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On a singular limit for a class of solutions of the simplified Wheeler-Dewitt equation with a massless single scalar field

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RIASSUNTO: *Si ottiene una soluzione ψ_∞ di una equazione di Schrödinger come limite, quando $p \to \infty$, di $\widetilde{\psi}_p(x,y) = e^{ic_p^2 y}\, x^{c_p}\, \psi_p(x, c_p y)$, $c_p = \frac{1}{2}(p-1)$, $p \neq 1$, essendo ψ_p è una soluzione particolare dell'equazione di Wheeler–DeWitt. Si studia il comportamento asintotico (in y) di ψ_∞.*

ABSTRACT: *In this paper we consider a special class ψ_p of solutions of the Wheeler–DeWitt equation (cf. [5], [3], [1]) and we study the limit, when $p \to \infty$, of the associated functions $\widetilde{\psi}_p(x,y) = e^{ic_p^2 y}\, x^{c_p}\, \psi_p(x, c_p y)$, $c_p = \frac{1}{2}(p-1)$, $p \neq 1$. We prove that, in an appropriate sense, $\widetilde{\psi}_p \underset{p\to\infty}{\longrightarrow} \psi_\infty$ and ψ_∞ verifies a Schrödinger equation. Finally we study the decay (in y) properties of ψ_∞.*

KEY WORDS: *Wheeler-DeWitt equation – Singular limit.*

A.M.S. CLASSIFICATION: 35B25 – 35Q40 – 83F05

1 – Introduction and main results

The Wheeler–DeWitt equation in the minisuperspace model with a massless single scalar field $y \in \mathbb{R}$ can be written as follows (cf [5], [3], [1])

$$\frac{\partial^2 \psi}{\partial y^2} - x^2 \frac{\partial^2 \psi}{\partial x^2} - px \frac{\partial \psi}{\partial x} + x^4 \psi = 0 \; , \tag{1.1}$$

where $x \in \mathbb{R}_+$ is a scale factor, $p \in \mathbb{R}$ is a given constant (p reflects the factor-ordering ambiguity) and $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ is the wave function of the universe for the minisuperspace model. In quantum gravity, the dynamics of universe is determined by its possible quantized states which are described by a wave function ψ which depends on the geometry of three dimensional compact manifolds (here represented by x, radius of universe) and the value of matter fields on these manifolds (here represented by a single scalar field y, without mass). The Cauchy problem (for data at $y = 0$) for the equation (1.1) has been studied in [1] by means of the following transformation: $u(z, y) = x^{\frac{p-1}{2}}\psi(x, y)$, with $z = \log x \in \mathbb{R}$. The equation (1.1) becomes, with $V(z) = e^{4z}$,

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + \frac{1}{4}(p-1)^2 u + Vu = 0 \,. \tag{1.2}$$

Then, with

$$H_V^1 = \left\{ v \in H^1(\mathbb{R}) \mid V^{\frac{1}{2}} v \in L^2(\mathbb{R}) \right\}$$

and

$$X = \left\{ v \in H_V^1 \mid \frac{d^2 v}{dz^2} - Vv \in L^2 \right\} ,$$

we proved in [1] the following results:

THEOREM 1. *Assume* $(u_0, u_1) \in X \times H_V^1$. *Then there exists an unique solution* $u \in C^2(\mathbb{R}; L^2) \cap C^1(\mathbb{R}; H_V^1) \cap C(\mathbb{R}; X)$ *of the equation* (1.2) *such that* $u(z, 0) = u_0(z)$, $\frac{\partial u}{\partial y}(z, 0) = u_1(z)$, $z \in \mathbb{R}$.

THEOREM 2. *Assume* $(u_0, u_1) \in X_1 \times X$, *where* $X_1 = \{v \in X \mid \frac{d^2 v}{dz^2} - Vv \in H_V^1\}$. *Then we have*

$$\lim_{y \to \infty} \int_{\mathbb{R}} V(z) \left[|u|^2(z, y) + \left| \frac{\partial u}{\partial y} \right|^2 (z, y) \right] dz = 0 \,,$$

where u *is the solution of the corresponding Cauchy problem for the equation* (1.2).

