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Second order non linear non variational parabolic systems

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Dedicated to Professor Francesco Guglielmino with our deepest esteem and gratitude, on his 65th birthday

RIASSUNTO: *Sia* $u \in L^2\big(-T,0,H^2(\Omega,\mathbb{R}^N)\big) \cap H^1\big(-T,0,L^2(\Omega,\mathbb{R}^N)\big)$ *(N intero* ≥ 1*) una soluzione in* $Q = \Omega \times (-T,0)$ *del sistema non lineare non variazionale*

$$-a\big(X,u,Du,H(u)\big) + \frac{\partial u}{\partial t} = b(X,u,Du)\,,$$

dove $a(X,u,p,\xi)$ *e* $b(X,u,p)$ *sono vettori di* $\mathbb{R}^N$ *misurabili in* X *e continui nelle altre variabili. Si dimostra che se* $b(X,u,p)$ *ha andamento lineare e* $a(X,u,p,\xi)$ *è di classe* C^1 *in* ξ *e verifica la condizione (A) e, unitamente a* $\frac{\partial a}{\partial \xi}$*, certe condizioni di continuità, allora il vettore* Du *è parzialmente hölderiano in* Q *con ogni esponente* $\alpha < 1$.

ABSTRACT: *Let* $u \in L^2\big(-T,0,H^2(\Omega,\mathbb{R}^N)\big) \cap H^1\big(-T,0,L^2(\Omega,\mathbb{R}^N)\big)$ *(N integer* ≥ 1*) be a solution in* $Q = \Omega \times (-T,0)$ *to the non linear non variational system*

$$-a\big(X,u,Du,H(u)\big) + \frac{\partial u}{\partial t} = b(X,u,Du)\,,$$

where $a(X,u,p,\xi)$ *and* $b(X,u,p)$ *are vectors in* $\mathbb{R}^N$ *measurable in* X *and continuous in the other variables. We prove that if* $b(X,u,p)$ *has a linear growth and* $a(X,u,p,\xi)$ *is of class* C^1 *in* ξ *and satisfies the condition (A) and, together with* $\frac{\partial a}{\partial \xi}$*, certain continuity*

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conditions, then the vector Du is partially Hölder continuous in Q for every exponent $\alpha < 1$.

KEY WORDS: *Nonlinear parabolic systems.*

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1 – Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n > 2$, of class C^2 and let $x = (x_1, x_2, \dots, x_n)$ be a generic point in it. Q denotes the cylinder $\Omega \times (-T, 0)$ with $T > 0$ and X the point (x,t) of $\mathbb{R}^n_x \times \mathbb{R}_t$.

The symbols $(\cdot|\cdot)_k$ and $\|\cdot\|_k$ denote the scalar product and the norm in $\mathbb{R}^k$, respectively. We shall omit the index k wherever there is no ambiguity.

If $u\colon Q \to \mathbb{R}^N$, N integer ≥ 1, we set

$$D_i u = \frac{\partial u}{\partial x_i},$$

$$Du = (D_1 u, D_2 u, \dots, D_n u), \qquad H(u) = \{D_i D_j u\} = \{D_{ij} u\},$$

$$i, j = 1, 2, \dots, n;$$

Du is a vector in $\mathbb{R}^{nN}$ and $H(u)$ is an element of $\mathbb{R}^{n^2N}$.

In what follows we denote by $p = (p_1, p_2, \dots, p_n)$, $p_i \in \mathbb{R}^N$, a generic vector of $\mathbb{R}^{nN}$ and by $\xi = \{\xi_{ij}\}$, $i, j = 1, 2, \dots, n$, $\xi_{ij} \in \mathbb{R}^N$, a generic element of $\mathbb{R}^{n^2N}$.

We consider in the cylinder Q the following second order non linear non variational system:

$$-a(X, u, Du, H(u)) + \frac{\partial u}{\partial t} = b(X, u, Du), \tag{1.1}$$

where $a(X, u, p, \xi)$ and $b(X, u, p)$ are vectors of $\mathbb{R}^N$, measurable in X, continuous in (u, p, ξ) and (u, p) respectively, satisfying the conditions:

$$a(X, u, p, 0) = 0; \tag{1.2}$$

(A) *there exist three positive constants* α, γ *and* δ *with* $\gamma + \delta < 1$, *such that,* $\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$, $\forall \tau, \eta \in \mathbb{R}^{n^2N}$ *and for almost every* $X \in Q$ *we* *have*[(1)]

$$\left\|\sum_{i=1}^{n}\tau_{ii}-\alpha\big[a(X,u,p,\tau+\eta)-a(X,u,p,\eta)\big]\right\|^2\le$$

$$\le\gamma\|\tau\|^2+\delta\left\|\sum_{i=1}^{n}\tau_{ii}\right\|^2 ;$$

(1.3) *there exists a constant c such that,* $\forall u\in\mathbb{R}^N$, $\forall p\in\mathbb{R}^{nN}$ *and for almost every* $X\in Q$ *we have*

$$\|b(X,u,p)\|\le c(1+\|u\|+\|p\|)\,.$$

We shall denote by $W^p(Q,\mathbb{R}^N)$ and $W_0^p(Q,\mathbb{R}^N)$ the functional spaces:

$$W^p(Q,\mathbb{R}^N)=\left\{u\colon u\in L^p(-T,0,H^{2,p}(\Omega,\mathbb{R}^N)),\frac{\partial u}{\partial t}\in L^p(Q,\mathbb{R}^N)\right\},$$

$$W_0^p(Q,\mathbb{R}^N)=\Big\{u\in W^p(Q,\mathbb{R}^N)\colon u\in L^p(-T,0,H_0^{1,p}(\Omega,\mathbb{R}^N)),$$

$$u(x,-T)=0\Big\},$$

where $H^{s,p}(\Omega,\mathbb{R}^N)$ and $H_0^{s,p}(\Omega,\mathbb{R}^N)$, s integer ≥ 0, $p\in[1,+\infty)$, are the usual Sobolev spaces[(2)].

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(1)From condition (A), assuming $\eta=0$, it follows, $\forall u\in\mathbb{R}^N$, $\forall p\in\mathbb{R}^{nN}$, $\forall\tau\in\mathbb{R}^{n^2N}$ and for almost all $X\in Q$:

$$\|a(X,u,p,\tau)\|\le c\|\tau\|\,.$$

Moreover one can show that, if the vector $a(X,u,p,\xi)$ is of class C^1 with respect to ξ, with derivatives $\dfrac{\partial a}{\partial\xi_{ij}}$ bounded, then the operator $a(X,u,p,\xi)$ is elliptic (see [6]). Sufficient conditions that ensure the hypothesis (A) are stated in [4] and [5].

(2)If s, j are integers ≥ 0 and $p\in[1,+\infty)$,

$$|u|_{j,p,\Omega}=\left[\int_\Omega\left(\sum_{|\alpha|=j}\|D^\alpha u\|^2\right)^{p/2}dx\right]^{1/p},\ \|u\|_{s,p,\Omega}=\left\{\sum_{j=0}^{s}|u|^p_{j,p,\Omega}\right\}^{1/p},$$

where $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}\dots D_n^{\alpha_n}$, $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$, $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_n$, α_i integer ≥ 0.

Particularly

$$|u|_{0,p,\Omega}=\|u\|_{0,p,\Omega}=\|u\|_{L^p(\Omega,\mathbb{R}^N)}\,.$$

$W^p(Q,\mathbb{R}^N)$ and $W_0^p(Q,\mathbb{R}^N)$ are Banach spaces with the following norm:

$$\|u\|_{p,Q} = \left[\int_Q \left(\|u\|^p + \|Du\|^p + \|H(u)\|^p + \left\|\frac{\partial u}{\partial t}\right\|^p\right) dX\right]^{1/p} .$$

By a solution to the system (1.1) we mean a vector $u \in W^2(Q,\mathbb{R}^N)$ satisfying (1.1) for almost all $X \in Q$. In this work we prove a partial Hölder continuity result for the spatial gradient of these solutions; this result is similar to the one proved by S. Campanato in [7] in the elliptic case.

