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The homogeneous equations of singular projective hypersurfaces with embedded tangent bundle

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RIASSUNTO: *Qui vengono classificate (ottenendone equazioni esplicite) tutte le possibili ipersuperfici di* $\mathbf{P}^n$ *il cui fascio tangente è un sottofibrato della restrizione del fibrato tangente di* $\mathbf{P}^n$ *ed ha* $\mathbf{O}_X$ *(per qualche k) come quoziente. Queste ipersuperfici esistono solo in caratteristica positiva.*

ABSTRACT: *Here we classify (giving the explicit equations) all possible hypersurfaces,* X*, of* $\mathbf{P}^n$ *which are singular, but nevertheless their tangent sheaf being a subbundle of the restriction of the tangent bundle of* $\mathbf{P}^n$ *with some* $\mathbf{O}_X(k)$ *as quotient.* X *exists only in positive characteristic.*

KEY WORDS: *Normal bundle - Projective varieties - Hypersurfaces - Normal sheaf - Tangent sheaf - Singular variety.*

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In [5] it was shown the existence in positive characteristic of several examples of singular varieties with locally free tangent sheaf (which, by definition, is the dual of the sheaf of regular 1-forms). Furthermore, the same explicit examples show that we may find (in positive characteristic) singular varieties, X, embedded in a smooth variety Z and such that the tangent sheaf TX is not only free but also a subbundle of TZ. Of course, if X is singular, the quotient bundle $(TZ \setminus X)/TX$ is never the normal bundle $N_{X,Z}$ of X in Z. In particular, if $\dim(Z) = \dim(X) + 1$, the inclusion of $(TZ \setminus X)/TX$ into $N_{X,Z}$ induces an effective Cartier divisor, D, $D \neq \emptyset$, and $D = \mathrm{Sing}(X)$ (hence in this case X is always singular in pure codimension 1). Several (global) such examples were considered (from various point of views and in connection with very different problems) in [1], [2] and [3]. Here we consider again the matter and prove the following result.

THEOREM 0.1. *Fix an algebraically closed base field* $\mathbf{K}$*; set* $p := \mathrm{char}(\mathbf{K}) \geq 0$. *Let* X *be an integral hypersurface in* $\mathbf{P}^n$, *say* $X = \{f = 0\}$ *with* f *irreducible homogeneous polynomial. Assume that* X *is singular and that the tangent sheaf of* X *is subbundle of* $T\mathbf{P}^n/X$ *with quotient bundle isomorphic to* $\mathbf{O}_X(k)$ *for some integer* k. *Then* $p > 0$, $p/\deg(f)$ *and there are homogeneous polynomials* u *and* v *such that*

$$f = u^p h + v^p \tag{1}$$

and $k = p(\deg(v) - \deg(u))$. *Viceversa, if* $p > 0$, f *is given by* (1) *with* $\deg(u) > 0$ *and at each point of* $X := \{f = 0\}$ *one of the partial derivatives of* h *does not vanish, then* X *has as tangent sheaf* TX *a subbundle of* $T\mathbf{P}^n/X$ *with* $(T\mathbf{P}^n/X)/TX \cong \mathbf{O}_X(k)$ *for* $k = p(\deg(v) - \deg(u))$.

The short proof of 0.1 consists only of simple manipulations of polynomials. However the result seems to be a nice, small, complete piece of classification around the funny behaviour in positive characteristic of varieties from a projective and differential point of view.

In 1.1 we give a general method to construct (even in the global case) huge linear families of pairs (X, Z) (with Z smooth and $\dim(Z) = 2\dim(X)$) as in the first part of the introduction. This construction is the extension of the construction given in [2] (the case "X a curve"). It is useful to obtain examples of pairs (X, Z) with X having some specific property (either "bad" or "good", as we want). The existence of the examples described by eq. (1) shows (even more) some elementary reasons for the failure of the "standard" analytic approach to the local study of hypersurface singularities.

We would be extremely interested in knowing local results which could show that something deeper than formal manipulation of polynomials and derivatives is behind these kind of examples.

PROOF OF 0.1. It is obvious that every integral hypersurface with equation given by (1) and satisfying the condition on the partial derivatives of h has the property we wanted. Furthermore, the effective Cartier divisor of X which measures the difference between the normal bundle of X in $\mathbf{P}^n$ and $(T\mathbf{P}^n \setminus X)/TX$ is the restriction to X of the multiple hypersurface $\{u^p = 0\}$.

Now we prove the other implication. We fix homogeneous coordinates $x_0, \dots, x_n$ on $\mathbf{P}^n$. D_i will denote the partial derivative with respect to x_i. Set $d := \deg(f)$. By the definition of k there is an inclusion of $\mathbf{O}_X(k)$ into the normal bundle $\mathbf{O}_X(d)$ with zero locus, D, not empty (hence an effective Cartier divisor) and $D = \mathrm{Sing}(X)$. Since $k > 0$, this inclusion is induced by a unique homogeneous polynomial of degree $d-k$; let g be such a polynomial.

