tau} \mathbf{\Omega}(\tau) d\tau = \mathbf{1} + \int_0^{\tau} \mathbf{\Omega}(\tau') d\tau' + \int_0^{\tau} \mathbf{\Omega}(\tau') \left(\int_0^{\tau'} \mathbf{\Omega}(\tau'') d\tau'' \right) d\tau' + \ldots \tag{60}$$

For instance let us choose a specific integration path $p_0 p' p'' p$. The path consists of three subsequent parts of curvilinear co-ordinate lines: along $p_0 p'$ there is $\theta^2 = 0$, $\theta^3 = 0$, along $p' p''$ there is $\theta^1 = \text{const}$, $\theta^3 = 0$, along $p'' p$ there is $\theta^1 = \text{const}$, $\theta^2 = \text{const}$.

Solving (59) along this integration path and taking into account that the general solution of (57) is given in a form $\mathbf{R}(\mathcal{P})\mathbf{R}_0$, where $\mathbf{R}_0$ is an arbitrary constant in space, proper orthogonal tensor, we obtain

$$\mathbf{R}(\bar{\sigma}, t) = \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1 \mathbf{R}_0 \tag{61}$$

with the following subsequent integrals

$$\begin{aligned}
\mathbf{R}_1 &= \mathbf{1} + \int_0^{\theta^1} \boldsymbol{\Omega}_1(\xi, 0, 0, t) d\xi + \\
&+ \int_0^{\theta^1} \boldsymbol{\Omega}_1(\xi.0.0, t) \Big(\int_0^{\xi} \boldsymbol{\Omega}_1(\xi', 0, 0, t) d\xi' \Big) d\xi + \dots \\
\mathbf{R}_2 &= \mathbf{1} + \int_0^{\theta^2} \boldsymbol{\Omega}_2(\theta, \eta, 0, t) d\xi + \\
&+ \int_0^{\theta^2} \boldsymbol{\Omega}_2(\theta^1.\eta.0, t) \Big(\int_0^{\eta} \boldsymbol{\Omega}_2(\theta^1, \eta', 0, t) d\eta' \Big) d\eta + \dots \\
\mathbf{R}_3 &= \mathbf{1} + \int_0^{\theta^3} \boldsymbol{\Omega}_3(\theta^1, \theta^2, \zeta, t) d\xi + \\
&+ \int_0^{\theta^3} \boldsymbol{\Omega}_3(\theta^1.\theta^2, \zeta', t) \Big(\int_0^{\zeta} \boldsymbol{\Omega}_3(\theta^1, \theta^2, \zeta', t) d\zeta' \Big) d\zeta + \dots
\end{aligned} \tag{62}$$

From the other hand, for the spin field $\boldsymbol{\Omega}(\theta^i, t)$ which remain compatible with spatial $\boldsymbol{\Omega}_j$ and satisfies $(39)_1$ the solution of

$$\mathbf{R}(\bar{\sigma}, t) = \boldsymbol{\Omega}(\bar{\sigma}, t)\mathbf{R}(\bar{\sigma}, 0) \tag{63}$$

takes the form of the path-ordered exponential

$$\mathbf{R}_t = \mathbf{1} + \int_0^t \boldsymbol{\Omega}(\theta^i, \tau) + \int_0^t \boldsymbol{\Omega}(\theta^i, \tau) \Big(\int_0^{\tau} \boldsymbol{\Omega}(\theta^i, \tau') d\tau' \Big) d\tau + \dots \tag{64}$$

determined with an arbitrary constant in time orthogonal tensor $\mathbf{R}_{\bar{\sigma}}$

$$\mathbf{R}(\bar{\sigma}, t) = \mathbf{R}_t \mathbf{R}_{\bar{\sigma}} \tag{65}$$

with the additional restriction that the boundary rotation $\mathbf{R}_0(t) = \mathbf{R}(\sigma = 0, t)$ and the initial rotation $\mathbf{R}_{\bar{\sigma}} = \mathbf{R}(\sigma = \sigma, t = 0)$.