The study of the local L^q-regularity of the derivatives $D_{ij}u$, $\dfrac{\partial u}{\partial t}$ is preliminary in order to achieve the partial Hölder regularity of the gradient; to this study, of interest in itself, is devoted section n. 3.

2 – Preliminary lemmas

LEMMA 2.1. *If* $u \in W^p(Q(X^0,\sigma),\mathbb{R}^N)$(3), $1 \le p < +\infty$, *and if* $P_{Q(X^0,\sigma)}$ *is the polynomial vector in* x, *of degree* ≤ 1 *such that*

$$\int_{Q(X^0,\sigma)} D^\alpha(u - P_{Q(X^0,\sigma)})dX = 0, \qquad \forall\alpha\colon |\alpha| \le 1, \tag{2.1}$$

then

$$\int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^p dX \le c\sigma^{2p} \int_{Q(X^0,\sigma)} \left(\|H(u)\|^p + \left\|\frac{\partial u}{\partial t}\right\|^p\right) dX\,, \tag{2.2}$$

where c *does not depend on* σ.

(3) $Q(X^0,\sigma)$ denotes the cylinder of $\mathbb{R}^n_x \times \mathbb{R}_t$: $B(x^0,\sigma) \times (t^0 - \sigma^2, t^0)$, where $X^0 = (x^0,t^0) \in \mathbb{R}^n_x \times \mathbb{R}_t$, $\sigma > 0$, $B(x^0,\sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\}$.

PROOF. Having fixed t a.e. in $(t^0-\sigma^2, t^0)$, let us denote by $P_{B(x^0,\sigma)}(t)$ the polynomial vector in x, of degree ≤ 1, such that

$$\int_{B(x^0,\sigma)} D^\alpha(u(x,t) - P_{B(x^0,\sigma)}(t))dx = 0, \quad \forall\alpha\colon |\alpha| \leq 1 .$$

We easily get

$$\begin{aligned}\int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^p dX \leq c \int_{Q(X^0,\sigma)} \|u - P_{B(x^0,\sigma)}\|^p dX + \\ + c \int_{Q(X^0,\sigma)} \|P_{B(x^0,\sigma)} - P_{Q(X^0,\sigma)}\|^p dX .\end{aligned} \tag{2.3}$$

The "Poincarè estimate" (3.20) of Chapter I of [2] ensures, for t a.e. in $(t^0 - \sigma^2, t^0)$:

$$\int_{B(x^0,\sigma)} \|u(x,t) - P_{B(x^0,\sigma)}(t)\|^p dx \leq c\sigma^{2p} \int_{B(x^0,\sigma)} \|H(u)\|^p dx$$

and hence, by integration on $(t^0 - \sigma^2, t^0)$

$$\int_{Q(X^0,\sigma)} \|u - P_{B(x^0,\sigma)}\|^p dX \leq c\sigma^{2p} \int_{Q(X^0,\sigma)} \|H(u)\|^p dX . \tag{2.4}$$

Now let us consider the last integral in the right hand side of (2.3).

From the definition of the polynomial vectors $P_{B(x^0,\sigma)}$ and $P_{Q(X^0,\sigma)}$ it easily follows, $\forall x \in B(x^0,\sigma)$ and for t a.e. in $(t^0 - \sigma^2, t^0)$[(4)]:

$$\begin{aligned}\|P_{B(x^0,\sigma)} - P_{Q(X^0,\sigma)}\|^p \leq c\sigma^p \sum_{i=1}^n \left\|(D_i u(x,t))_{B(x^0,\sigma)} - (D_i u)_{Q(X^0,\sigma)}\right\|^p + \\ + c\left\|(u(x,t))_{B(x^0,\sigma)} - u_{Q(X^0,\sigma)}\right\|^p ,\end{aligned}$$

(4) If $E \subset \mathbb{R}^k$ is a measurable set with positive measure and $f \in L^1(E, \mathbb{R}^h)$, we set: $f_E = \fint_E f\, dx = \dfrac{\int_E f\, dx}{\text{meas}E}$.

from which

$$\int\limits_{Q(X^0,\sigma)} \|P_{B(x^0,\sigma)} - P_{Q(X^0,\sigma)}\|^p dX \leq$$

$$(2.5) \qquad \leq c\sigma^p \int\limits_{Q(X^0,\sigma)} \sum_{i=1}^{n} \left\|(D_i u)_{B(x^0,\sigma)} - (D_i u)_{Q(X^0,\sigma)}\right\|^p dX +$$

$$+ c \int\limits_{Q(X^0,\sigma)} \|u_{B(x^0,\sigma)} - u_{Q(X^0,\sigma)}\|^p dX \,.$$

Taking into account the technique used to obtain the lemma 2.II of [9], we get

$$(2.6) \qquad \begin{aligned} &\int\limits_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|(D_i u)_{B(x^0,\sigma)} - (D_i u)_{Q(X^0,\sigma)}\|^p dX \leq \\ &\leq c\sigma^p \int\limits_{Q(X^0,\sigma)} \left(\|H(u)\|^p + \left\|\frac{\partial u}{\partial t}\right\|^p\right) dX \,. \end{aligned}$$

We have moreover

$$\int\limits_{Q(X^0,\sigma)} \|u_{B(x^0,\sigma)} - u_{Q(X^0,\sigma)}\|^p dX \leq$$

$$\leq \frac{1}{\sigma^2} \int\limits_{t^0-\sigma^2}^{t^0} dt \int\limits_{t^0-\sigma^2}^{t^0} d\xi \int\limits_{B(x^0,\sigma)} \|u(x,t) - u(x,\xi)\|^p dx \,.$$

On the other hand, from $u \in H^{1,p}\left(t^0 - \sigma^2, t^0, L^p(B(x^0,\sigma), \mathbb{R}^N)\right)$, it easily follows for t, ξ a.e. in $(t^0 - \sigma^2, t^0)$:

$$\int\limits_{B(x^0,\sigma)} \|u(x,t) - u(x,\xi)\|^p dx \leq \sigma^{2(p-1)} \int\limits_{Q(X^0,\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^p dX$$

and hence

$$\int_{Q(X^0,\sigma)} \|u_{B(x^0,\sigma)} - u_{Q(X^0,\sigma)}\|^p dX \leq \sigma^{2p} \int_{Q(X^0,\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^p dX. \tag{2.7}$$

From (2.5), in virtue of (2.6) and (2.7), we get:

$$\begin{aligned} &\int_{Q(X^0,\sigma)} \|P_{B(x^0,\sigma)} - P_{Q(X^0,\sigma)}\|^p dX \leq \\ &\leq c\sigma^{2p} \int_{Q(X^0,\sigma)} \left(\|H(u)\|^p + \left\|\frac{\partial u}{\partial t}\right\|^p\right) dX, \end{aligned} \tag{2.8}$$

that is the estimate we need.

The estimate (2.2) follows from (2.3), (2.4) and (2.8).

LEMMA 2.2. *If $u \in W^2(Q(X^0,\sigma), \mathbb{R}^N)$, if $\sigma \in (0,2)$ and if $P_{Q(X^0,\sigma)}$ is the polynomial vector in x, of degree ≤ 1, verifying* (2.1), *then*

$$\begin{aligned} &\sigma^{-2} \int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^2 dX + \int_{Q(X^0,\sigma)} \|D(u - P_{Q(X^0,\sigma)})\|^2 dX \leq \\ &\leq c \left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^{\frac{2(n+2)}{n+4}} + \left\|\frac{\partial u}{\partial t}\right\|^{\frac{2(n+2)}{n+4}} \right) dX \right]^{\frac{n+4}{n+2}}, \end{aligned} \tag{2.9}$$

where c does not depend on σ.