In this paper we will consider a special class of solutions ψ_p of the equation (1.1) depending on the parameter $p\,(p \neq 1)$ verifying, for example,

$$\frac{\partial \psi_p}{\partial y}(x,0) = -\frac{i}{2}(p-1)\psi_p(x,0) = -\frac{i}{2}(p-1)x^{-\frac{1}{2}(p-1)}\xi_p(x).$$

We prove that, if $\xi_p \longrightarrow \xi_{\pm\infty}\,(p \to \pm\infty)$ in an appropriate sense, then $\psi_p \longrightarrow \xi_{\pm\infty}\,(p \to \pm\infty)$, where $\psi_{\pm\infty}$ are solutions of a singular Schrödinger equation with initial (that is for $y = 0$) data $\xi_{\pm\infty}$. Then we study the decay (in y) properties of the solutions of this Schrödinger equation. We obtain a decay for $x^2\psi_{\pm\infty}$ in the $L^q\left(\mathbf{R}_+; \frac{dx}{x}\right)$ norm for each $q \in [2, +\infty]$, that is, for large values of y, the wave functions $\psi_{\pm\infty}$ are very small for large values of x. Now, with domain $D(A_p) = X$, $p \in \mathbb{R}$, we introduce in $L^2(\mathbb{R})$ the following operator defined by

$$A_p v = -\frac{d^2 v}{dz^2} + c_p^2 v + Vv, \text{ where } c_p = \frac{1}{2}(p-1)\,. \tag{1.3}$$

This operator which is associated to the form

$$a(u,v) = \int_{\mathbf{R}} \frac{du}{dz}\frac{d\bar{v}}{dz}\,dz + c_p^2 \int_{\mathbf{R}} u\bar{v}\,dz + \int_{\mathbf{R}} Vu\bar{v}\,dz\,, \qquad u,v \in H_V^1\,, \tag{1.4}$$

is self-adjoint in $L^2(\mathbb{R})$ (cf [6], ch. VI). It is easy to see that $\rho(A_p)$ (resolvent set of A_p) is contained in $]-\infty, c_p^2[$. Furthermore, theorem 2 implies that if $\sigma_p(A_p)$ is the point spectrum of A_p then $\sigma_p(A_p) \cap [c_p^2, +\infty[= \emptyset$. Finally, by applying the theorem 5.7.1 of [8], we easily obtain that $\sigma_e(A_p) \supset [c_p^2, +\infty[$, where $\sigma_e(A_p)$ is the essential spectrum of A_p. Hence, denoting by $\sigma(A)$ the spectrum of A_p, we obtain:

PROPOSITION 1. *We have* $\sigma(A_p) = \sigma_e(A_p) = [c_p^2, +\infty[$ *and* $\sigma_p(A_p) = \emptyset$.□

Now, for each $p \neq 1$, let us consider a function $\xi_p : \mathbb{R}_+ \to \mathbb{C}$ such that the function $\tilde{\xi}_p : \mathbb{R} \to \mathbb{C}$ defined by

$$\tilde{\xi}_p(z) = \xi_p(e^z)\,,\ z \in \mathbb{R}\,, \tag{1.5}$$

verify

(1.6) $\widetilde{\xi}_p \in D(A_p) = X$, $\widetilde{\xi}_p \underset{p\to+\infty}{\longrightarrow} \widetilde{\xi}_{+\infty}$ (respectively $\widetilde{\xi}_p \underset{p\to-\infty}{\longrightarrow} \widetilde{\xi}_{-\infty}$) in X ,

X with the A_1 graph norm through this paper. Let us also consider, for each $p \neq 1$, $\delta_p \in \mathbb{R}$, $0 < \delta_p \leq 1$, such that $\delta_p \underset{p\to\infty}{\longrightarrow} 1$ and let u_p be the solution of the equation (1.2) for the initial data

$$u_p(z,0) = \widetilde{\xi}_p(z),\ \frac{\partial u_p}{\partial y}(z,0) = -i\,\delta_p\,c_p\,\widetilde{\xi}_p(z)\,, \qquad z \in \mathbb{R}\ . \tag{1.7}$$

This means, with $u_p(z,y) = x^{c_p}\,\psi_p(x,y)$, $z = \log x$:

$$x^{c_p}\,\psi_p(x,0) = \xi_p(x),\ x^{c_p}\,\frac{\partial \psi_p}{\partial y}(x,0) = -i\,\delta_p\,c_p\,\xi_p(x)\,, \qquad x \in \mathbb{R}_+\ .$$

As a consequence of a result of H.O. Fattorini (cf [2], ch. VII) we will prove the following theorem, where ∞ means $+\infty$ (or $-\infty$, respectively):