(a) By the definition of g for every i we have $D_i(f) = gh_i$ for some polynomial h_i. By the Euler's formula, we have $\Sigma x_i g h_i = \deg(f) f$. Since f is irreducible, p divides $\deg(f)$ and $\Sigma x_i h_i = 0$.

(b) Here we will check that $g = u^p$ for some u. The fact that $D_i(D_j(f)) = D_j(D_i(f))$ is equivalent to:

$$g(D_j(h_i) - D_i(h_j)) = h_j D_i(g) - h_i D_j(g). \tag{2}$$

Since $\{f, h_0, \dots, h_n\}$ have no common zero, up to a linear change of coordinates we may assume that for all i, j, the sequence (g, h_i, h_j) is a regular sequence. Since $\deg(D_j(h_i)) < \deg(h_i)$, by (2) we have $D_i(g) = 0$ for every i, i.e. $g = u^p$ for some homogeneous polynomial u. In particular $p \setminus (d-k)$, hence $p \setminus k$.

(c) Fix the integer i. Since $D_i(f) = u^p h_i$, no monomial of h_i in the variables $x_k's$ with non zero coefficient has an exponent of x_i congruent to -1 modulo p. This is exactly the condition needed to integrate formally monomial h_i, obtaining a primitive A_i with respect to D_i (in which no monomial with non zero coefficient has exponent in the variable x_i divisible by p). Now fix i and j with $i \neq j$. Since $D_i(u^p h_j) = D_j(u^p h_i)$, we have $D_i(h_j) = D_j(h_i)$. It is easy to check that this is equivalent to the fact every monomial with exponent > 0 with respect to both variables x_i and x_j

has the same coefficient in A_i and A_j. Hence is a homogeneous polynomial h (uniquely determined up to p-powers) of degree $d-k$ with $D_r(h) = h_r$ for every r, i.e. $f = u^p h + v^p$ for some v. Alternatively, the computations just made prove essentially the following elementary (and known) statement: a sequence, say $\{h_r\}_{0 \leq r \leq n}$, of homogeneous polynomials is of the form $\{D_r(h)\}$ for some polynomial h if and only if $D_r(h_s) = D_s(h_r)$ and $D_{rp-1}(h_r) = 0$ for every r and s. □

EXAMPLE 1.1. Here we show how the construct several examples (not necessary seen as varieties embedded in a fixed projective space). The construction is the natural extension of the construction given in [2] in the case of curves. Fix any smooth variety T (e.g. complete); set $n := \dim(T)$. Let E be a vector bundle on E with rank $(E) = n+1$. Set $\mathbf{P} := \mathbf{P}(E)$ and let $f : \mathbf{P} \to T$ be the projection. Fix a line bundle R on T and set $M := f^*(R)$. Fix any power q of p and an integer $s > 0$. Set $V := H^0(T, S^{sq}(E) \otimes R) \cong H^0(\mathbf{P}, \mathbf{O_P}(1) \otimes R)$. Consider the (not linear) map $t : S^s(E) \to S^{sq}(E)$ induced by the iterated relative Frobenius of f. Let $W \subseteq V$ be the vector subspace corresponding to t; W is a vector space because $\mathbf{K}$ is perfect; for instance if E is the direct sum of $n+1$ line bundles $A_0, \dots, A_n$, W is the direct sum of vector spaces $H^0(T, A_0{}^{a(0)} \otimes \dots \otimes A_n{}^{a(n)} \otimes M)$ in which each $a(i)$ is a power of q. Let $X \subset \mathbf{P}$ be any zero loci of a family of equations in W (seen as elements of $H^0(\mathbf{P}, \mathbf{O_P}(1) \otimes M)$). Assume $\dim(X) = n$, X reduced and $f(X) = T$. Then it is easy to see ("locally X is given by equations in which the fibers variables are q-powers") that TX is the locally free subbundle of $T\mathbf{P}$ which is the restriction to X of the relative tangent of f. Such a variety X certainly exists if R is ample enough: take as X the intersection of n general enough divisors. Furthermore the set of such X, for fixed R, s and q, forms a linear system whose dimension is in principle easily computable. If $E \cong r\mathbf{O}_T \oplus A$ for some integer r with $0 < r \leq n$, we may map birationally $\mathbf{P}$ into a projective space, Π, with image a cone with vertex of dimension $r-1$; in this way we find examples of embedded $X \subset \Pi$ which are strange. As in the case of curves considered in [2], form these linear families we may easily construct examples with certain behaviour, and/or if interested in a particular (T, E, M, s, q), study the singularities of the general member of such linear families (see [4], prop. 3, for the case of strange plane curves).

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