PROOF. Setting $q = \dfrac{2(n+2)}{n+4}$, it follows

$$u \in W^q(Q(X^0,\sigma), \mathbb{R}^N)$$

because $q < 2$ and $u \in W^2(Q(X^0,\sigma), \mathbb{R}^N)$. Then we are able to apply to the function $u - P_{Q(X^0,\sigma)}$ the lemma 3.3 of Chapter II of [13] (with

$q = \dfrac{2(n+2)}{n+4}$, $p = 2$, $l = 1$), and hence we get the estimates:

$$\int_{Q(X^0,\sigma)} \|D(u - P_{Q(X^0,\sigma)})\|^2 dX \leq$$

$$\leq c \left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^{\frac{2(n+2)}{n+4}} + \left\| \frac{\partial u}{\partial t} \right\|^{\frac{2(n+2)}{n+4}} \right) dX \right]^{\frac{n+4}{n+2}} +$$

$$+ c \left[\int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^{\frac{2(n+2)}{n+4}} dX \right]^{\frac{n+4}{n+2}},$$

$$\sigma^{-2} \int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^2 dX \leq c \left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^{\frac{2(n+2)}{n+4}} + \right.\right.$$

$$\left.\left. + \left\| \frac{\partial u}{\partial t} \right\|^{\frac{2(n+2)}{n+4}} \right) dX \right]^{\frac{n+4}{n+2}} +$$

$$+ c\sigma^{-4} \left[\int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^{\frac{2(n+2)}{n+4}} dX \right]^{\frac{n+4}{n+2}},$$

from which, in virtue of the assumption $\sigma < 2$, it follows

$$\sigma^{-2} \int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^2 dX + \int_{Q(X^0,\sigma)} \|D(u - P_{Q(X^0,\sigma)})\|^2 dX \leq$$

$$\text{(2.10)} \quad \leq c \left[\int_{Q(X^0,\sigma)} \left(\|H(u)\|^{\frac{2(n+2)}{n+4}} + \left\| \frac{\partial u}{\partial t} \right\|^{\frac{2(n+2)}{n+4}} \right) dX \right]^{\frac{n+4}{n+2}} +$$

$$+ c\sigma^{-4} \left[\int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^{\frac{2(n+2)}{n+4}} dX \right]^{\frac{n+4}{n+2}}.$$

On the other hand, taking into account the lemma 2.1 $\left(\text{written for } p = \frac{2(n+2)}{n+4} > 1\right)$, we get:

$$\int_{Q(X^0,\sigma)} \|u - P_{Q(X^0,\sigma)}\|^{\frac{2(n+2)}{n+4}} dX \le \tag{2.11}$$
$$\le c\sigma^{\frac{4(n+2)}{n+4}} \int_{Q(X^0,\sigma)} \left(\|H(u)\|^{\frac{2(n+2)}{n+4}} + \left\|\frac{\partial u}{\partial t}\right\|^{\frac{2(n+2)}{n+4}} \right) dX .$$

Then from (2.10) and (2.11) the estimate (2.9) follows.

LEMMA 2.3. *If $u \in W^2(Q(X^0,\sigma), \mathbb{R}^N)$, with $\sigma \in (0,1)$, then $\forall \tau \in (0,1)$:*

$$\int_{Q(X^0,\tau\sigma)} (\|u\|^2 + \|Du\|^2) dX \le$$
$$\le c\tau^{n+2} \int_{Q(X^0,\sigma)} (\|u\|^2 + \|Du\|^2) dX + \tag{2.12}$$
$$+ c\sigma^2 \int_{Q(X^0,\sigma)} \left(\|Du\|^2 + \|H(u)\|^2 + \right.$$
$$\left. + \left\|\frac{\partial u}{\partial t}\right\|^2 \right) dX ,$$

where c does not depend on σ and τ.

PROOF. From the estimate:

$$\|u\|^2 \le 2\|u_{Q(X^0,\sigma)}\|^2 + 2\|u - u_{Q(X^0,\sigma)}\|^2 ,$$

by integrating on $Q(X^0,\tau\sigma)$, $\tau \in (0,1)$, it follows:

$$\int_{Q(X^0,\tau\sigma)} \|u\|^2 dX \le c(\tau\sigma)^{n+2} \|u_{Q(X^0,\sigma)}\|^2 + 2 \int_{Q(X^0,\sigma)} \|u - u_{Q(X^0,\sigma)}\|^2 dX . \tag{2.13}$$

On the other hand, taking into account the definition of $u_{Q(X^0,\sigma)}$ and the Hölder inequality, we get:

$$(2.14)\quad \|u_{Q(X^0,\sigma)}\|^2 \leq \frac{c}{\sigma^{2n+4}}\left(\int_{Q(X^0,\sigma)} \|u(X)\|dX\right)^2 \leq \frac{c}{\sigma^{n+2}} \int_{Q(X^0,\sigma)} \|u(X)\|^2 dX.$$

From (2.13), (2.14) and by means of Poincarè estimate (2.6) of [9] we obtain:

$$(2.15)\quad \begin{aligned} \int_{Q(X^0,\tau\sigma)} \|u\|^2 dX &\leq c\tau^{n+2} \int_{Q(X^0,\sigma)} \|u\|^2 dX + \\ &+ c\sigma^2 \int_{Q(X^0,\sigma)} \|Du\|^2 dX + c\sigma^4 \int_{Q(X^0,\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^2 dX\,. \end{aligned}$$

Using the same procedure for the vector $D_i u$, $i = 1, 2, \dots, n$, we achieve:

$$\begin{aligned} \int_{Q(X^0,\tau\sigma)} \|D_i u\|^2 dX &\leq c\tau^{n+2} \int_{Q(X^0,\sigma)} \|D_i u\|^2 dX + \\ &+ 2 \int_{Q(X^0,\sigma)} \|D_i u - (D_i u)_{Q(X^0,\sigma)}\|^2 dX\,, \end{aligned}$$

from which, in virtue of the Poincarè estimate (2.7) of [9], we get:

$$(2.16)\quad \begin{aligned} \int_{Q(X^0,\tau\sigma)} \|D_i u\|^2 dX &\leq c\tau^{n+2} \int_{Q(X^0,\sigma)} \|D_i u\|^2 dX + \\ &+ c\sigma^2 \int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX\,, \qquad \forall i = 1, 2, \dots, n\,. \end{aligned}$$

From (2.15), (2.16) and being $\sigma < 1$, the estimate (2.12) follows.

3 – L^q-local regularity for the matrix $H(u)$

Let $u \in W^2(Q, \mathbb{R}^N)$ be a solution in Q to the non variational system

$$-a(X, u, Du, H(u)) + \frac{\partial u}{\partial t} = b(X, u, Du), \tag{3.1}$$

being $a(X, u, p, \xi)$ and $b(X, u, p)$ vectors of $\mathbb{R}^N$ fulfilling assumptions (1.2), (1.3) and (A).

In this section we shall show a "Caccioppoli's type inequality", from which we will be able to derive, by means of a well known lemma due to Gehring - Giaquinta - G. Modica, a L^q-local regularity result for the matrix $H(u)$, where u is a solution to the system (3.1); this result, of interest in itself, is preliminary to the study of the partial Hölder continuity of the vector Du.

