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On certain inequalities in weighted Orlicz spaces

A. FIORENZA(*)

RIASSUNTO: *Detta* Φ *una N-funzione sia* (B_Φ) *la classe, introdotta da Bagby, dei pesi* $w : \mathbb{R}^n \to]0,\infty[$ *tali che la funzione massimale secondo Hardy-Littlewood verifichi la disuguaglianza debole con peso w, e sia* (A_Φ) *la classe, introdotta da Kerman-Torchinsky, dei pesi w tali che sia verificata la disuguaglianza forte con peso w. Si dimostra che ogni* $w \in (B_\Phi)$ *è una misura "doubling" e che se* $\tilde{\Phi}$ *è sottomoltiplicativa, allora le classi* (A_Φ) *e* (B_Φ) *coincidono.*

ABSTRACT: *For* Φ *an N-function let* (B_Φ) *be the class, introduced by Bagby, of the weights w such that the Hardy-Littlewood maximal function verifies the w-weighted weak inequality, and* (A_Φ) *be the class, introduced by Kerman - Torchinsky, of the weights such that the w-weighted strong inequality is verified. It is shown that any* $w \in (B_\Phi)$ *is doubling and if* $\tilde{\Phi}$ *is submultiplicative, then the classes* $(A_\Phi), (B_\Phi)$ *are the same.*

KEY WORDS: *Integral inequalities - Maximal function - Weighted Orlicz spaces.*

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1 – Notations

Let Φ be an N-function, i.e. a function of the type $\Phi(t) = \int_0^t \varphi(s)ds$ where $\varphi : [0,\infty[\to \mathbf{R}$ is continuous, nondecreasing, and such that

$$\varphi(s) > 0 \quad \forall s > 0, \quad \varphi(0) = 0, \quad \lim_{s\to\infty} \varphi(s) = +\infty.$$

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Let φ^{-1} be the generalized inverse of φ defined by $\varphi^{-1}(t) = \sup\{s : \varphi(s) \leq t\}$, and $\widetilde{\Phi}(t) = \int_0^t \varphi^{-1}(s)ds$. The function Φ is said to verify the Δ_2 condition, and we will write $\Phi \in \Delta_2$, if there exists $c > 0$ such that $\Phi(2t) \leq c\Phi(t) \quad \forall t \geq 0$ (see [7]).

The letters u, w will denote "weight" functions, i.e. measurable positive functions on $\mathbf{R}^n$; we will say that weight functions define "doubling" measures (see [8]) if there exists $c > 0$ such that $w(2Q) \leq cw(Q) \quad \forall Q$ cube in $\mathbf{R}^n$, where $w(Q) = \int_Q wdx$ and $2Q$ denotes the cube having the same center as Q, but double edges length. We will denote by $| Q |$ the Lebesgue measure of Q.

Following [1], we define

$$\| f \|_{\Phi,w,Q} = \inf\left\{\lambda > 0 : \frac{1}{w(Q)} \int_Q \Phi\left(\frac{| f |}{\lambda}\right) wdx \leq 1\right\};$$

for sake of simplicity, we will write $\| f \|$ in place of $\| f \|_{\widetilde{\Phi},w,Q}$.

2 – Main Results

The weight w is said to verify the condition (B_Φ) if one of the two following equivalent conditions holds:

$$(B_\Phi^1) \qquad \exists\, c > 0 : \left(\frac{1}{| Q |}\int_Q wdx\right)\left\|\frac{1}{w}\right\| \leq c \qquad \forall Q \subset \mathbf{R}^n$$

$$(B_\Phi^2) \quad \exists\, c > 0 : \frac{1}{w(Q)}\int_Q \widetilde{\Phi}\left(\frac{cw(Q)}{| Q |\, w(x)}\right) w(x)dx \leq 1 \qquad \forall Q \subset \mathbf{R}^n.$$

First of all, we will show that if $w \in (B_\Phi)$, then w is doubling.