Theorem 3. *Let, for each $p \neq 1$, v_p defined by*

$$v_p(z,y) = e^{ic_p^2 y}\,u_p(z,c_p y)\ . \tag{1.8}$$

Then, for each $M \in \mathbb{R}_+$, we have

$$v_p \underset{p\to\infty}{\longrightarrow} v_\infty \ \text{in}\ C([-M,M];X)\ ,$$

where $v_\infty \in C^1(\mathbb{R};L^2) \cap C(\mathbb{R};X)$ is the unique solution of the Schrödinger equation

$$i\,\frac{\partial v_\infty}{\partial y} + \frac{1}{2}\Big(\frac{\partial^2 v_\infty}{\partial z^2} - V v_\infty\Big) = 0\,, \qquad z \in \mathbb{R},\ y \in \mathbb{R}\ , \tag{1.9}$$

with the initial condition

$$v_\infty(z,0) = \widetilde{\xi}_\infty(z)\,,\ z \in \mathbb{R}\ . \tag{1.10}$$

Now, let $\widetilde{\psi}_p(x,y) = v_p(z,y) = e^{ic_p^2 y} x^{c_p} \psi_p(x, c_p y)$, $p \neq 1$, with $z = \log x$, and put

$$\psi_\infty(x,y) = v_\infty(z,y) \text{ and } \xi_\infty(x) = \widetilde{\xi}_\infty(z) \ .$$

We obtain

$$\begin{cases} i\,\dfrac{\partial \psi_\infty}{\partial y} + \dfrac{1}{2}\Big(x^2\,\dfrac{\partial^2 \psi_\infty}{\partial x^2} + x\,\dfrac{\partial \psi_\infty}{\partial x} - x^4\,\psi_\infty\Big) = 0\,,\ x \in \mathbb{R}_+,\ y \in \mathbb{R}\ , \\ \psi_\infty(x,0) = \xi_\infty(x)\,,\ x \in \mathbb{R}_+\ , \end{cases}$$

and $\widetilde{\psi}_p \underset{p\to\infty}{\longrightarrow} \psi_\infty$ in $C([-M,M]; L^2(\mathbb{R}_+; d\mu))$ with $d\mu = \frac{1}{x}\,dx$, $\forall M > 0$.

Finally, we prove the following decay result for the equation (1.9):

THEOREM 4. *Assume* $v_0 \in D(A_1^2) = \{u \in X \mid \dfrac{d^2u}{dz^2} - Vu \in X\}$ *and let* $v \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; D(A_1^2))$ *be the unique solution of the Schrödinger equation* (1.9) *for the initial* ($y = 0$) *data* v_0. *Then, for each* $q \in [2, +\infty]$, *we have*

$$\lim_{y\to\infty} \Big\| V^{\frac{1}{2}}(\widehat{z})\, v(\widehat{z}, y) \Big\|_{L^q(\mathbb{R})} = 0\ .$$

We can speculate that the approximation result in theorem 3 is related to the theory of the "small quantum subsystems of the universe" (cf [9]).

2 – Proof of Theorem 3

Let us consider, for each $p \neq 1$, the function v_p defined by (1.8) and put $\varepsilon_p = \dfrac{1}{2c_p^2} \underset{p\to\infty}{\longrightarrow} 0$. It is easy to see, by (1.2), (1.7) and theorem 1, that $v_p \in C^2(\mathbb{R}; L^2) \cap C^1(\mathbb{R}; H_V^1) \cap C(\mathbb{R}; X)$ and that v_p and $w_p = e^{-ic_p^2 y} v_p$

verify

$$(2.1) \quad c_p^{-2}\,\frac{\partial^2 w_p}{\partial y^2} - \frac{\partial^2 w_p}{\partial z^2} + c_p^2\, w_p + V\, w_p = 0\,,\ z \in \mathbb{R},\ y \in \mathbb{R}\,,$$

$$(2.2) \quad \begin{cases} i\,\dfrac{\partial v_p}{\partial y} - \varepsilon_p\,\dfrac{\partial^2 v_p}{\partial y^2} + \dfrac{1}{2}\Big(\dfrac{\partial^2 v_p}{\partial z^2} - V v_p\Big) = 0\,,\ z \in \mathbb{R},\ y \in \mathbb{R}\,, \\ v_p(z,0) = \widetilde{\xi}_p(z),\ \varepsilon_p\,\dfrac{\partial v_p}{\partial y}(z,0) = \dfrac{i}{2}(1-\delta_p)\,\widetilde{\xi}_p(z)\,,\ z \in \mathbb{R}\,. \end{cases}$$

Furthermore, we have $-\dfrac{d^2}{dz^2} + V = A_1$ with domain $D(A_1) = X$ in $L^2(\mathbb{R})$ which is (cf § 1) self-adjoint and $\sigma(A_1) = [0,+\infty[$ (prop. 1). Hence, we can apply the results of H.O. FALTORINI in the ch. VII of [2] (since $\delta_p \underset{p\to\infty}{\longrightarrow} 1$) and we obtain

$$v_p \underset{p\to\infty}{\longrightarrow} v_\infty \text{ in } C([-M,M];L^2),\ \forall M > 0\,,$$

where $v_\infty \in C^1(\mathbb{R};L^2)\cap C(\mathbb{R};X)$ is the unique solution of the Schrödinger equation (1.9) with the initial data (1.10).