LEMMA 3.1. *If $u \in W^2(Q, \mathbb{R}^N)$ is a solution to the system* (3.1) *and if the assumptions* (1.2), (1.3) *and* (A) *are fulfilled, then, $\forall Q(X^0, 2\sigma) \subset\subset Q$, we have:*

$$\int_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \le$$

$$\le c\sigma^{-2} \left\{ \sigma^{-2} \int_{Q(X^0,2\sigma)} \|u - P_{Q(X^0,2\sigma)}\|^2 dX + \right. \tag{3.2}$$

$$\left. + \int_{Q(X^0,2\sigma)} \|D(u - P_{Q(X^0,2\sigma)})\|^2 dX \right\} + c \int_{Q(X^0,2\sigma)} \|b(X, u, Du)\|^2 dX,$$

where c does not depend on σ and $P_{Q(X^0,2\sigma)}$ is the polynomial vector in x, of degree ≤ 1, such that

$$\int_{Q(X^0,2\sigma)} D^\alpha (u - P_{Q(X^0,2\sigma)}) dX = 0, \qquad \forall \alpha \colon |\alpha| \le 1.$$

PROOF. Let us fix a cylinder $Q(X^0, 2\sigma) \subset\subset Q$ and let $\vartheta(x)$, $\rho(t)$ be two real functions of class $C_0^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R})$ respectively, such that:

$$0 \le \vartheta \le 1, \quad \vartheta = 1 \text{ in } B(x^0, \sigma), \quad \vartheta = 0 \text{ in } \mathbb{R}^n \setminus B(x^0, 2\sigma)\,, \tag{3.3}$$

$$|D^\alpha \vartheta| \le c\sigma^{-|\alpha|} \quad \text{for all multi - indices } \alpha\,, \tag{3.4}$$

$$\begin{gathered} 0 \le \rho \le 1, \quad \rho = 1 \text{ for } t \ge t^0 - \sigma^2\,, \\ \rho = 0 \text{ for } t \le t^0 - (2\sigma)^2, \quad |\rho'(t)| \le c\sigma^{-2}\,. \end{gathered} \tag{3.5}$$

Denoting by $P_{Q(X^0,2\sigma)}$ the polynomial vector in x, of degree ≤ 1, such that

$$\int_{Q(X^0,2\sigma)} D^\alpha(u - P_{Q(X^0,2\sigma)})dX = 0, \qquad \forall\alpha\colon |\alpha| \le 1 \tag{3.6}$$

and setting $\mathcal{U}(X) = \vartheta(x)\rho(t)(u - P_{Q(X^0,2\sigma)})$, it results

$$H(u - P_{Q(X^0,2\sigma)}) = H(u), \quad \frac{\partial(u - P_{Q(X^0,2\sigma)})}{\partial t} = \frac{\partial u}{\partial t}\,, \tag{3.7}$$

$$\mathcal{U} \in W_0^2(Q(X^0, 2\sigma), \mathbb{R}^N)\,, \tag{3.8}$$

$$H(\mathcal{U}) = H(u), \quad \frac{\partial \mathcal{U}}{\partial t} = \frac{\partial u}{\partial t} \quad \text{in } Q(X^0, \sigma)\,. \tag{3.9}$$

Now from (3.1) and from the assumption (1.2), it follows[5]:

$$\begin{aligned} \vartheta\rho\Delta u - \alpha\vartheta\rho\frac{\partial u}{\partial t} &= \vartheta\rho\Big\{\Delta u - \alpha\big[a(X,u,Du,H(u)) - a(X,u,Du,0)\big]\Big\} - \\ &\quad - \alpha\vartheta\rho b(X,u,Du), \quad \text{in } Q\,, \end{aligned}$$

[5] α is the positive constant that appears in the condition (A).

from which, making use of the condition (A) (with $\tau = H(u)$ and $\eta = 0$), one gets:

$$\vartheta\rho\left\|\Delta u - \alpha\frac{\partial u}{\partial t}\right\| \le \vartheta\rho\left[\gamma\|H(u)\|^2 + \delta\|\Delta u\|^2\right]^{1/2} + \\ + \alpha\vartheta\rho\|b(X,u,Du)\|, \quad \text{in } Q. \tag{3.10}$$

On the other hand:

$$\begin{cases} \Delta\mathcal{U} = \vartheta\rho\Delta u + A(u - P_{Q(X^0,2\sigma)}) \\ H(\mathcal{U}) = \vartheta\rho H(u) + B(u - P_{Q(X^0,2\sigma)}) \\ \dfrac{\partial\mathcal{U}}{\partial t} = \vartheta\rho'(u - P_{Q(X^0,2\sigma)}) + \vartheta\rho\dfrac{\partial u}{\partial t} \end{cases} \tag{3.11}$$

where

$$\begin{cases} A(u - P_{Q(X^0,2\sigma)}) = \Delta\vartheta\rho(u - P_{Q(X^0,2\sigma)}) + \\ \quad +2\rho\sum_i D_i\vartheta \cdot D_i(u - P_{Q(X^0,2\sigma)}), \\ B(u - P_{Q(X^0,2\sigma)}) = \{\rho D_{ij}\vartheta \cdot (u - P_{Q(X^0,2\sigma)}) + \\ \quad +\rho D_i\vartheta \cdot D_j(u - P_{Q(X^0,2\sigma)}) + \rho D_j\vartheta \cdot D_i(u - P_{Q(X^0,2\sigma)})\}. \end{cases} \tag{3.12}$$

From (3.11) and (3.10) it follows:

$$\left\|\Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\right\| \le \vartheta\rho\left\|\Delta u - \alpha\frac{\partial u}{\partial t}\right\| + \|A(u - P_{Q(X^0,2\sigma)})\| + \\ + \alpha\vartheta|\rho'|\|u - P_{Q(X^0,2\sigma)}\| \le \vartheta\rho\left[\gamma\|H(u)\|^2 + \delta\|\Delta u\|^2\right]^{1/2} + \\ + \alpha\vartheta\rho\|b(X,u,Du)\| + \|A(u - P_{Q(X^0,2\sigma)})\| + \alpha\vartheta|\rho'|\|u - P_{Q(X^0,2\sigma)}\|,$$

from which, $\forall\epsilon > 0$ and for a.e. X in Q:

$$\left\|\Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\right\|^2 \le (1+\epsilon)\vartheta^2\rho^2\left[\gamma\|H(u)\|^2 + \delta\|\Delta u\|^2\right] + \\ + c(\epsilon,\alpha)\left\{\|b(X,u,Du)\|^2 + \|A(u - P_{Q(X^0,2\sigma)})\|^2 + \\ + (\rho')^2\|u - P_{Q(X^0,2\sigma)}\|^2\right\}. \tag{3.13}$$

Taking again into account (3.11), we obtain:

$$\vartheta^2\rho^2\|H(u)\|^2 \le (1+\epsilon)\|H(\mathcal{U})\|^2 + c(\epsilon)\|B(u - P_{Q(X^0,2\sigma)})\|^2,$$

$$\vartheta^2\rho^2\|\Delta u\|^2 \le (1+\epsilon)\|\Delta\mathcal{U}\|^2 + c(\epsilon)\|A(u - P_{Q(X^0,2\sigma)})\|^2.$$

These estimates allow us to estimate the right hand side of (3.13):

$$\left\|\Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\right\|^2 \le (1+\epsilon)^2\gamma\|H(\mathcal{U})\|^2 + (1+\epsilon)^2\delta\|\Delta\mathcal{U}\|^2 + c(\epsilon,\alpha,\gamma,\delta)\cdot$$

$$\cdot\left\{\|b(X,u,Du)\|^2 + \|A(u - P_{Q(X^0,2\sigma)})\|^2 + \|B(u - P_{Q(X^0,2\sigma)})\|^2 + \right.$$

$$\left. + (\rho')^2\|u - P_{Q(X^0,2\sigma)}\|^2\right\}.$$

By integrating on $Q(X^0, 2\sigma)$, taking into account lemmas 2.3 and 2.4 of [8] and the hypothesis (3.5), we get:

$$\int_{Q(X^0,2\sigma)} \left\|\Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\right\|^2 dX \le$$

$$\le (1+\epsilon)^2(\gamma+\delta)\int_{Q(X^0,2\sigma)} \left\|\Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\right\|^2 dX +$$

$$+ c(\epsilon,\alpha,\gamma,\delta)\left\{\int_{Q(X^0,2\sigma)} \|b\|^2 dX + \int_{Q(X^0,2\sigma)} \left(\|A(u - P_{Q(X^0,2\sigma)})\|^2 + \right.\right.$$

$$\left.\left. + \|B(u - P_{Q(X^0,2\sigma)})\|^2\right) dX + \sigma^{-4}\int_{Q(X^0,2\sigma)} \|u - P_{Q(X^0,2\sigma)}\|^2 dX\right\}.$$