We have the following result by BAGBY (see [1]):

THEOREM 1. *Suppose* $w \in (B_\Phi)$. *If* $\widetilde{\Phi}$ *is submultiplicative, i.e. verifies the condition*

$$(\Delta') \qquad \exists\, c : \widetilde{\Phi}(st) \leq c\widetilde{\Phi}(s)\widetilde{\Phi}(t) \qquad \forall s, t > 0,$$

then there is a $\delta > 0$ *such that* $w \in (B_\Psi)$ *for* $\widetilde{\Psi}(t) = \left[\widetilde{\Phi}(t)\right]^{1+\delta}$.

In [1] it is also shown that the condition (B_Φ) is equivalent to the weak type inequality

$$w\left(\{x \in \mathbf{R}^n : Mf(x) > \lambda\}\right) \leq \int_{\mathbf{R}^n} \Phi(c\frac{|f|}{\lambda})wdx.$$

We now recall the following definition: under the assumption Φ, $\tilde{\Phi} \in \Delta_2$, the weight w is said to verify the condition (A_Φ) if

$$\exists c : \left(\frac{1}{|Q|}\int_Q \epsilon w dx\right) \varphi\left(\frac{1}{|Q|}\int_Q \varphi^{-1}\left(\frac{1}{\epsilon w}\right)dx\right) \leq c, \quad \forall Q \subset \mathbf{R}^n, \ \forall \epsilon > 0.$$

About (A_Φ) condition it is well-known the following result obtained by KERMAN-TORCHINSKY (see [4]):

THEOREM 2. *If w satisfies the (A_Φ) condition, then there is a $\delta > 0$ such that w verifies (A_Ψ) for* $\overset{-1}{\psi}(t) = \left[\varphi^{-1}(t)\right]^{1+\delta}$.

In [4] it is also proved that if $\Phi, \tilde{\Phi} \in \Delta_2$, then $w \in (A_\Phi)$ iff the maximal operator verifies the strong type inequality

$$\int_{\mathbf{R}^n} \Phi(Mf)\, wdx \leq c\int_{\mathbf{R}^n} \Phi(|f|)wdx$$

The condition (A_Φ) is a generalization of the well-known condition (A_p) of MUCKENHOUPT ([6],[8],[5]) which characterize weights w such that the maximal Hardy-Littlewood operator is bounded in $L^p(\mathbf{R}^n, wdx)$, $1 < p < \infty$.

Bagby remarked that the assumption in theorem 1 that $\tilde{\Phi}$ is submultiplicative is essential to get the assertion: in fact, it is sufficient to consider the function Φ such that $\tilde{\Phi}(t) = \left(\frac{t}{2+t}\right)^2$ $\forall\, 0 \leq t \leq 1$, $\tilde{\Phi}(t) = \left(\frac{t}{3+\log t}\right)^2$ $\forall t > 1$. In the same assumptions of theorem 1 we will show that the class (B_Φ) is equivalent to (A_Φ), and so if $\Phi \in \Delta_2$ we can deduce theorem 1 by simply applying the result of Kerman-Torchinsky.

With the assumption $\Phi, \tilde{\Phi} \in \Delta_2$, from the results of [1], [4], we have $w \in (A_\Phi) \Rightarrow w \in (B_\Phi)$: we will give a direct proof of this assuming only $\widetilde{\Phi} \in \Delta_2$, without assuming that $\Phi \in \Delta_2$ and without using maximal functions. Similarly, without assuming that $\Phi \in \Delta_2$ and without using maximal functions, we will prove also the following

PROPOSITION 1. *If* $\widetilde{\Phi} \in (\Delta')$, *then* $w \in (B_\Phi) \Rightarrow w \in (A_\Phi)$.

We will give two proofs of the Proposition 1, the first of which makes use of the following proposition (for related results, see [2], [10]).