Furthermore, since $\widetilde{\xi}_p \in D(A_1) = X$ and $\widetilde{\xi}_p \underset{p\to\infty}{\longrightarrow} \widetilde{\xi}_\infty$ in $D(A_1)$ for the graph norm (and hence $v_p(\widehat{z},0) \underset{p\to\infty}{\longrightarrow} \widetilde{\xi}_\infty$, $\varepsilon_p\,\dfrac{\partial v_p}{\partial y}(\widehat{z},0) \underset{p\to\infty}{\longrightarrow} 0$ in $D(A_1)$ for the graph norm), we can apply the operator commutation techniques used in the proof of theorem 1.1 in [7] in order to obtain

$$A_1\, v_p \underset{p\to\infty}{\longrightarrow} A_1\, v_\infty \text{ in } C([-M,M];L^2),\ \forall M > 0\,,$$

and this achieves the proof of theorem 3. □

3 – Proof of Theorem 4

For $u \in D(A_1^2)$ we get $-\dfrac{d^3u}{dz^3} + V\dfrac{du}{dz} + 4Vu = \dfrac{d}{dz}(A_1 u) \in L^2$. In particular, if $\varphi \in \mathcal{D}(\mathbb{R})$ we deduce, by integration by parts,

$$\operatorname{Re}\int_{\mathbb{R}} \frac{d\varphi}{dz}\Big(-\frac{d^3\overline{\varphi}}{dz^3} + V\frac{d\overline{\varphi}}{dz} + 4V\overline{\varphi}\Big)\,dz + 8\int_{\mathbb{R}} V|\varphi|^2\,dz =$$

$$= \int_{\mathbb{R}} \Big|\frac{d^2\varphi}{dz^2}\Big|^2\,dz + \int_{\mathbb{R}} V\Big|\frac{d\varphi}{dz}\Big|^2\,dz\,,$$

and so, by the density of $\mathcal{D}(\mathbb{R})$ in $D(A_1^2)$ with the norm $(\|u\|_{H^1}^2 + \|V^{\frac{1}{2}}u\|_{L^2}^2 + \|A_1 u\|_{H^1}^2)^{\frac{1}{2}}$ (proof similar to the one of lemma 3.1 in [1]) and by standard arguments, the quantity

$$E\Big(\frac{du}{dz}\Big) = \frac{1}{4}\Big[\int_{\mathbb{R}}\Big|\frac{d^2u}{dz^2}\Big|^2 dz + \int_{\mathbb{R}} V\Big|\frac{du}{dz}\Big|^2 dz\Big]$$

is finite. Hence $\frac{du}{dz} \in H_V^1$ and so $u \in H^2$, $Vu \in L^2$, $-\frac{d^3u}{dz^3} + V\frac{du}{dz} \in L^2$. We conclude that $\frac{du}{dz} \in D(A_1)$.

We start with the proof of the

PROPOSITION 2. *Under the assumptions of theorem 4 we have*

$$\lim_{y\to\infty}\Big\|V^{\frac{1}{2}}(\hat{z})\, v(\hat{z},y)\Big\|_{L^2(\mathbb{R})} = 0\,. \tag{3.1}$$

PROOF. We have, with $V(z) = e^{4z}$,

$$i\frac{\partial v}{\partial y} + \frac{1}{2}\Big(\frac{\partial^2 v}{\partial z^2} - Vv\Big) = 0\,. \tag{3.2}$$

Multiplying the equation (3.2) by $\frac{\partial \overline{v}}{\partial z}$, taking the real part and integrating in $z \in \mathbb{R}$, we obtain (since $\frac{d}{dz}V = 4V$ and the integration by parts is justified by $\frac{dv}{dz} \in D(A_1)$):

$$\operatorname{Re} i\int_{\mathbb{R}} \frac{\partial v}{\partial y}\frac{\partial \overline{v}}{\partial z}\, dz + \int_{\mathbb{R}} V|v|^2\, dz = 0\,. \tag{3.3}$$