It is important to note that the above formulae, in the case of a rigid body motion reduce to the well-known, from analytical mechanic [15], relation on a rotation expressed by a given spin tensor $\boldsymbol{\Omega}$. Also, due to the rigid body analogy, one can use the specific algorithms which have been worked out for the time-dependent equations $\dot{\mathbf{R}} = \boldsymbol{\Omega}(t)\mathbf{R}$. For the general form of $\boldsymbol{\Omega}(t)$ an effective analytical method of constructing the solution was proposed by VOLTERRA [17]. The Volterra's solution, at present known as "product integral", has been study also in the exhaustive papers by BIRKHOFF [16] and RASCH [14].

The second step leads to an expression of the actual position vector of the point under consideration in terms of the already known rotation and stretch tensors. On the position vector $\bar{\mathbf{p}}(\bar{\sigma}, t)$, referring a placement of particle p at time t, we put the following differential equation

$$\partial_i \bar{\mathbf{p}} = \mathbf{V}(\bar{\sigma}, t)\mathbf{R}(\bar{\sigma}, t)\mathbf{g}_i(\bar{\sigma}, 0) \tag{66}$$

being similar to (57). Due to the already fulfilled condition $\partial_{[i}\partial_{j]}\bar{\mathbf{p}} = 0$, we can integrate (66) along our integration path $p_0 p' p'' p$ which reads

$$\begin{aligned}\bar{\mathbf{p}} =& \bar{\mathbf{p}}_0 + \int_0^{\bar{\sigma}} \mathbf{V}(\sigma)\mathbf{R}(\sigma) d\mathbf{p}(\sigma) = \\ =& \bar{\mathbf{p}}_0 + \int_{\theta_0}^{\theta^1} \bar{\mathbf{g}}_1(\xi, 0, 0) d\xi + \int_{\theta_0}^{\theta^2} \bar{\mathbf{g}}_2(\theta^1, \eta, 0) d\eta + \int_{\theta_0}^{\theta^3} \bar{\mathbf{g}}_3(\theta^1, \theta^2, \zeta) d\zeta\end{aligned} \tag{67}$$

with $\bar{\mathbf{p}}_0$ as the constant position vector at p_0. From the above formula follows that only subsequent parts of (62) are requiring to calculation of (67), namely

$$\begin{aligned}\bar{\mathbf{p}} = \bar{\mathbf{p}}_0 &+ \left[\int_{\theta_0}^{\theta^1} \mathbf{V}(\xi, 0, 0, t)\mathbf{R}(\xi, 0, 0, t)\mathbf{g}_1(\xi, 0, 0) d\xi\right]\mathbf{R}_0 + \\ &+ \left[\int_{\theta_0}^{\theta^2} \mathbf{V}(\theta^1, \eta, 0)\mathbf{R}(\theta^1, \eta, 0)\mathbf{g}_2(\theta^1, \eta, 0) d\eta\right]\mathbf{R}_1 + \mathbf{R}_0 + \\ &+ \left[\int_{\theta_0}^{\theta^3} \mathbf{V}(\theta^1, \theta^2, \zeta)\mathbf{R}(\theta^1, \theta^2, \zeta)\mathbf{g}_3(\theta^1, \theta^2, \zeta) d\zeta\right]\mathbf{R}_2\mathbf{R}_1 + \mathbf{R}_0\end{aligned} \tag{68}$$

Notice that in the case of an infinitesimal static deformation the formulae (68) reduced to the already known ones introduced by CESÀRO [10] in 1906. The field $\bar{\mathrm{p}}$, likes $\mathbf{R}$, is dependent of the field ν, therefore, the compatibility conditions (20) unable us to looking for a position vector as a solution of $\partial_t \mathrm{p} = \nu \mathbf{R}$ in the form

$$\bar{\mathrm{p}} = \bar{\mathrm{p}}_{\bar{\sigma}} + \int_0^t \nu(\sigma.\tau)\mathbf{R}(\sigma,\tau)d\tau \tag{69}$$

where $\bar{\mathrm{p}}_{\bar{\sigma}}$ plays the role of naught initial position vector of the particle p. Here formulae (64) and (69) define the space-time analogue of Cesaro ansatz. Same examples of using these formulae one can finds in the paper by KRYLOV [4].

11 – The nonintegrable phase factors in a multiply-connected domain

Now, we are able to adopt Dirac's concept of the nonintegrable phase factors [13] to a reconsideration, within the non-linear context, of Volterra's model of linear defects [9].