PROPOSITION 2. *If* $\widetilde{\Phi} \in (\Delta')$, *then there exist positive constants* $c', c"$ *such that we have:*

(i) $$\frac{1}{w(Q)}\int_Q \widetilde{\Phi}(|\, u(x)\,|)w(x)dx \le \widetilde{\Phi}(c' \,\|\, u \,\|) \quad \forall u \in L_{\widetilde{\Phi}}(wdx) \quad \forall Q \subset \mathbf{R}^n$$

(ii) $$w \in (B_\Phi) \Rightarrow \varphi\left(\frac{1}{|\,Q\,|}\int_Q \varphi^{-1}\left(\frac{1}{w}\right)dx\right) \le c" \left\|\frac{1}{w}\right\| \quad \forall Q \subset \mathbf{R}^n.$$

We remark that an immediate consequence of the above statements is the following

COROLLARY 1. *If* $\Phi \in \Delta_2$, $\widetilde{\Phi} \in (\Delta')$, *then* $\exists \delta > 0$ *such that* $w \in (B_\Phi)$ *implies* $w \in (B_\Psi)$ *with* $\overset{-1}{\psi}(t) = \left[\varphi^{-1}(t)\right]^{1+\delta}$.

3 – Proofs

The fact that $w \in (B_\Phi)$ implies w doubling is consequence of the following chain of inequalities, in which the constant c is given by (B^1_Φ):

$$\frac{1}{c} < \frac{1}{c}\cdot\frac{3^n\,|\,Q\,|}{|\,2Q\,|} = \frac{1}{c}\cdot\frac{w(2Q)}{|\,2Q\,|}\int_{2Q}\frac{1}{w(x)}\cdot 3^n\chi_Q(x)\cdot\frac{w(x)}{w(2Q)}dx \le$$

$$\le \frac{1}{c}\cdot\frac{w(2Q)}{|\,2Q\,|}\left\|\frac{1}{w}\right\|_{\widetilde{\Phi},w,2Q} \|\, 2\cdot 3^n\chi_Q \,\|_{\Phi,w,2Q} \le$$

$$\frac{1}{c}\cdot\frac{w(2Q)}{|2Q|}\cdot\frac{c\,|2Q|}{w(2Q)}\cdot\inf\left\{\lambda>0:\int_{2Q}\Phi\left(\frac{2\cdot 3^n\chi_Q(x)}{\lambda}\right)w(x)dx\le w(2Q)\right\}=$$

$$=\inf\left\{\lambda>0:\int_{Q}\Phi\left(\frac{2\cdot 3^n}{\lambda}\right)w(x)dx\le w(2Q)\right\}\qquad \forall Q\subset\mathbf{R}^n$$

and therefore

$$\int_Q \Phi(2\cdot 3^n c)w(x)dx > w(2Q)\qquad \forall Q\subset\mathbf{R}^n.$$

We prove now that $w\in(A_\Phi)$ implies $w\in(B_\Phi)$.
For any $\epsilon>0$

$$\text{(1)}\qquad \int_Q \varphi^{-1}\left(\frac{1}{\epsilon w(x)}\right)dx\le |Q|\,\varphi^{-1}\left(\frac{c}{\epsilon}\frac{|Q|}{w(Q)}\right)\qquad \forall Q\subset\mathbf{R}^n;$$

on the other hand, being $\varphi^{-1}(0+)=0$, there exists $c_0>q$ such that

$$\text{(2)}\qquad \tilde{\Phi}\left(\frac{c}{c_0}\right)<\frac{c}{c_0},$$

where q is such that $t\varphi^{-1}(t)\le q\tilde{\Phi}(t)\,\forall t\ge 0$ (the existence of such a q follows from the assumption $\tilde{\Phi}\in\Delta_2$). We have $\tilde{\Phi}(t)\le t\varphi^{-1}(t)$, and therefore:

$$\int_Q \tilde{\Phi}\left(\frac{w(Q)}{c_0\,|Q|\,w(x)}\right)w(x)dx\le\int_Q\varphi^{-1}\left(\frac{w(Q)}{c_0\,|Q|\,w(x)}\right)\frac{w(Q)}{c_0\,|Q|\,w(x)}w(x)dx$$

$$=\frac{w(Q)}{c_0\,|Q|}\int_Q\varphi^{-1}\left(\frac{w(Q)}{c_0\,|Q|\,w(x)}\right)dx.$$