Since $v \in D(A_1^2)$ we get $\frac{\partial v}{\partial y} \in D(A_1)$ and we can write

$$\operatorname{Re} i\int_{\mathbb{R}} \frac{\partial v}{\partial y}\frac{\partial \overline{v}}{\partial z}\, dz = -\frac{d}{dy}\operatorname{Im}\int_{\mathbb{R}} v\frac{\partial \overline{v}}{\partial z}\, dz - \operatorname{Re} i\int_{\mathbb{R}} v\frac{\partial^2 \overline{v}}{\partial y\,\partial z}\, dz\,.$$

Hence, by (3.3) we deduce

$$(3.4)\qquad -\frac{d}{dy}\,\mathrm{Im}\int_{\mathbb{R}} v\,\frac{\partial\overline{v}}{\partial z}\,dz = -\int_{\mathbb{R}} V|v|^2\,dz + \mathrm{Re}\; i\int_{\mathbb{R}} v\,\frac{\partial^2\overline{v}}{\partial y\,\partial z}\,dz\;.$$

From (3.2) we derive

$$-i\,\frac{\partial^2\overline{v}}{\partial y\,\partial z} = -\frac{1}{2}\left(\frac{\partial^3\overline{v}}{\partial z^3} - V\frac{\partial\overline{v}}{\partial z} - 4V\overline{v}\right) \in L^2\;.$$

We get

$$\begin{aligned}\mathrm{Re}\; i\int_{\mathbb{R}} v\,\frac{\partial^2\overline{v}}{\partial y\,\partial z}\,dz &= \frac{1}{2}\,\mathrm{Re}\int_{\mathbb{R}} v\left[\frac{\partial^3\overline{v}}{\partial z^3} - V\frac{\partial\overline{v}}{\partial z} - 4V\overline{v}\right]dz\\ &= -\int_{\mathbb{R}} V|v|^2\,dz\;,\end{aligned}$$

by integration by parts with a density argument in $D(A_1^2)$.

Hence, by (3.4) we obtain

$$(3.5)\qquad \frac{d}{dy}\,\mathrm{Im}\int_{\mathbb{R}} v\,\frac{\partial\overline{v}}{\partial z}\,dz = 2\int_{\mathbb{R}} V|v|^2\,dz\;.$$

Now, multiplying the equation (3.2) by $\frac{\partial\overline{v}}{\partial y}$, taking the real part and integrating in $z \in \mathbb{R}$ we obtain, first for $v \in D(A_1^2)$ and after, by density, for $v \in D(A_1)$,

$$(3.6)\qquad E(v(y)) = \frac{1}{4}\int_{\mathbb{R}}\left[\left|\frac{\partial v}{\partial z}\right|^2 + V|v|^2\right]dz \equiv E(v_0)\,,\ \forall\, y \in \mathbb{R}\;.$$

By (3.5) and (3.6) we obtain, for $y \geq 0$,

$$(3.7)\qquad \int_0^y\int_{\mathbb{R}} V|v|^2\,dz\,ds \leq 2E(v_0) + \frac{1}{2}\int_{\mathbb{R}} |v_0|^2\,dz$$

(since, by the conservation of charge, $\int_{\mathbb{R}} |v(y)|^2\,dz = \int_{\mathbb{R}} |v_0|^2\,dz$).

The same arguments apply to $\frac{\partial v}{\partial y}$ (first for $v \in D(A_1^3)$, and hence $\frac{\partial v}{\partial y} \in D(A_1^2)$, and after, by density, for $v \in D(A_1^2)$). Hence, we obtain

$$(3.8)\quad \int_0^y \int_{\mathbf{R}} V\left[|v|^2 + \left|\frac{\partial v}{\partial y}\right|^2\right] dz\, ds \leq$$
$$\leq 2E(v_0) + 2E\left(\frac{\partial v}{\partial y}(0)\right) + \frac{1}{2}\int_{\mathbf{R}} |v_0|^2\, dz + \frac{1}{2}\int_{\mathbf{R}} \left|\frac{\partial v}{\partial y}(0)\right|^2 dz\,, \ \forall y \geq 0\,.$$