Now, to simplify matters radically in the following developments, we shall restrict our attention to the static fields and, correspondingly, we assume that the given fields is the function of three co-ordinates only.

The Volterra theorem assures, in the simply connected body, a unit solution of (57) with an accuracy of the arbitrary value of $\mathbf{R}_0$. It means that the path-ordered exponential (59) is independent from a choose of a path of integration. It can be checked by choose of a loop C with the begin and end in the same point p_0. Then from $\mathbf{R}_0 = \mathbf{R}(\mathcal{P})\mathbf{R}_0$ follows that

$$\mathcal{P} \exp \oint \boldsymbol{\Omega}_i d\theta^i = \mathbf{1} \tag{70}$$

The above can be easily proved because of every loops in the simple connected domain are shrinkable to a point.

In general the path integral taken on an unshrinkable loop (C_1 or C_2

fig. 1) indicates that the so-called nonintegrable phase factor [29]

$$\tilde{\Phi} = \mathcal{P} \exp \oint \Omega_i d\theta^i$$

describes an internal sources of fields as the topologically charged, extended objects of $0-, 1-, 2-, \ldots, n$-dimension.

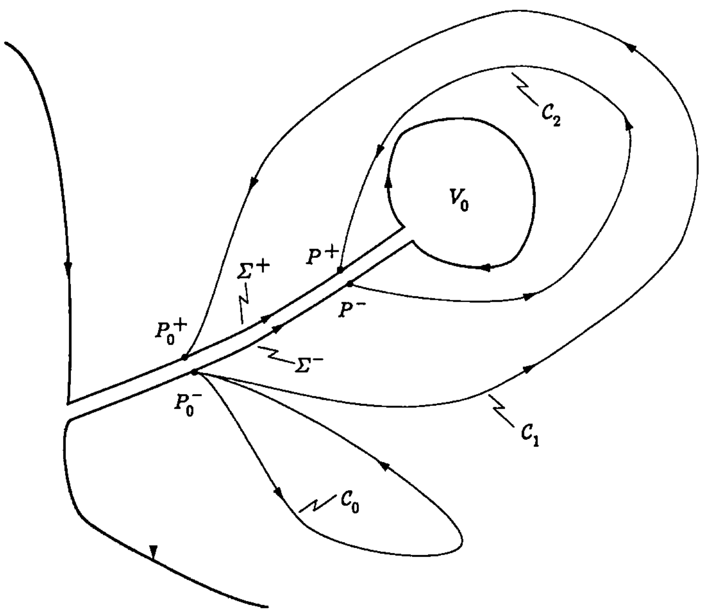

Fig. 1 Doubly-connected region with unshrinkable loops C_1, C_2

Notice also that the path integral taken on any path from p_0 to p (59) constitutes a fundamental concept such as the gauge invariant fields. A formulation of the abelian theory in terms of gauge invariant fields has been given by Mandelstam and generalized for non-Abelian case by BIALYNICKI-BIRULA (see [29] and the works cited there).

For the multiply-connected body the solution (59) became a path-dependent even if $\mathcal{F}_{\alpha\beta} = 0$ everywhere in the region of body (fig.1). It follows from a topological property of the region: from two loops C_0 and

C_1 only C_0 is shrinkable to a point. However, the loop integral taken along the unshrinkable loop C_1 or C_2 represents a nonintegrable phase factor.

Let us consider a multiply-connected region which can be reduced to a simply connected one by introducing a set of non-intersecting "cuts" or "barriers". For instance the doubly connected region, on fig. 1, is cutting along the barrier Σ and disconnecting the region by means of an additional positive Σ^+ and negative Σ^- side.

In the simply connected region we can employ the formulae provided in the preceding chapter to obtain single-valued fields $\mathbf{R}(\theta^i)$, $\bar{\mathbf{p}}(\theta^i)$. Indeed, we can integrate (57) along the path p_0p twice; one time on Σ^+ and second one on Σ^-

$$\begin{aligned} \mathbf{R}^-(p) &= \left[\mathcal{P}\exp\int_{p_0}^{p}\Omega_i d\theta^i\right]\mathbf{R}^-(p_0) \\ \mathbf{R}^+(p) &= \left[\mathcal{P}\exp\int_{p_0}^{p}\Omega_i d\theta^i\right]\mathbf{R}^+(p_0) \end{aligned} \tag{71}$$