The assumption $\gamma + \delta < 1$ allows us to choose a suitable ϵ in such a way that:

$$\int_{Q(X^0,2\sigma)} \left\|\Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\right\|^2 dX \le \tag{3.14}$$

$$\leq c(\epsilon,\alpha,\gamma,\delta)\Bigg\{\int\limits_{Q(X^0,2\sigma)}\|b\|^2dX+$$

$$+\int\limits_{Q(X^0,2\sigma)}\left(\left\|A(u-P_{Q(X^0,2\sigma)})\right\|^2+\left\|B(u-P_{Q(X^0,2\sigma)})\right\|^2\right)dX+$$

$$+\sigma^{-4}\int\limits_{Q(X^0,2\sigma)}\|u-P_{Q(X^0,2\sigma)}\|^2dX\Bigg\}$$

and hence, by lemma 2.3 of [8] and estimates (3.9):

$$\int\limits_{Q(X^0,\sigma)}\left(\|H(u)\|^2+\alpha^2\left\|\frac{\partial u}{\partial t}\right\|^2\right)dX\leq$$

(3.15)
$$\leq c(\epsilon,\alpha,\gamma,\delta)\Bigg\{\int\limits_{Q(X^0,2\sigma)}\|b\|^2dX+ + \int\limits_{Q(X^0,2\sigma)}\left(\left\|A(u-P_{Q(X^0,2\sigma)})\right\|^2+\left\|B(u-P_{Q(X^0,2\sigma)})\right\|^2\right)dX+ +\sigma^{-4}\int\limits_{Q(X^0,2\sigma)}\|u-P_{Q(X^0,2\sigma)}\|^2dX\Bigg\}.$$

On the other hand, by virtue of (3.12) and (3.4), we have

$$\int\limits_{Q(X^0,2\sigma)}\left(\left\|A(u-P_{Q(X^0,2\sigma)})\right\|^2+\left\|B(u-P_{Q(X^0,2\sigma)})\right\|^2\right)dX\leq$$

(3.16)
$$\leq c\sigma^{-2}\Bigg\{\sigma^{-2}\int\limits_{Q(X^0,2\sigma)}\|u-P_{Q(X^0,2\sigma)}\|^2dX+ + \int\limits_{Q(X^0,2\sigma)}\left\|D(u-P_{Q(X^0,2\sigma)})\right\|^2dX\Bigg\}.$$

The estimate (3.2) easily follows from (3.15) and (3.16).
From (3.2), by virtue of lemma 2.2, we deduce:

LEMMA 3.2. *If* $u \in W^2(Q, \mathbb{R}^N)$ *is a solution to the system* (3.1) *and if assumptions* (1.2), (1.3) *and* (A) *hold, then,* $\forall Q(X^0, 2\sigma) \subset\subset Q$, *with* $\sigma < 1$, *it results:*

$$\fint_{Q(X^0,\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \leq$$

$$\leq c \left[\fint_{Q(X^0,2\sigma)} \left(\|H(u)\|^{\frac{2(n+2)}{n+4}} + \left\| \frac{\partial u}{\partial t} \right\|^{\frac{2(n+2)}{n+4}} \right) dX \right]^{\frac{n+4}{n+2}} + \tag{3.17}$$

$$+ c \fint_{Q(X^0,2\sigma)} \|b\|^2 dX \,,$$

where c does not depend on σ.

Now, for reader's convenience, we recall the following result by Gehring - Giaquinta - G. Modica (see: [12], Proposition 5.1 and [2], Chap. II, Lemma 10.I)

LEMMA 3.3. *If U and G are non negative functions on Q with*

$$U \in L^r(Q), \qquad G \in L^s(Q), \qquad 1 < r < s \,,$$

and if, for every $Q(X^0, 2\sigma) \subset\subset Q$, *with* $\sigma < 1$, *it results:*

$$\fint_{Q(X^0,\sigma)} U^r dX \leq c \left(\fint_{Q(X^0,2\sigma)} U \, dX \right)^r + c \fint_{Q(X^0,2\sigma)} G^r dX, \ c > 1 \,,$$

then there exists $\epsilon \in (0, s - r]$ *such that* $U \in L^p_{loc}(Q)$, $\forall p \in [r, r + \epsilon)$ *and, for every* $Q(X^0, 2\sigma) \subset\subset Q$, *with* $\sigma < 1$, *it results:*

$$\left(\fint_{Q(X^0,\sigma)} U^p dX \right)^{1/p} \leq K \left(\fint_{Q(X^0,2\sigma)} U^r dX \right)^{1/r} + K \left(\fint_{Q(X^0,2\sigma)} G^p dX \right)^{1/p} ,$$

where the constants K and ϵ depend only on c, r, s and n.

By means of the estimate (3.17) and of the above - mentioned lemma 3.3, written for[6]

$$U = \left(\|H(u)\| + \left\|\frac{\partial u}{\partial t}\right\|\right)^{\frac{2(n+2)}{n+4}},$$

$$G = \|b\|^{\frac{2(n+2)}{n+4}}, \qquad r = \frac{n+4}{n+2}, \qquad s = \frac{n+4}{n},$$

we achieve the following

THEOREM 3.1. *If $u \in W^2(Q, \mathbb{R}^N)$ is a solution to the system (3.1) and if the assumptions (1.2), (1.3) and (A) hold, then there exists $\bar{q} \in \left(2, \frac{2(n+2)}{n}\right]$ such that, $\forall q \in [2, \bar{q})$,*

$$u \in W^q_{loc}(Q, \mathbb{R}^N),$$

and, $\forall Q(X^0, 2\sigma) \subset\subset Q$, with $\sigma < 1$, one has:

$$\left[\fint_{Q(X^0,\sigma)} \left(\|H(u)\|^q + \left\|\frac{\partial u}{\partial t}\right\|^q\right) dX\right]^{1/q} \leq$$

$$\leq c \left[\fint_{Q(X^0,2\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX\right]^{1/2} +$$

$$+ c \left[\fint_{Q(X^0,2\sigma)} \|b(X, u(X), Du(X))\|^q dX\right]^{1/q},$$

---

[6] If $u \in W^2(Q, \mathbb{R}^N)$, by virtue of lemma 3.3 of Chap. II of [13] (see also lemma 2.III of [9]), one has:

$$u \in L^{\frac{2(n+2)}{n}}\left(-T, 0, H^{1,\frac{2(n+2)}{n}}(\Omega, \mathbb{R}^N)\right),$$

and hence, for (1.3):

$$G \in L^s(Q).$$

where c does not depend on σ.