Put $\epsilon=\frac{c_0|Q|}{w(Q)}$ in (1), we have:

$$\int_Q \tilde{\Phi}\left(\frac{w(Q)}{c_0\,|Q|\,w(x)}\right)w(x)dx\le\frac{w(Q)}{c_0\,|Q|}\,|Q|\,\varphi^{-1}\left(\frac{cw(Q)}{c_0\,|Q|}\frac{|Q|}{w(Q)}\right)$$

$$=w(Q)\frac{1}{c}\frac{c}{c_0}\varphi^{-1}\left(\frac{c}{c_0}\right)\le q\,w(Q)\frac{1}{c}\tilde{\Phi}\left(\frac{c}{c_0}\right)<\frac{q}{c_0}w(Q)<w(Q)\quad\forall Q\subset\mathbf{R}^n$$

from which the assertion follows. □

We will give now two proofs of the Proposition 1, the first of which makes use of an inequality relating a Jensen mean with an Orlicz norm (see [10]), that will be proved here in the more general case of the weighted Orlicz spaces.

PROOF OF THE PROPOSITION 2. (i) - Let us denote in this proof by c the constant of the condition (Δ') satisfied by $\widetilde{\Phi}$. We remark that for any $K > 0$ it is possible to choice K' such that $\frac{1}{K'} \leq \overset{-1}{\widetilde{\Phi}}\left(\frac{1}{cK}\right)$, and so such that

$$K\widetilde{\Phi}(t) = K\widetilde{\Phi}\left(\frac{1}{K'}K't\right) \leq cK\widetilde{\Phi}\left(\frac{1}{K'}\right)\widetilde{\Phi}(K't) \leq \widetilde{\Phi}(K't) \quad \forall t \geq 0$$

By convexity of $\widetilde{\Phi}$ and the fact that $\widetilde{\Phi} \in (\Delta')$ implies $\widetilde{\Phi} \in \Delta_2$, there exists $c_1 \geq 1$ such that

$$\widetilde{\Phi}(s) \leq s\overset{-1}{\varphi}(s) \leq c_1\widetilde{\Phi}(s) \quad \forall s \geq 0,$$

so if we put $c_2 = cc_1$, we have

$$\overset{-1}{\varphi}(st) \leq \frac{c_1}{st}\widetilde{\Phi}(st) \leq c\frac{c_1}{st}\widetilde{\Phi}(s)\widetilde{\Phi}(t) \leq c_2\overset{-1}{\varphi}(s)\overset{-1}{\varphi}(t) \quad \forall s, t > 0.$$

Now, if $u \equiv 0$ the assertion is obvious. If $u \not\equiv 0$, let c' be the constant obtained by c using the remark above. Then $c\widetilde{\Phi}(t) \leq \widetilde{\Phi}(c't)$ $\forall t \geq 0$, and therefore

$$\frac{1}{w(Q)}\int_Q \widetilde{\Phi}(|\, u(x) \,|)w(x)dx = \frac{1}{w(Q)}\int_Q \widetilde{\Phi}\left(\|\, u \,\| \frac{|\, u(x) \,|}{\|\, u \,\|}\right) w(x)dx$$

$$\leq c\widetilde{\Phi}(\|\, u \,\|)\frac{1}{w(Q)}\int_Q \widetilde{\Phi}\left(\frac{|\, u(x) \,|}{\|\, u \,\|}\right) w(x)dx \leq c\widetilde{\Phi}(\|\, u \,\|) \leq \widetilde{\Phi}(c' \,\|\, u \,\|).$$