Since Ω_i remains continuous across the barrier Σ the values of both path-integral are equal, and for an arbitrary chosen points on Σ we obtain

$$\Phi = \left[\mathbf{R}^-(p_0)\right]^{-1}\mathbf{R}^-(p_0) = \left[\mathbf{R}^+(p)\right]^{-1}\mathbf{R}^+(p) \tag{72}$$

the nonintegrable phase factor Φ in the form of a constant orthogonal tensor defined now to be

$$\begin{aligned} \Phi &= \mathbf{R}^{-1}(p_0)\mathcal{P}\exp\oint_{C_1}\omega_i(\sigma)d\theta^i(\sigma)d\sigma\mathbf{R}(p_0) = \\ &= \mathbf{R}^{-1}\mathcal{P}\exp\oint_{C_2}\omega_i(\sigma)d\theta^i(\sigma)d\sigma\mathbf{R}(p) \end{aligned} \tag{73}$$

a measure of a jump of the rotation across the barrier Σ with the value independent of choose of the starting point of path-integration.

From above follows that the values of path-integral calculated for the different unshrinkable loops are, in general, the self-similar tensors [15].

If the nonintegrable phase factor $\mathbf{\Phi}$ to express by a rotation vector using the SIGNORINI representation [18] for instance

$$\begin{aligned} \mathbf{\Phi} &= (1+\mathbf{d}^2/4)^{-1}\left[(1-\mathbf{d}^2/4)\mathbf{1}+\frac{1}{2}\mathbf{d}\otimes\mathbf{d}-\mathbf{1}\times\mathbf{d}\right] \\ \mathbf{d} &= 2(1+\operatorname{tr}\mathbf{\Phi})^{-1}\boldsymbol{\varepsilon}\cdot\mathbf{\Phi}^{-1}, \qquad \mathbf{d}=2\operatorname{tg}\frac{\alpha}{2}\mathbf{e} \end{aligned} \tag{74}$$

then the vector $\mathbf{d}$ expresses the jump of the rotation vector across the barrier Σ and eq. (73) is the nonlinear extension of Volterra's formula. Indeed, after linearization of (61) together with $\mathbf{R}\simeq\mathbf{1}+\boldsymbol{\varphi}\times\mathbf{1}$, $E_{ij}\simeq e_{ij}$, $V_{ij}\simeq\delta_{ij}+e_{ij}$, $\boldsymbol{\Omega}_j\simeq(e_{ij,k}-e_{kj,i})\mathbf{g}^i\otimes\mathbf{g}^k$, $\omega_j\simeq\varepsilon^{ikl}e_{kj,i}\mathbf{g}_l$ we obtain instead of (73)

$$\mathbf{d}\simeq\varphi(p^+)-\varphi(p^-)=\oint_{C_1}\omega_j(\sigma)d\theta^j(\sigma)d\sigma \tag{75}$$

Volterra's expression on the jump of the rotation vector across any barrier [11].

Similarly, a jump of the displacement vector across the barrier can be calculated using the expression (68) for the simply connected region. Integrating $\bar{\mathbf{p}}_{,i}=\mathbf{V}\mathbf{R}\mathbf{g}_i$ along path p_0p twice; on the positive and the negative sides of Σ we have

$$\bar{\mathbf{p}}^+=\mathbf{\Phi}\bar{\mathbf{p}}^-+\mathbf{b} \tag{76}$$

the formula expressed by a constant vector $\mathbf{b}$ of a nonintegrable translation phase factor. Because the action of $\mathbf{\Phi}$ on $\bar{\mathbf{p}}^-$ can be written down to be

$$\mathbf{\Phi}\bar{\mathbf{p}}^-=\bar{\mathbf{p}}^-+(1+\mathbf{d}^2/4)^{-1}\mathbf{d}\times(\bar{\mathbf{p}}^-+\frac{1}{2}\mathbf{d}\times\bar{\mathbf{p}}^-)$$

then the jump of the displacement ($\mathbf{u}^\pm=\bar{\mathbf{p}}^\pm-\mathbf{p}$) across Σ is expressed as follows

$$\mathbf{u}^+-\mathbf{u}^-=(1+\mathbf{d}^2/4)^{-1}\mathbf{d}\times(\bar{\mathbf{p}}^-+\frac{1}{2}\mathbf{d}\times\bar{\mathbf{p}}^-)+\mathbf{b} \tag{77}$$

which after linearization ($\bar{\mathbf{p}}^\pm\simeq\mathbf{p}$) leads to the well known Volterra solution

$$\mathbf{u}^+-\mathbf{u}^-\simeq\mathbf{b}+\mathbf{d}\times\mathbf{p} \tag{78}$$

depending only of the position vector $\mathbf{p}$ of an arbitrary point of Σ.