4 – Partial Hölder continuity of the vector Du

Let $u \in W^2(Q, \mathbb{R}^N)$ be a solution in Q to the non variational system

$$-a(X, u, Du, H(u)) + \frac{\partial u}{\partial t} = b(X, u, Du)\,, \tag{4.1}$$

where $a(X, u, p, \xi)$ and $b(X, u, p)$ are vectors of $\mathbb{R}^N$ with the following properties:

(4.2) *$b(X, u, p)$ is measurable in X, continuous in (u, p) and such that*

$$\|b(X, u, p)\| \le c(1 + \|u\| + \|p\|), \quad \forall u \in \mathbb{R}^N, \quad \forall p \in \mathbb{R}^{nN}$$

and for X a.e. in Q;

(4.3) *$a(X, u, p, \xi)$ is continuous in (X, u, p), of class C^1 in ξ, with derivatives $\frac{\partial a}{\partial \xi_{ij}}$[7] uniformly continuous and bounded in $Q \times \mathbb{R}^N \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2N}$ and such that:*

$$a(X, u, p, 0) = 0\,,$$

(A) there exist three positive constants α, γ and δ, with $\gamma + \delta < 1$ such that, $\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$, $\forall \tau, \eta \in \mathbb{R}^{n^2N}$ and for almost every $X \in Q$:

$$\left\| \sum_{i=1}^{n} \tau_{ii} - \alpha\big[a(X, u, p, \tau + \eta) - a(X, u, p, \eta)\big] \right\|^2 \le$$

$$\le \gamma \|\tau\|^2 + \delta \left\| \sum_{i=1}^{n} \tau_{ii} \right\|^2 ,$$

(7) $\frac{\partial a(X, u, p, \xi)}{\partial \xi_{ij}} = \left\{ \frac{\partial a^h(X, u, p, \xi)}{\partial \xi^k_{ij}} \right\}$, $h, k = 1, 2, \dots, N$.

(*B*) *there exists a non negative function* $\omega(t)$, *defined for* $t \geq 0$, *continuous, bounded, concave, non decreasing with* $\omega(0) = 0$ *such that* $\forall X$, $Y \in Q$, $\forall u, v \in \mathbb{R}^N$, $\forall p, \bar{p} \in \mathbb{R}^{nN}$ *and* $\forall \xi, \tau \in \mathbb{R}^{n^2N}$:

$$\|a(X,u,p,\xi) - a(Y,v,\bar{p},\xi)\| \leq \omega(d^2(X,Y)+ + \|u-v\|^2 + \|p-\bar{p}\|^2) \cdot \|\xi\|^{(8)},$$

$$\left\|\frac{\partial a(X,u,p,\xi)}{\partial \xi} - \frac{\partial a(X,u,p,\tau)}{\partial \xi}\right\| \leq \omega(\|\xi - \tau\|^2)^{(9)}.$$

Let us start by showing the following

LEMMA 4.1. *If* $u \in W^2(Q, \mathbb{R}^N)$ *is a solution to the system* (4.1) *and if assumptions* (4.2) *and* (4.3) *hold, then,* $\forall Q(X^0,\sigma) \subset\subset Q$, *with* $\sigma < 2$, $\forall \tau \in (0,1)$ *and* $\forall \epsilon \in (0,n]$, *it results:*

$$\Phi(u,X^0,\tau\sigma) \leq A\Phi(u,X^0,\sigma)\Bigg\{\tau^{n+2-\epsilon} + \sigma^2 + \left[\omega(c\sigma^{-n}\Phi(u,X^0,\sigma))\right]^{1-\frac{2}{q}} + \left[\omega\left(\fint_{Q(X^0,\sigma)} \|H(u) - (H(u))_{Q(X^0,\sigma)}\|^2 dX\right)\right]^{1-\frac{2}{q}}\Bigg\}, \tag{4.4}$$

where $q \in (2, \bar{q})^{(10)}$ *and*

$$\Phi(u,X^0,\sigma) = \sigma^{n+2} + \int_{Q(X^0,\sigma)} \left(\|u\|^2 + \|Du\|^2 + \|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX. \tag{4.5}$$

(8) $d(X,Y) = \max\left\{\|x-y\|, |t-\tau|^{1/2}\right\}$, $X = (x,t)$, $Y = (y,\tau)$.

(9) $\dfrac{\partial a(X,u,p,\eta)}{\partial \xi} = \left\{\dfrac{\partial a(X,u,p,\eta)}{\partial \xi_{ij}}\right\}$, $i,j = 1,2,\dots,n$.

(10) $\bar{q}$ is the constant (> 2) that appears in theorem 3.1.

PROOF. Let us fix the cylinder $Q(X^0,\sigma)$, with $\sigma < 1$, such that $Q(X^0,2\sigma) \subset\subset Q$ and let us set:

$$\frac{\partial\tilde{a}(X,u,p,\eta)}{\partial\xi_{ij}} = \left\{\int_0^1 \frac{\partial a^h(X,u,p,t\eta)}{\partial\xi_{ij}^k}dt\right\}, \quad h,k = 1,2,\dots,N,$$

$$\frac{\partial\tilde{a}(X,u,p,\eta)}{\partial\xi} = \left\{\frac{\partial\tilde{a}(X,u,p,\eta)}{\partial\xi_{ij}}\right\}, \qquad i,j = 1,2,\dots,n.$$

In $Q(X^0,\sigma)$ the system (4.1) can also be written in the following form[11]

$$(4.6)\qquad \begin{aligned} &-a(X^0,u_\sigma,(Du)_\sigma,H(u)) + \frac{\partial u}{\partial t} = \Big[a(X,u,Du,H(u)) - \\ &-a(X^0,u_\sigma,(Du)_\sigma,H(u))\Big] + b(X,u,Du) = \mathcal{B}_1 + b(X,u,Du). \end{aligned}$$

On the other hand, denoting by $a^h(X^0,u_\sigma,(Du)_\sigma,\eta)$, $h = 1,2,..,N$, the hth component of the vector $a\ (X^0,u_\sigma,(Du)_\sigma,\eta)$, one gets:

$$\begin{aligned} a^h(X^0,u_\sigma,(Du)_\sigma,\eta) &= a^h(X^0,u_\sigma,(Du)_\sigma,\eta) - a^h(X^0,u_\sigma,(Du)_\sigma,0) = \\ &= \sum_{ij=1}^n \sum_{k=1}^N \int_0^1 \frac{\partial a^h(X^0,u_\sigma,(Du)_\sigma,t\eta)}{\partial\xi_{ij}^k}dt\eta_{ij}^k, \quad h = 1,2,\dots,N, \end{aligned}$$

from which

$$a(X^0,u_\sigma,(Du)_\sigma,\eta) = \sum_{ij=1}^n \frac{\partial\tilde{a}(X^0,u_\sigma,(Du)_\sigma,\eta)}{\partial\xi_{ij}}\eta_{ij}.$$

Hence, from (4.6), the system (4.1) can be written in the following form:

$$-\sum_{ij=1}^n \frac{\partial\tilde{a}(X^0,u_\sigma,(Du)_\sigma,H(u))}{\partial\xi_{ij}}D_{ij}u + \frac{\partial u}{\partial t} = \mathcal{B}_1 + b(X,u,Du),$$

(11) $u_\sigma = u_{Q(X^0,\sigma)}$, $(Du)_\sigma = (Du)_{Q(X^0,\sigma)}$, $\big(H(u)\big)_\sigma = \big(H(u)\big)_{Q(X^0,\sigma)}$.

or, equivalently

$$-\sum_{ij=1}^{n}\frac{\partial\tilde{a}\Big(X^0,u_\sigma,(Du)_\sigma,\big(H(u)\big)_\sigma\Big)}{\partial\xi_{ij}}D_{ij}u+\frac{\partial u}{\partial t}=$$

$$=\sum_{ij=1}^{n}\left(\frac{\partial\tilde{a}(X^0,u_\sigma,(Du)_\sigma,H(u))}{\partial\xi_{ij}}-\frac{\partial\tilde{a}\Big(X^0,u_\sigma,(Du)_\sigma,\big(H(u)\big)_\sigma\Big)}{\partial\xi_{ij}}\right)\cdot \tag{4.7}$$

$$\cdot D_{ij}u+\mathcal{B}_1+b(X,u,Du)=\mathcal{B}_2+\mathcal{B}_1+b(X,u,Du)\,.$$