(ii) - Since $w \in (B_\Phi)$, there exists $H > 0$ such that $\|\frac{1}{w}\| \leq \frac{H|Q|}{w(Q)}$; applying the remark made above to the function $\overset{-1}{\varphi}$, it follows that there exist $K' > 0$ such that $c_1 c' H \overset{-1}{\varphi}(t) \leq \overset{-1}{\varphi}(K't)$ $\forall t \geq 0$. Put $c'' = K'c'$,by (i) applied to $u = w^{-1}$ it follows that

$$\frac{1}{|\, Q \,|}\int_Q \overset{-1}{\varphi}\left(\frac{1}{w}\right) dx \leq \frac{c_1}{|\, Q \,|}\int_Q \widetilde{\Phi}\left(\frac{1}{w}\right) w dx \leq \frac{c_1 w(Q)}{|\, Q \,|}\widetilde{\Phi}\left(c' \left\|\frac{1}{w}\right\|\right) \leq$$

$$\leq c_1 c' \frac{w(Q)}{|Q|} \left\| \frac{1}{w} \right\| \varphi^{-1}\left(c' \left\| \frac{1}{w} \right\|\right) \leq c_1 c' H \varphi^{-1}\left(c' \left\| \frac{1}{w} \right\|\right)$$
$$\leq \varphi^{-1}\left(K'c' \left\| \frac{1}{w} \right\|\right) = \varphi^{-1}\left(c'' \left\| \frac{1}{w} \right\|\right),$$

from which the assertion follows. □

Now we can prove the Proposition 1. From *(ii)* it follows that

$$\left(\frac{1}{|Q|}\int_Q w dx\right) \varphi\left(\frac{1}{|Q|}\int_Q \varphi^{-1}\left(\frac{1}{w}\right) dx\right) \leq$$
$$\leq c'' \left(\frac{1}{|Q|}\int_Q w dx\right) \left\| \frac{1}{w} \right\| \leq c'' H$$

from which the assertion follows for $\epsilon = 1$. The same proof applies to ϵw, and so the assertion is proved in the general case.

Let us give now another

PROOF OF THE PROPOSITION 1. We have seen that φ^{-1} is submultiplicative, and so

$$\frac{\frac{1}{|Q|}\int_Q \varphi^{-1}\left(\frac{1}{w}\right) dx}{\varphi^{-1}\left(\frac{H|Q|}{w(Q)}\right)} \leq c_2 \frac{\frac{1}{|Q|}\int_Q \varphi^{-1}\left(\frac{H|Q|}{w(Q)}\right) \varphi^{-1}\left(\frac{w(Q)}{H|Q|w(x)}\right) dx}{\varphi^{-1}\left(\frac{H|Q|}{w(Q)}\right)}$$
$$= cc_1 \frac{1}{|Q|}\int_Q \varphi^{-1}\left(\frac{w(Q)}{H\,|Q|\,w(x)}\right) dx,$$

where H is such that $\|\frac{1}{w}\| \leq \frac{H|Q|}{w(Q)}$, and so such that

$$\frac{1}{c_1 H\,|Q|}\int_Q \varphi^{-1}\left(\frac{w(Q)}{H\,|Q|\,w(x)}\right) dx \leq \frac{1}{w(Q)}\int_Q \tilde{\Phi}\left(\frac{w(Q)}{H\,|Q|\,w(x)}\right) w(x) dx$$
$$\leq 1.$$

Then we have, by the remark of the proof of Proposition 2 (i) applied to φ^{-1}:

$$\frac{1}{|Q|}\int_Q \varphi^{-1}\left(\frac{1}{w}\right) dx \leq cc_1^2 H \varphi^{-1}\left(\frac{H\,|Q|}{w(Q)}\right) \leq \varphi^{-1}\left(\frac{1}{\varphi\left(\frac{1}{c^2 c_1^3 H}\right)} H \frac{|Q|}{w(Q)}\right)$$

i.e.