Now, the nonintegrable translational phase factor $\mathbf{b}$, likes the rotation one $\mathbf{\Phi}$, is expressed by the path-integral along C_1

$$\mathrm{b} = \oint_{C_1} \mathbf{V}(\sigma)\left[\mathcal{P}\exp\int_0^{\sigma} \mathbf{\Omega}_i(\sigma')d\theta^i(\sigma')d\sigma'\right]\mathbf{R}_0 d\mathbf{p}(\sigma) \tag{79}$$

Note that the above solutions in a natural way can be used to simulating of the translational and rotational defects in crystal. But then the empty region v_0 (fig.1) describes the highly distorted one ($\mathcal{F}_{ij} \neq 0$) and physically represents the dislocation/disclination core.

REFERENCES

[1] R. KIRCHHOFF: *Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes*, J. reine angew. Math. (Crelle), **56** 385-313 (1859).

[2] R. KIRCHHOFF: *Ueber die Bewegung eines Rotainoskörpers in einer Flüssigkeit*, J. reine angew. Math. (Crelle), **71**, 237-262 (1869).

[3] W. HESS: *Ueber die Biegung und Drillung eines unendlich dünnen elastischen Stabes, dessen eines Ende von einem Kräftepaar angegriffen wird*, Math. Ann. **23**, 181-212 (1884).

[4] E. COSSERAT - F. COSSERAT: *Sur la théorie de l'élasticité*, Ann. Toulouse **10**, 1-116 (1896).

[5] A.N. KRYLOV: *A general theory of the oscillations of a ship on waves*, Tran Inst. nav. archit. London, **40**, 135-190 (1898).

[6] C. RIQUIER: *Sur une question fondamentale de calcul integral*, Acta Math. **23**, 203-332 (1900).

[7] O. MANVILLE: *Sur la deformation finite d'um milieu continu*, Mém. de la Soc. des Sci. Phys. et Nat. de Bordeaux, **2**, 83-162 (1904).

[8] R. MARCOLONGO: *Le formule del Saint-Venant per le deformazioni finite*, Rend. Circ. Nat. Palermo **19**, 151-155.

[9] V. VOLTERRA: *Sull'equilibrio dei corpi elastici più volte connessi*, Il nuovo Cimento, Cap. I; Teoremi generali sull'equilibrio, Ser V, **T10**, 361-373 (1905), Cap. II; Le distorsioni, Ser. V, **T10**, 374-385 (1905), Cap. III, Gli Sforzi, Ser. V, **T11**, 5-12 (1906), Cap. IV; Distorsioni e sforzi in un corpo ciclico simmetrico, Ser. V, **t11**, 12-20 (1906), Cap. V; Cilindro cavo di rivoluzione - distorsione d'ordine 6,

Ser. V, t11, 144-161 (1906), Cap. VI; Cilindro cavo di rivoluzione, distorsione d'ordine 2, Ser. V, T11, 205-221 (1906), Cap. VII; Cilindro cavo di rivoluzione - distorsioni d'ordini 1,3,4,5., Ser. V, T11, 338.347 (1906).

[10] E. CESARO: *Sulle formule del Volterra fondamentali nella teoria delle distorsioni elastiche*, Nuovo Cimento (5), **12**, 143-154 (1906).

[11] V. VOLTERRA: *Sur l'équilibre des corps élastiques multiplement connexes,*, Ann. École Norm. Sup. (3) **24**, 401-517 (1907).

[12] V. CRUDELI: *Sopra le deformazioni finite, Le equazioni del De Saint-Venant*, Rend. Acc. Lincei **2**, 306-308 (1911).

[13] P.A.M. DIRAC: *Quantised singularities in the electromagnetic field*, Proc. Roy. Soc. London **A133**, 60-72 (1931).