Letting w to be the solution in $Q(X^0,\sigma)$ to the Cauchy - Dirichlet problem

$$\begin{cases} w\in W_0^2(Q(X^0,\sigma),\mathbb{R}^N) \\ -\sum_{ij=1}^{n}\frac{\partial\tilde{a}\left(X^0,u_\sigma,(Du)_\sigma,\left(H(u)\right)_\sigma\right)}{\partial\xi_{ij}}D_{ij}w+\frac{\partial w}{\partial t}=\mathcal{B}_2+\mathcal{B}_1\,, \end{cases} \tag{4.8}$$

it results, in $Q(X^0,\sigma)$, $u=w+v$, where $v\in W^2(Q(X^0,\sigma),\mathbb{R}^N)$ is solution to the linear system

$$-\sum_{ij=1}^{n}\frac{\partial\tilde{a}\Big(X^0,u_\sigma,(Du)_\sigma,\big(H(u)\big)_\sigma\Big)}{\partial\xi_{ij}}D_{ij}v+\frac{\partial v}{\partial t}=b(X,u,Du)\,. \tag{4.9}$$

Now the following estimate (see [1]) holds for v:

$$\int_{Q(X^0,\tau\sigma)}\left(\|H(v)\|^2+\left\|\frac{\partial v}{\partial t}\right\|^2\right)dX\le$$

$$\le c\tau^{n+2}\int_{Q(X^0,\sigma)}\left(\|H(v)\|^2+\left\|\frac{\partial v}{\partial t}\right\|^2\right)dX+ \tag{4.10}$$

$$+c\int_{Q(X^0,\sigma)}\|b(X,u,Du)\|^2dX,\quad \forall\tau\in(0,1)\,,$$

from which, setting

$$F(u,X^0,\sigma)=\sigma^{n+2}+\int_{Q(X^0,\sigma)}(\|u\|^2+\|Du\|^2)dX\,,$$

in virtue of assumption (4.2), it follows:

$$\begin{aligned}(4.11)\qquad &\int_{Q(X^0,\tau\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX \leq \\ &\leq c\tau^{n+2} \int_{Q(X^0,\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX + \\ &+ cF(u, X^0, \sigma), \qquad \forall \tau \in (0,1)\,,\end{aligned}$$

where c does not depend on X^0, τ and σ.

On the other hand, thanks to lemma 2.3, we have:

$$(4.12)\quad F(u, X^0, \tau\sigma) \leq c\tau^{n+2} F(u, X^0, \sigma) + c\sigma^2 \Phi(u, X^0, \sigma), \quad \forall \tau \in (0,1)\,.$$

The estimates (4.11) and (4.12) allow us to apply the lemma 1.II of Chap. I of [2] and hence we obtain $\forall \tau \in (0,1)$ and $\forall \epsilon \in (0, n]$

$$\begin{aligned}(4.13)\qquad &\int_{Q(X^0,\tau\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX \leq \\ &\leq c\tau^{n+2-\epsilon} \int_{Q(X^0,\sigma)} \left(\|H(v)\|^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) dX + \\ &+ c\Phi(u, X^0, \sigma)(\tau^{n+2-\epsilon} + \sigma^2)\,.\end{aligned}$$

As for w the following estimate holds:

$$\begin{aligned}(4.14)\qquad &\int_{Q(X^0,\sigma)} \left(\|H(w)\|^2 + \left\|\frac{\partial w}{\partial t}\right\|^2\right) dX \leq \\ &\leq c \int_{Q(X^0,\sigma)} \|\mathcal{B}_1\|^2 dX + c \int_{Q(X^0,\sigma)} \|\mathcal{B}_2\|^2 dX\,.\end{aligned}$$

Now let us estimate the integrals in the right hand side of (4.14).

From assumption (4.3) - (B), from the L^q_{loc}-regularity results given in theorem 3.1 and in virtue of the Poincarè estimates (2.6) and (2.7) of

[9], we get:

$$
\begin{aligned}
&\int_{Q(X^0,\sigma)} \|\mathcal{B}_1\|^2 dX \leq \\
&\leq \int_{Q(X^0,\sigma)} \omega^2\Big(\sigma^2 + \|u-u_\sigma\|^2 + \|Du-(Du)_\sigma\|^2\Big)\|H(u)\|^2 dX \leq \\
&\leq c\sigma^{n+2}\Bigg(\ \fint_{Q(X^0,\sigma)} \|H(u)\|^q dX\Bigg)^{2/q}\Bigg(\ \fint_{Q(X^0,\sigma)} \omega\Big(\sigma^2+\|u-u_\sigma\|^2+ \\
&+\|Du-(Du)_\sigma\|^2\Big)dX\Bigg)^{1-\frac{2}{q}} \leq c\sigma^{n+2}\Bigg\{\ \fint_{Q(X^0,2\sigma)} \Bigg(\|H(u)\|^2+ \\
&+\left\|\frac{\partial u}{\partial t}\right\|^2\Bigg)dX + \Bigg(\ \fint_{Q(X^0,2\sigma)} \|b\|^q dX\Bigg)^{2/q}\Bigg\}\cdot \\
&\cdot\Bigg[\omega\Bigg(\ \fint_{Q(X^0,\sigma)} \Big(\sigma^2+\|u-u_\sigma\|^2+\|Du-(Du)_\sigma\|^2\Big)dX\Bigg)\Bigg]^{1-\frac{2}{q}} \leq \\
&\leq c\Bigg\{\int_{Q(X^0,2\sigma)} \Bigg(\|H(u)\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2\Bigg)dX+ \\
&+\sigma^{(n+2)\left(1-\frac{2}{q}\right)}\Bigg(\int_{Q(X^0,2\sigma)} \|b\|^q dX\Bigg)^{\frac{2}{q}}\Bigg\}\cdot\Big[\omega\big(c\sigma^{-n}\Phi(u,X^0,\sigma)\big)\Big]^{1-\frac{2}{q}}.
\end{aligned}
\tag{4.15}
$$

From assumption (4.2) it follows:

$$
\begin{aligned}
\Bigg(\int_{Q(X^0,2\sigma)} \|b\|^q dX\Bigg)^{\frac{2}{q}} \leq c\Bigg\{\sigma^{\frac{2}{q}(n+2)} + \Bigg(\int_{Q(X^0,2\sigma)} \|u\|^q dX\Bigg)^{\frac{2}{q}} + \\
+\Bigg(\int_{Q(X^0,2\sigma)} \|Du\|^q dX\Bigg)^{\frac{2}{q}}\Bigg\}.
\end{aligned}
\tag{4.16}
$$

On the other hand, since $2 < q < \dfrac{2(n+2)}{n}$, using a well-known embedding result (see e.g. [13], Chap. II, Lemma 3.3), one gets:

$$\left(\int_{Q(X^0,2\sigma)} \|u\|^q dX\right)^{\frac{2}{q}} \leq$$

$$\leq c\sigma^{(n+2)\left(\frac{2}{q}-1\right)} \left\{\sigma^4 \int_{Q(X^0,2\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX + \right. \tag{4.17}$$

$$\left. + \int_{Q(X^0,2\sigma)} \|u\|^2 dX \right\},$$

$$\left(\int_{Q(X^0,2\sigma)} \|Du\|^q dX\right)^{\frac{2}{q}} \leq$$

$$\leq c\sigma^{\frac{2}{q}(n+2)-n} \left\{\int_{Q(X^0,2\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX + \right. \tag{4.18}$$

$$\left. + \int_{Q(X^0,2\sigma)} \|u\|^2 dX \right\}.$$

Then from (4.16), (4.17) and (4.18) we deduce

$$\sigma^{(n+2)\left(1-\frac{2}{q}\right)} \left(\int_{Q(X^0,2\sigma)} \|b\|^q dX\right)^{\frac{2}{q}} \leq$$

$$\leq c \left\{\sigma^{n+2} + (1+\sigma^2) \int_{Q(X^0,2\sigma)} \|u\|^2 dX + \right.$$

$$\left. +(\sigma^2+\sigma^4) \int_{Q(X^0,2\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX \right\},$$

from which, recalling the meaning of $\Phi(u, X^0, \sigma)$ and that $\sigma < 1$, it follows:

$$\sigma^{(n+2)\left(1-\frac{2}{q}\right)} \left(\int\limits_{Q(X^0,2\sigma)} \|b\|^q dX \right)^{\frac{2}{q}} \leq c\Phi(u, X^0, 2\sigma) . \tag{4.19}$$

