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Notes on α-almost paracompact subsets

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RIASSUNTO: *Si analizzano le relazioni tra la nozione di insieme α^*-paracompatto e quella di insieme α-quasi paracompatto. Si dimostra che le due nozioni sono equivalenti in ogni spazio che possieda una base localmente σ-finita.*

ABSTRACT: *In this paper, we investigate the relationship between α^*-paracompact subsets and α-almost paracompact subsets. The main result is that locally almost paracompactness and almost paracompactness are equivalent in a space with a σ-locally finite base.*

KEY WORDS: *α-almost paracompact set – α^*-paracompact set – Preopen – Semipreopen.*

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1 – Introduction

In [3], Kovačević introduced the notion of α-almost paracompact subsets in a topological space and defined locally almost paracompact spaces by utilizing these subsets. Quite recently, the authors [2] of the present paper have introduced the notion of α^*-paracompact subsets which is strictly weaker than that of α-almost paracompact subsets. In this paper, we investigate the relationship between these two notions. The main result of the present paper is that locally almost paracompactness and almost paracompactness are equivalent in a space with a σ-locally finite base (Theorem 1).

2 – Preliminaries

Throughout the present paper, X means a topological space, $Cl(A)$ and $\text{Int}(A)$ denote the closure and the interior of a subset A of X, respectively. A subset A of X is said to be *nowhere dense* (resp. *rare* or *condense*) if $\text{Int}(Cl(A)) = \phi$ (resp. $\text{Int}(A) = \phi$). A subset A is said to be *semi-preopen* [1] (resp *semi-open* [4], *preopen* [5], *α-open* [6]) if $A \subset Cl(\text{Int}(Cl(A)))$ (resp. $A \subset Cl(\text{Int}(A))$, $A \subset \text{Int}(Cl(A))$, $A \subset \text{Int}(Cl(\text{Int}(A)))$). A subset A is said to be *regular closed* if $A = Cl(\text{Int}(A))$.

REMARK 1. The following are well-known:
(i) both regular closeness and openness imply semi-openness;
(ii) both semi-openness and preopenness imply semi-preopenness; and
(iii) α-openness implies both semi-openness and preopenness.

DEFINITION 1. *A subset A of X is said to be α-almost paracompact* [3] *if every X-open cover of A has an X-locally finite X-open refinement such that X-closures of its elements cover A. If the set X is α-almost paracompact, then the space X is said to be almost paracompact* [7].

DEFINITION 2. *A space X is said to be locally almost paracompact* [3] *if each point of X has an open neighborhood U such that $Cl(U)$ is α-almost paracompact.*

DEFINITION 3. *A subset A of X is said to be α^*-paracompact* [2] *if for each cover $\mathcal{V}$ of A by X-open sets, there exist*
(i) *an X-locally finite family $\mathcal{U}$ of X-open sets which refines $\mathcal{V}$ and*
(ii) *nowhere dense subset N of X*
such that $A \subset \cup\{U | U \in \mathcal{U}\} \cup N$.

REMARK 2. Every α-almost paracompact subset is α^*-paracompact, but not conversely by Proposition 3.1 and Remark 3.2 of [2].

LEMMA 1. *A subset A of X is semi-preopen if and only if $A = Cl(G) - R$, where G is open in X and R is a rare set of X.*

PROOF.

Necessity: Suppose that A is semi-preopen. Put $G = \text{Int}(Cl(A))$ and $R = Cl(\text{Int}(Cl(A))) - A$. Then G is open in X, $\text{Int}(R) = \text{Int}(Cl(A)) - Cl(A) = \phi$, and $A = Cl(G) - R$.

Sufficiency: Let $A = Cl(G) - R$, where G is open and R is rare. Then, we have $G = G \cap Cl(X - R) \subset Cl(G - R)$ and hence $Cl(G) = Cl(G - R) \subset Cl(Cl(G) - R) \subset Cl(G)$. Therefore, we obtain $Cl(A) = Cl(G)$ and hence $A \subset Cl(\text{Int}(Cl(A)))$. This shows that A is semi-preopen.

3 – α^*-paracompactness and α-almost paracompactness

In this section we investigate the relationship between α^*-paracompact subsets and α-almost paracompact subsets.