Now, put $Q(y) = \int_{\mathbf{R}} V|v|^2\, dz$. Following a technique introduced by R.T. Glassey in [4], we get $\frac{d}{dy}Q(y) = 2\int_{\mathbf{R}} V\,\mathrm{Re}(\frac{\partial v}{\partial y}\,\overline{v})\, dz$ and so, with $0 \leq y_1 \leq \tau \leq y+1$, $y \geq 0$:

$$\frac{1}{2}Q(\tau) - \frac{1}{2}Q(y_1) = \int_{y_1}^{\tau} \int_{\mathbf{R}} V\,\mathrm{Re}\left(\frac{\partial v}{\partial y}\,\overline{v}\right) dz\, ds$$
$$\leq \int_{y_1}^{y+1} \int_{\mathbf{R}} V\left[|v|^2 + \left|\frac{\partial v}{\partial y}\right|^2\right] dz\, ds\,.$$

Integrating again in $\tau \in [y_1, y+1]$ we deduce

$$[(y+1)-y_1]Q(y_1) \leq \int_{y_1}^{y+1} Q(\tau)\, d\tau + 2[(y+1)-y_1]\int_{y_1}^{y+1}\int_{\mathbf{R}} V\left[|v|^2 + \left|\frac{\partial v}{\partial y}\right|^2\right] dz\, ds$$

and so, with $y_1 = y$,

$$Q(y) \leq \int_y^{y+1} Q(\tau)\, d\tau + 2\int_y^{y+1} V\left[|v|^2 + \left|\frac{\partial v}{\partial y}\right|^2\right] dz\, ds$$
$$\leq 3\int_y^{y+1} V\left[|v|^2 + \left|\frac{\partial v}{\partial y}\right|^2\right] dz\, ds \underset{p\to+\infty}{\longrightarrow} 0\,,$$

by (3.8). By the reversibility in y, this achieves the proof of proposition 2. □

Now, we will prove the following

PROPOSITION 3. *Under the assumptions of theorem 4 we have*

$$
\begin{aligned}
E\Big(\frac{\partial v}{\partial z}(y)\Big) \le c_0 \Big[& E(v_0) + \int_{\mathbf{R}} |v_0|^2\,dz + E\Big(\frac{\partial v}{\partial y}(0)\Big) + \\
& + \int_{\mathbf{R}} \Big|\frac{\partial v}{\partial y}(0)\Big|^2 dz + E\Big(\frac{\partial v}{\partial z}(0)\Big) + \\
& + \int_{\mathbf{R}} \Big|\frac{\partial v}{\partial z}(0)\Big|^2 dz \Big] , \ \forall y \in \mathbb{R} ,
\end{aligned}
\tag{3.9}
$$

where c_0 *is a positive constant.*

PROOF. Let $u = \dfrac{\partial v}{\partial z}$. We have, from (3.2),

$$
i\,\frac{\partial u}{\partial y} + \frac{1}{2}\Big(\frac{\partial^2 u}{\partial z^2} - Vu - 4Vv\Big) = 0 .
\tag{3.10}
$$

Multiplying by $\overline{u}$, taking the imaginary part and integrating in $z \in \mathbb{R}$ we obtain by density, since $u \in D(A_1)$,

$$
\frac{d}{dy}\int_{\mathbf{R}} |u|^2\,dz = 4\,\mathrm{Im}\int_{\mathbf{R}} Vv\overline{u}\,dz .
$$

Hence, for each $\varepsilon > 0$, we obtain, for all $y \ge 0$,

$$
\begin{aligned}
\int_{\mathbf{R}} |u|^2(y)\,dz \le & \int_{\mathbf{R}} |u(0)|^2\,dz + 4\,\mathrm{Im}\int_0^y \int_{\mathbf{R}} Vv\overline{u}\,dz\,ds \le \\
\le & \int_{\mathbf{R}} |u(0)|^2\,dz + \varepsilon \int_0^y \int_{\mathbf{R}} V|u|^2\,dz\,ds + \\
& + c(\varepsilon)\int_0^y \int_{\mathbf{R}} V|v|^2\,dz\,ds .
\end{aligned}
\tag{3.11}
$$

Now, suppose $v \in D(A_1^3)$. We get $\dfrac{\partial v}{\partial y} \in D(A_1^2)$ and so $\dfrac{\partial}{\partial z}(\dfrac{\partial v}{\partial y}) \in D(A_1)$. Hence, $\dfrac{\partial^2 u}{\partial z\,\partial y} = \dfrac{\partial^3 v}{\partial z^2\,\partial y} \in L^2$. Multiplying the equation (3.10) by $\dfrac{\partial \overline{u}}{\partial y}$ and integrating we obtain (since $\displaystyle\int_{\mathbf{R}} (\frac{\partial^2 u}{\partial z^2} - Vu)\,\frac{\partial \overline{u}}{\partial y}\,dz = -\int_{\mathbf{R}} \frac{\partial u}{\partial z}\,\frac{\partial^2 \overline{u}}{\partial z\,\partial y}\,dz -$