Hence (4.15) and (4.19) ensure that

$$\int\limits_{Q(X^0,\sigma)} \|\mathcal{B}_1\|^2 dX \leq c\Phi(u, X^0, 2\sigma) \left[\omega\left(c\sigma^{-n}\Phi(u, X^0, 2\sigma)\right)\right]^{1-\frac{2}{q}} . \tag{4.20}$$

Similarly we obtain:

$$\begin{aligned} &\int\limits_{Q(X^0,\sigma)} \|\mathcal{B}_2\|^2 dX \leq \\ &\leq c\Phi(u, X^0, 2\sigma) \left[\omega \left(\fint\limits_{Q(X^0,2\sigma)} \left\| H(u) - (H(u))_{2\sigma} \right\|^2 dX \right) \right]^{1-\frac{2}{q}} \end{aligned} \tag{4.21}$$

and finally from (4.14), (4.20) and (4.21), we get

$$\begin{aligned} &\int\limits_{Q(X^0,\sigma)} \left(\|H(w)\|^2 + \left\| \frac{\partial w}{\partial t} \right\|^2 \right) dX \leq \\ &\leq c\Phi(u, X^0, 2\sigma) \left\{ \left[\omega\left(c\sigma^{-n}\Phi(u, X^0, 2\sigma)\right)\right]^{1-\frac{2}{q}} + \right. \\ &\left. + \left[\omega \left(\fint\limits_{Q(X^0,2\sigma)} \left\| H(u) - (H(u))_{2\sigma} \right\|^2 dX \right) \right]^{1-\frac{2}{q}} \right\} . \end{aligned} \tag{4.22}$$

Since $u = v + w$, from the estimates (4.13) for v and (4.22) for w, we obtain $\forall\tau \in (0,1)$, $\forall\epsilon \in (0,n]$

$$\int_{Q(X^0,\tau\sigma)} \left(\|H(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX \leq$$

$$\leq c\Phi(u,X^0,2\sigma)\Bigg\{\tau^{n+2-\epsilon} + \sigma^2 + \left[\omega(c\sigma^{-n}\Phi(u,X^0,2\sigma))\right]^{1-\frac{2}{q}} + \tag{4.23}$$

$$+ \left[\omega\left(\fint_{Q(X^0,2\sigma)} \left\|H(u) - (H(u))_{2\sigma}\right\|^2 dX\right)\right]^{1-\frac{2}{q}}\Bigg\}.$$

Adding the estimates (4.23) and (4.12) we achieve, $\forall\tau \in (0,1)$ and $\forall\epsilon \in (0,n]$:

$$\Phi(u,X^0,\tau\sigma) \leq c\Phi(u,X^0,2\sigma)\Bigg\{\tau^{n+2-\epsilon} + \sigma^2 + \left[\omega(c\sigma^{-n}\Phi(u,X^0,2\sigma))\right]^{1-\frac{2}{q}} +$$

$$+ \left[\omega\left(\fint_{Q(X^0,2\sigma)} \left\|H(u) - (H(u))_{2\sigma}\right\|^2 dX\right)\right]^{1-\frac{2}{q}}\Bigg\}.$$

Finally this estimate is trivially true for $\tau \in [1,2)$.
Thus the lemma is proved.
Let us set

$$Q_1 = \left\{X \in Q\colon \lim_{\sigma\to 0}{}'' \fint_{Q(X,\sigma)} \left\|H(u) - (H(u))_{Q(X,\sigma)}\right\|^2 dY > 0\right\},$$

$$Q_2 = \left\{X \in Q\colon \lim_{\sigma\to 0}{}' \sigma^{-n}\Phi(u,X,\sigma) > 0\right\}.$$

It results:

$$\text{meas } Q_1 = 0$$

and (see [11], Theorem 2)

$$\mathcal{H}_n(Q_2) = 0,$$

where $\mathcal{H}_n$ is the n-dimensional Hausdorff measure with respect to the parabolic metric $d(X,Y)$.

Hence the set $Q_1 \cup Q_2$ has measure zero.

Now reasoning exactly as in theorem 5.I of [3] it is easy to prove

LEMMA 4.2. *If $u \in W^2(Q, \mathbb{R}^N)$ is a solution to the system (4.1) and if the assumptions (4.2) and (4.3) hold, then for every fixed $\epsilon \in (0,1)$, it is possible to associate to every $X^0 \in Q \setminus (Q_1 \cup Q_2)$ a cylinder $Q(X^0, R_{X^0}) \subset Q \setminus Q_2$ and a positive number σ_ϵ such that*

$$\Phi(u, Y, \tau\sigma_\epsilon) \leq (1+A)\tau^{n+2-2\epsilon}\Phi(u, Y, \sigma_\epsilon), \ \forall \tau \in (0,1), \ \forall Y \in Q(X^0, R_{X^0})$$

and hence:

$$(4.24) \qquad \begin{aligned} &H(u) \in L^{2,n+2-2\epsilon}(Q(X^0, R_{X^0}), d, \mathbb{R}^{n^2N}), \\ &\frac{\partial u}{\partial t} \in L^{2,n+2-2\epsilon}(Q(X^0, R_{X^0}), d, \mathbb{R}^N)^{(12)}, \end{aligned}$$

$$(4.25) \qquad Du \in \mathcal{L}^{2,n+4-2\epsilon}(Q(X^0, R_{X^0}), d, \mathbb{R}^{nN})^{(13)}.$$

(12) $L^{2,n+2-2\epsilon}(Q(X^0, R_{X^0}), d, \mathbb{R}^k)$ (k integer > 0) denotes the space of those functions $v \in L^2(Q(X^0, R_{X^0}), \mathbb{R}^k)$ such that

$$\sup_{\substack{Y \in Q(X^0, R_{X^0}) \\ \rho > 0}} \left\{ \rho^{-(n+2-2\epsilon)} \int_{Q(Y,\rho)} \|v\|^2 dX \right\} < +\infty.$$

(13) $\mathcal{L}^{2,n+4-2\epsilon}(Q(X^0, R_{X^0}), d, \mathbb{R}^{nN})$ denotes the space of those functions $v \in L^2(Q(X^0, R_{X^0}), \mathbb{R}^{nN})$ for which

$$\sup_{\substack{Y \in Q(X^0, R_{X^0}) \\ \rho > 0}} \left\{ \rho^{-(n+4-2\epsilon)} \int_{Q(Y,\rho)} \|v - v_{Q(Y,\rho)}\|^2 dX \right\} < +\infty.$$

If $v \in \mathcal{L}^{2,n+4-2\epsilon}(Q(X^0, R_{X^0}), d, \mathbb{R}^{nN})$, then v is $(1-\epsilon)$ - Hölder continuous (with respect to the parabolic metric) in $Q(X^0, R_{X^0})$ (see [10]).

From lemma 4.2 the following result of partial Hölder continuity for Du easily follows:

THEOREM 4.1. *If* $u \in W^2(Q, \mathbb{R}^N)$ *is a solution to the system* (4.1) *and if the hypotheses* (4.2) *and* (4.3) *are fulfilled, then there exists a set* Q_0*, closed in* Q*, with*[14]*:*

$$Q_2 \subset Q_0 \subset Q_1 \cup Q_2$$

such that

$$Du \in C^{0,\alpha}(Q \setminus Q_0, d, \mathbb{R}^{nN}), \qquad \forall \alpha < 1 .$$

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(14) In particular, meas $Q_0 = 0$.

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