PROPOSITION 1. *A semi-preopen set A of X is α^*-paracompact if only if it is α-almost paracompact.*

PROOF. Let A be a semi-preopen α^*-paracompact set of X. For any X-open cover $\mathcal{V}$ of A, there exist an X-open X-locally finite refinement $\mathcal{U}$ of $\mathcal{V}$ and a nowhere dense set N such that $A - N \subset \cup\{U | U \in \mathcal{U}\}$. Since $\mathcal{U}$ is locally finite, we have

$$\text{Int}(Cl(A)) \subset \text{Int}\Big[Cl(N) \cup Cl(\bigcup_{U \in \mathcal{U}} U)\Big] \subset \text{Int}(Cl(N)) \cup Cl(\bigcup_{U \in \mathcal{U}} U) = \\ = \bigcup_{U \in \mathcal{U}} Cl(U).$$

Therefore, we obtain $A \subset Cl\big(\text{Int}(Cl(A))\big) \subset \bigcup\{Cl(U) | U \in \mathcal{U}\}$. This shows that A is α-almost paracompact. It follows from [2, Proposition 3.1] that every α-almost paracompact subset is α^*-paracompact.

COROLLARY 1. *The following are equivalent for a space X:*

(a) *X is locally almost paracompact;*

(b) *for each $x \in X$, there exists an open neighborhood U of x such that $Cl(U)$ is α^*-paracompact;*

(c) *for each $x \in X$, there exists a preopen set U containing x such that $Cl(U)$ is α^*-paracompact.*

PROOF. The implications $(a) \implies (b)$ and $(b) \implies (c)$ easily follow from Proposition 1.

$(c) \implies (a)$: For each $x \in X$, there exists a preopen set U containing x such that $Cl(U)$ is α^*-paracompact. Therefore, we have $x \in U \subset \mathrm{Int}\,(Cl(U))$ and $Cl\Big(\mathrm{Int}\,(Cl(U))\Big) = Cl(U)$. Moreover, $Cl(U)$ is regular closed and hence it is α-almost paracompact by Proposition 1.

COROLLARY 2. *The union of an X-locally finite family of semi-preopen α-almost paracompact sets is α-almost paracompact.*

PROOF. It is shown in [2, Proposition 3.2]that the union of X-locally finite family of α^*-paracompact subsets is α^*-paracompact. Any union of semi-preopen sets is semi-preopen [1, Theorem 2.5] and hence Proposition 1 completes the proof.

COROLLARY 3. (Kovačević [3]) *The union of an X-locally finite family of α-almost paracompact is α-almost paracompact.*

PROPOSITION 2. *If A is an α^*-paracompact set of X, then $Cl(\mathrm{Int}\,(A))$ is α-almost paracompact.*

PROOF. Let A be α^*-paracompact. Then, by Proposition 3.3 of [2] $Cl(A)$ is α^*-paracompact and hence $Cl(\mathrm{Int}(A))$ is also α^*-paracompact by Proposition 3.4 of [2]. Since $Cl(\mathrm{Int}(A))$ is semi-preopen, it follows from Proposition 1 that $Cl(\mathrm{Int}(A))$ is α-almost paracompact.

PROPOSITION 3. *A closed set A of X is α^*-paracompact if and only if $A = R \cup N$, where R is regular closed α-almost paracompact and N is closed nowhere dense.*

PROOF. We shall prove only the necessity. Suppose that A is closed and α^*-paracompact. If A is nowhere dense, then there is nothing to prove. If $\mathrm{Int}(A) \neq \phi$, then let $R = Cl(\mathrm{Int}(A))$ and N be the boundary of A. Then, by Proposition 2, R is α-almost paracompact, N is closed nowhere dense, and $A = R \cup N$.