$\int_{\mathbf{R}} Vu \frac{\partial \overline{u}}{\partial y}\, dz$, by density), taking the real part:

$$\frac{1}{2}\frac{d}{dy}\left[\int_{\mathbf{R}}\left|\frac{\partial u}{\partial z}\right|^2 dz + \int_{\mathbf{R}} V|u|^2\, dz\right] + 4\,\mathrm{Re}\int_{\mathbf{R}} Vv\frac{\partial \overline{u}}{\partial y}\, dz = 0\,. \tag{3.12}$$

Furthermore we have, integrating by parts,

$$\begin{aligned}\mathrm{Re}\int_{\mathbf{R}} Vv\frac{\partial^2 \overline{v}}{\partial y\,\partial z}\, dz &= -\,\mathrm{Re}\int_{\mathbf{R}} V\frac{\partial v}{\partial z}\frac{\partial \overline{v}}{\partial y}\, dz - \mathrm{Re}\int_{\mathbf{R}} 4Vv\frac{\partial \overline{v}}{\partial y}\, dz = \\ &= -\,\mathrm{Re}\int_{\mathbf{R}} Vu\frac{\partial \overline{v}}{\partial y}\, dz - 2\frac{d}{dy}\int_{\mathbf{R}} V|v|^2\, dz\,.\end{aligned}$$

Hence, by (3.12) we obtain, for each $\delta > 0$ and for all $y \geq 0$,

$$\begin{aligned}E(u(y)) \leq E(u(0)) + c_0\, E(v_0) + \delta\int_0^y\int_{\mathbf{R}} V|u|^2\, dz + \\ + c(\delta)\int_0^y\int_{\mathbf{R}} V\left|\frac{\partial v}{\partial y}\right|^2 dz\end{aligned} \tag{3.13}$$

and this inequality can be extended, by density, to $v \in D(A_1^2)$.

Now multiply equation (3.10) by $\frac{\partial \overline{u}}{\partial z}$ and take the real part. By integration (and once again by density, since $u \in D(A_1)$) we obtain

$$\mathrm{Re}\; i\int_{\mathbf{R}} \frac{\partial u}{\partial y}\frac{\partial \overline{u}}{\partial z}\, dz + \int_{\mathbf{R}} V|u|^2\, dz - 2\,\mathrm{Re}\int_{\mathbf{R}} Vv\frac{\partial \overline{u}}{\partial z}\, dz = 0 \tag{3.14}$$

and

$$\begin{aligned}\mathrm{Re}\int_{\mathbf{R}} Vv\frac{\partial \overline{u}}{\partial z}\, dz &= -\int_{\mathbf{R}} V|u|^2\, dz - \mathrm{Re}\int_{\mathbf{R}} 4Vv\frac{\partial \overline{v}}{\partial z}\, dz \\ &= -\int_{\mathbf{R}} V|u|^2\, dz + 8\int_{\mathbf{R}} V|v|^2\, dz\,.\end{aligned}$$

We get, by (3.14),

$$\mathrm{Re}\; i\int_{\mathbf{R}} \frac{\partial u}{\partial y}\frac{\partial \overline{u}}{\partial z}\, dz + 3\int_{\mathbf{R}} V|u|^2\, dz - 16\int_{\mathbf{R}} V|v|^2\, dz = 0\,. \tag{3.15}$$

Furthermore we have, if $v \in D(A_1^3)$,

$$\operatorname{Re} i \int_{\mathbf{R}} \frac{\partial u}{\partial y} \frac{\partial \overline{u}}{\partial z}\, dz = -\frac{d}{dy} \operatorname{Im} \int_{\mathbf{R}} u \frac{\partial \overline{u}}{\partial z}\, dz - \operatorname{Re} i \int_{\mathbf{R}} u \frac{\partial^2 \overline{u}}{\partial y\, \partial z}\, dz \,. \tag{3.16}$$

Hence, by (3.15), (3.16) we obtain

$$\begin{aligned} -\frac{d}{dy} \operatorname{Im} \int_{\mathbf{R}} u \frac{\partial \overline{u}}{\partial z}\, dz &= -3 \int_{\mathbf{R}} V|u|^2\, dz + \\ &+ 16 \int_{\mathbf{R}} V|v|^2\, dz + \operatorname{Re} i \int_{\mathbf{R}} u \frac{\partial^2 \overline{u}}{\partial y\, \partial z}\, dz \,. \end{aligned} \tag{3.17}$$