[14] G. RASCH: *Zur Theorie und Anwendung des Produktintegrals*, J. reine und angew. Math. **171**, 65-119 (1934).

[15] I.A. LAPPO-DANILEVSKI: *Mèmories sur la théorie des systèmes dès équations différentielles linéaires*, vol 1, Tr. Fiz-math. Inst. Steclov'a Otd. Math, Moskva (1934).

[16] G. BIRKHOFF: *On product integration*, J. Math. Phys. **16**, 104-132 (1937).

[17] V. VOLTERRA: *Opérations infinitésimales lineéaries*, Paris (1938).

[18] A. SIGNORINI: *Trasformazioni termoelastiche finite, Memoria 1*, Ann. Mat. Pura Appl. (4), **22**, 33-143 (1943).

[19] C.N. YANG - R.L. MILLS: *Conservation of isotropic spin and isotropic gauge invariance*, Phys. Rev. **96**, 191-195 (1954).

[20] G. FERRARESE: *Sulla velocità angolare nei moti rigidi e la rotazione locale nelle deformazioni finite*, Rend. Mat. V, **18**, 169-177 (1959).

[21] C. TRUESDELL - R.A. TOUPIN: *The Classical Field Theory*, Hand. der Physik III/1, 226-793, (1960).

[22] G. FERRARESE: *Sulle equazioni di moto di un sistema soggetto a un vincolo anolonomo mobile*, Rend. Mat. V, **22**, 351-370 (1963).

[23] G. FERRARESE: *Sulla forma intrinseca delle condizioni di congruenza per deformazioni finite*, Atti. Acad. naz. Lincei, Rend. CL. Fis. Mat. Natur ser **8**, **36**, 629-638 (1964).

[24] G. FERRARESE: *Condizioni di congruenza per deformazioni finite in riferimenti anolonomi*, Rend. Mat. V, **23**, 66-83 (1964).

[25] TH. LEHMANN: *Formäderungen eines klassischen Kontinuums in vierdimensionaler Darstellung*, in: Proc. 19 Int. Congress of Appl. Mech., Ed. H. Görtler, Springer-Verlag, 376-382 (1964).

[26] G. GRIOLI: *Questioni di compatibilità per i continui Cosserat*, Symposia Mathematica I.N.D.A.M. 271-287 (1968).

[27] G. FERRARESE: *Sulla compatibilità dei continui alla Cosserat*, Rend. Mat. VI, **4** 151-174 (1971).

[28] V.A. SHAMINA: *Determinations of the displacement vector from the components of the deformation tensor in nonlinear continuum mechanics*, Mekhanika Tverdogo Tela, Izv. AN SSSR, 9,(1), 14-22 (1974).

[29] T.T. WU, C.N. YANG: *Concept of nonintegrable phase factors and global formulation of gauge fields*, Phys. Rev. **D12**, 3845-3857 (1975).

[30] G. FERRARESE: *Sulla formulazione intrinseca della dinamica dei continui alla Cosserat*, Ann. Mat. Pura Appl. **108**, 109-124 (1976).

[31] E. KOSSECKA - R. DEWIT: *Disclination kinematics*, Arch. Mech. Stos. **29**, 633-651 (1977).

[32] J.K. DENIS: *On the analysis of rotation and stress rate in deforming bodies*, Acta Mech. **32**, 217-232 (1979).

[33] W. PIETRASZKIEWCZ: *Determination of displacements from gives strains in the non-linear continuum mechanics*, ZAMM, **62**, 154-156 (1982).

[34] W. PIETRASZKIEWCZ - J. BADUR: *On non-classical forms of compatibility conditions in continuum mechanics*, in: Trends in Application of Pure Math. to Mech. Vol IV, ed. J. Brilla, Pitman Ad. Pub. Program, 197-227 (1983).

[35] J. BADUR: *A Yang-Mills type equations for the compatibility conditions*, Lett. Appl. Engng Sci. **27**, 1439-1442 (1989).

[36] G. FERRARESE - L. STAZI - C. CATTANI - D. BINI: *Kinematics of relativistic deformations and Thomas precession*, Il Nuovo Cimento 105B, 1131-1140 (1990).

[37] J. BADUR: *Pure gauge theory of the Cosserat surface*, Int. J. Engng Sci. **31** (1993), 41-59.

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