PROPOSITION 4. *Let X be a Hausdorff space. A subset A of X is α^*-paracompact if and only if $A = R \cup N$, where R is regular closed α-almost paracompact and N is nowhere dense.*

PROOF. We shall prove only the necessity. Suppose that A is an α^*-paracompact set in a Hausdorff space X. It follows from Propositions 3.6 and 3.7 of [2] that A is closed α^*-paracompact in the space X_α, where X_α denotes the space on X having the family of all α-open sets as the topology. Then by Proposition 3, $A = Cl_\alpha(G) \cup N$, where $Cl_\alpha(G)$ denote the closure of an α-open set G with respect to X_α, is the union of a regular closed α-almost paracompact set in X_α with a nowhere dense set in X_α. It follows from Lemma 3.2 and Proposition 3.7 of [2] that $Cl_\alpha(G)$ is α^*-paracompact in X and N is nowhere dense in X. Since G is α-open in X, we have $Cl(G) = Cl(\text{Int}(G))$ and by [1, Theorem 1.5] $Cl_\alpha(G) = G \cup Cl(\text{Int}(Cl(G))) = Cl(\text{Int}(G))$. Therefore, $Cl_\alpha(G)$ is regular closed and α-almost paracompact in X by Proposition 1.

PROPOSITION 5. *If A is an α^*-paracompact set of X, then $A = R \cup N$, where R is a relatively α-almost paracompact semi-preopen set and N is a nowhere dense set.*

PROOF. Let A be an α^*-paracompact set. It follows from [2, Proposition 3.3] that $Cl(A)$ is α^*-paracompact. Therefore, by Proposition 3 we have $Cl(A) = Cl(G) \cup N^*$, where $Cl(G)$ is regular closed α-almost paracompact and N^* is closed nowhere dense. Put $R_A = Cl(A) - A$, then R_A is a rare set. We have $A = Cl(A) - (Cl(A) - A) = (Cl(G) \cup N^*) \cap (X - R_A) = (Cl(G) - R_A) \cup (N^* - R_A)$. By Lemma 1, $R = Cl(G) - R_A$ is semi-preopen and $N = N^* - R_A$ is nowhere dense. Moreover, we have $Cl(R) = Cl(Cl(G) - R_A) = Cl(G)$ since $G = G \cap Cl(X - R_A) \subset Cl(G \cap (X - R_A)) \subset Cl(Cl(G) - R_A)$. Therefore, $Cl(R)$ is α-almost paracompact. This completes the proof.

4 – Locally almost paracompact spaces

THEOREM 1. *Locally almost paracompactness and almost paracompactness are equivalent in a space with a σ-locally finite base.*

PROOF. Let X be a locally almost paracompact space with a σ-locally finite base $\bigcup\{\mathcal{B}_n | n \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of positive integers. For each $n \in \mathbb{N}$, let us put

$$\mathcal{B}_n^* = \{B \in \mathcal{B}_n | B \subset U_x \quad \text{for some} \quad x \in X\},$$

where U_x denotes a fixed basic neighborhood of x such that $Cl(U_x)$ is α-almost paracompact. Now, we set

$$G_n = \bigcup_{k=1}^{n} \left[\bigcup_{B \in \mathcal{B}_k^*} B \right] \quad \text{for each} \quad n \in \mathbb{N}.$$

Then G_n is open for each $n \in \mathbb{N}$, $G_n \subset G_{n+1}$, and $X = \bigcup\{G_n | n \in \mathbb{N}\}$. Since $Cl(B)$ is α^*-paracompact for each $B \in \mathcal{B}_k^*$ and $\mathcal{B}_k$ in locally finite, it follows from [2, Proposition 3.2] that $Cl(G_n) = \bigcup\{Cl(B) | B \in \mathcal{B}_k^*, 1 \leq k \leq n\}$ is α^*-paracompact for each $n \in \mathbb{N}$. Since $X - \text{Int}\,(Cl(G_n))$ is regular closed, $Cl(G_{n+1}) - \text{Int}\,(Cl(G_n))$ is α^*-paracompact [2, Proposition 3.4]. Moreover, the family $\{Cl(G_{n+1} - \text{Int}\,(Cl(G_n)) | n \in \mathbb{N}\} \cup \{Cl(G_1)\}$ is locally finite.

Therefore, $X = \bigcup\{Cl(G_{n+1}) - \text{Int}\,(Cl(G_n)) | n \in \mathbb{N}\} \bigcup Cl(G_1)$ is α^*-paracompact and hence α-almost paracompact since X is open. This shows that X is almost paracompact.

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