Now, from the equation (3.10), we deduce, with $v \in D(A_1^3)$ and by density,

$$\begin{aligned} \operatorname{Re} i \int_{\mathbf{R}} u \frac{\partial^2 \overline{u}}{\partial y\, \partial z}\, dz &= \frac{1}{2} \operatorname{Re} \int_{\mathbf{R}} u \left[\frac{\partial^3 \overline{u}}{\partial z^3} - V \frac{\partial \overline{u}}{\partial z} - 8\, V\overline{u} - 16\, V\overline{v} \right] dz \\ &= \int_{\mathbf{R}} V|u|^2\, dz - 4 \int_{\mathbf{R}} V|u|^2\, dz - 8 \operatorname{Re} \int_{\mathbf{R}} V \frac{\partial v}{\partial z} \overline{v}\, dz \\ &= -3 \int_{\mathbf{R}} V|u|^2\, dz + 16 \int_{\mathbf{R}} V|v|^2\, dz \,. \end{aligned}$$

Hence, by (3.17) we obtain

$$\frac{d}{dy} \operatorname{Im} \int_{\mathbf{R}} u \frac{\partial \overline{u}}{\partial z}\, dz = 6 \int_{\mathbf{R}} V|u|^2\, dz - 32 \int_{\mathbf{R}} V|v|^2\, dz \,. \tag{3.18}$$

This implies

$$\begin{aligned} \int_0^y \int_{\mathbf{R}} V|u|^2\, dz\, ds &\le c_1 \Big(\int_{\mathbf{R}} |u(0)|^2\, dz + E(u(0)) \Big) + \\ &+ c_2 \int_0^y \int_{\mathbf{R}} V|v|^2\, dz\, ds + \\ &+ c_3 \int_{\mathbf{R}} |u|^2\, dz + c_4\, E(u(y)) \,, \ \forall\, y \ge 0 \,. \end{aligned} \tag{3.19}$$

This inequality can be extended to $v \in D(A_1^2)$, by density. The result is now a consequence of (3.8), (3.11), (3.13) and (3.19): from (3.19), (3.8) and (3.11) we obtain, with $\varepsilon = \frac{1}{2} c_3^{-1}$,

$$\int_0^y \int_{\mathbf{R}} V|u|^2 \, dz \, ds \leq c_5 + 2 c_4 E(u(y)) , \ \forall y \geq 0 , \tag{3.20}$$

with c_5 of the form in the right hand side of (3.9). Then, by (3.13), (3.8) and (3.20), putting $\delta = \frac{1}{4} c_4^{-1}$, we achieve the proof of proposition 3 (the case $y \leq 0$ is obtained by reversibility). □

We can now complete the proof of theorem 4:

We have $\frac{\partial}{\partial z}(V^{\frac{1}{2}} v) = V^{\frac{1}{2}} \frac{\partial v}{\partial z} + 2V^{\frac{1}{2}} v \in L^2(\mathbb{R})$, for each $y \in \mathbb{R}$, and so

$$\left\| \frac{\partial}{\partial z}(V^{\frac{1}{2}} v) \right\|_{L^2(\mathbf{R})} \leq 2\left(\left\| V^{\frac{1}{2}} \frac{\partial v}{\partial z} \right\|_{L^2(\mathbf{R})} + \left\| V^{\frac{1}{2}} v \right\|_{L^2(\mathbf{R})} \right) . \tag{3.21}$$

Now, let $q \in [2, +\infty]$. With

$$a = a(q) = \frac{1}{2} - \frac{1}{q} \ (\text{hence } 0 \leq a \leq \tfrac{1}{2}) ,$$

we have, by the inequality of Gagliardo–Nirenberg and (3.21),

$$\begin{aligned} \left\| V^{\frac{1}{2}} v \right\|_{L^q(\mathbf{R})} &\leq c(q) \left\| \frac{\partial}{\partial z}(V^{\frac{1}{2}} v) \right\|^a_{L^2(\mathbf{R})} \left\| V^{\frac{1}{2}} v \right\|^{1-a}_{L^2(\mathbf{R})} \\ &\leq 2^a c(q) \left(\left\| V^{\frac{1}{2}} \frac{\partial v}{\partial z} \right\|_{L^2(\mathbf{R})} + \left\| V^{\frac{1}{2}} v \right\|_{L^2(\mathbf{R})} \right)^a \left\| V^{\frac{1}{2}} v \right\|^{1-a}_{L^2(\mathbf{R})} . \end{aligned} \tag{3.22}$$

The theorem is now a consequence of propositions 2 and 3 and inequality (3.22). □

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