$$\frac{w(Q)}{|Q|}\varphi\left(\frac{1}{|Q|}\int_Q \varphi^{-1}\left(\frac{1}{w}\right)dx\right) \le \frac{H}{\varphi\left(\frac{1}{c^2c_1^3H}\right)},$$

from which the assertion follows for $\epsilon = 1$. The same proof applies to ϵw, and so the assertion is proved in the general case. □

We now show that if $\Phi \in \Delta_2$, $\widetilde{\Phi} \in \Delta'$, then $w \in (B_\Phi)$ implies that there exists $\delta > 0$ such that $w \in (B_\Psi)$ with $\widetilde{\Psi}(t) = [\widetilde{\Phi}(t)]^{1+\delta}$.

PROOF. By Corollary 1 there exists $\sigma > 0$ such that $w \in (B_{\Phi_1})$ with $\Phi_1(t) = \int_0^t \varphi_1(t)dt$, $\overset{-1}{\varphi_1}(t) = \left[\varphi^{-1}(t)\right]^{1+\sigma}$.

Now, let $p > 1$ such that $\frac{\widetilde{\Phi}(t)}{t^p}$ be nondecreasing (the existence of such a p follows from the assumption $\Phi \in \Delta_2$). Let $\delta = \sigma\left(1 - \frac{1}{p}\right)$ and $K = \left[\widetilde{\Phi}(1)\right]^{\frac{\sigma}{p}}$; by Hölder inequality we have

$$\widetilde{\Phi_1}(t) = \int_0^t \overset{-1}{\varphi_1}(s)ds = \int_0^t \left[\varphi^{-1}(s)\right]^{1+\sigma} ds \ge t^{-\sigma}\left[\widetilde{\Phi}(t)\right]^{1+\sigma}$$

$$= \left[\widetilde{\Phi}(t)\right]^{1+\sigma\left(1-\frac{1}{p}\right)}\left[\frac{\widetilde{\Phi}(t)}{t^p}\right]^{\frac{\sigma}{p}} \ge K\left[\widetilde{\Phi}(t)\right]^{1+\delta} \quad \forall t \ge 1$$

and therefore, putting $\widetilde{\Psi}(t) = \left[\widetilde{\Phi}(t)\right]^{1+\delta}$, we have $\widetilde{\Phi_1}(t) \ge K\widetilde{\Psi}(t)$ $\forall t \ge 1$. Then it follows that $\exists\, h : \Psi(t) \le h\Phi_1(t)$ near infinity, and therefore $w \in (B_{\Phi_1})$ implies $w \in (B_\Psi)$, from which the assertion follows. □

4 – A final remark

Let us now consider again the strong type inequality

$$\int_{\mathbf{R}^n} \Phi(Mf)\, w dx \le c\int_{\mathbf{R}^n} \Phi(f) w dx$$

which is true in the assumption $\Phi, \widetilde{\Phi} \in \Delta_2$ iff $w \in (A_\Phi)$. Let us drop the condition that $\widetilde{\Phi} \in \Delta_2$. If we take, for instance, $\Phi(t) = t$, then the inequality is generally false, even if $w = 1$ a.e., because of the Stein's result $Mf \in L^1 \Leftrightarrow f \in L \log L$ ([9]). Problems about conditions on Ψ to be assumed in order to obtain the inequality

$$\int_{Q_0} \Psi(Mf)\, dx \leq c \int_{Q_0} \Phi(f) dx$$

are treated in ([3]). We remark that applying Proposition 4.2 of [3] we can deduce the inequality

$$\int_{Q_0} \log |\, Mf \,|\, dx \leq c \int_{Q_0} |\, f \,|\, dx \tag{3}$$

and take the opportunity to advertise that the inequality (10) in [3] is wrong and should be replaced by (3).

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INDIRIZZO DELL'AUTORE:

Alberto Fiorenza -Dipartimento di Matematica e Applicazioni "Renato Caccioppoli" - Università Monte S. Angelo - via Cintia - 80126 NAPOLI (Italy)