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About reflexivity of the B^q_{ap} spaces of almost periodic functions

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RIASSUNTO: *In questo lavoro si dimostra la riflessività degli spazi di* BESICOVITCH B^q_{ap} *delle funzioni quasi periodiche. Inoltre si fornisce la seguente caratterizzazione dei funzionali lineari e continui su* B^q_{ap}, *per* $q \in]1,+\infty[$*: lo spazio duale di* B^q_{ap} *è lo spazio* $B^{q'}_{ap}$, *dove* $1/q' = 1 - 1/q$.

ABSTRACT: *The aim of this paper is to establish the reflexivity of the* BESICOVITCH *spaces* B^q_{ap} *of almost periodic functions, for* $q \in]1,+\infty[$ *and to give the following characterization of the bounded linear functionals on* B^q_{ap} *for* $q \in]1,+\infty[$*: the dual of* B^q_{ap} *is the space* $B^{q'}_{ap}$, *where* $1/q' = 1 - 1/q$.

KEY WORDS: *Uniformly almost periodic function - Almost periodic function - Dual space - Reflexivity.*

A.M.S. CLASSIFICATION: 42A75 - 46B10 - 46A25

1 – Introduction

Very recently A. AVANTAGGIATI, G. BRUNO and R. IANNACCI [2] began examining the BESICOVITCH spaces B^q_{ap} of B-almost periodic functions according to the definition introduced by A.S. BESICOVITCH (cf. [3], ch. II, §6) and studied subsequently by A.C. ZAANEN in an equivalent form (cf. [9], pp. 103 - 108). Identifing the elements of the B^q_{ap} spaces with their FOURIER series, some criteria have been given for expansions in FOURIER series. After having studied the BOHR transformation, they have introduced the FOURIER series of any bounded linear functionals on B^q_{ap} and they have stated duality theorems for some particular subspaces of B^q_{ap}. Moreover they have proved that, for any $q \in]1, +\infty[$, B^q_{ap} is isometrically embedded in the dual of $B^{q'}_{ap}$, where $\frac{1}{q'} = 1 - \frac{1}{q}$, by using the results contained in Lemma 3.1 and Lemma 3.2 in this paper [see Section 3]. Going on with these studies, in the present article we prove that, for any $q \in]1, +\infty[$ the complex Banach space B^q_{ap} is reflexive; moreover we establish for the B^q_{ap} space the analogue of RIESZ representation Theorem for L^q spaces. In order to obtain our results we use classical techniques of the theory of L^q spaces (cf. [8]).

There is a wide literature dealing with the uniformly almost periodic functions and their properties; let us mention L. AMERIO - G. PROUSE [1], A.S. BESICOVITCH [3], S. CINQUINI [4], C. CORDUNEANU [6], A.M. FINK [7] and A.C. ZAANEN [9].

The uniform almost periodicity has been generalized in various directions by several authors; the reader is referred to the bibliography contained in this paper.

We would like to point out that in this article we are concerned with the B-almost periodicity introduced by A.S. BESICOVITCH in [3].

This paper is organized as follows. In Section 2 we collect the notations and the definitions to be used throughout the paper. Dealing with uniformly almost periodic functions, in Sections 3 we formulate and we prove two lemmata that are the key tools to obtain the announced results. Section 4 is devoted to the reflexivity of the B^q_{ap} space, when $q \in]1, +\infty[$. Finally in Section 5 we are concerned with the characterization of bounded linear functionals of B^q_{ap} for $q \in]1, +\infty[$: we show that $(B^q_{ap})^*$ can be identified with $B^{q'}_{ap}$, where $\frac{1}{q} + \frac{1}{q'} = 1$ (RIESZ Representation Theorem).

2 – Notations and definitions

For any real number $T > 0$, let $L^q[-T, T]$ denote the classical space of measurable complex valued functions whose q-th power of the absolute value is LEBESGUE integrable.

Let $\mathcal{P}$ denote the complex vector space of all trigonometric polynomial $P(x)$ of the form

$$P(x) = \sum_{j=1}^{\nu} c_j e^{i\lambda_j x}, \qquad \forall\, x \in \mathbb{R},$$

where $\nu \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $\lambda_j \in \mathbb{R}$, with $\lambda_j \neq \lambda_i$ for $i \neq j$, are arbitrary.

We will use the following definition of uniformly almost periodic function (cf. [6]).

A complex valued function $f(x)$, defined on $\mathbb{R}$, is called uniformly almost periodic (u.a.p.) if for any $\varepsilon > 0$ there exists a trigonometric polynomial $P_\varepsilon(x)$ such that

$$|f(x) - P_\varepsilon(x)| < \varepsilon, \qquad \forall\, x \in \mathbb{R}.$$

Thus the space C^0_{ap} of all u.a.p. functions is the completion of $\mathcal{P}$ with respect to the L^∞ norm

$$\|P\|_\infty = \sup_{x\in\mathbb{R}} |P(x)|, \qquad \forall\, P \in \mathcal{P}.$$

Fixed $q \in [1, +\infty[$, for any $P \in \mathcal{P}$ we set

$$\|P\|_q = \lim_{T\to+\infty} \left(\frac{1}{2T} \int_{-T}^{T} |P(x)|^q dx \right)^{1/q}, \tag{2.1}$$

which is well posed since $|P(x)|^q$ is u.a.p. and there exists finite the mean value of any u.a.p. function.

DEFINITION 2.1. *For any fixed $q \in [1, +\infty)$ we shall denote by B^q_{ap} the completion of $\mathcal{P}$ with respect to the norm $\|\cdot\|_q$ defined by (2.1). These spaces are called Besicovitch's spaces (of almost periodic functions).*

Observe that B^q_{ap} is identical with the space obtained by completion of $\mathcal{P}$ with respect to the norm

$$\limsup_{T\longrightarrow+\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(x)|^q dx \right)^{1/q},$$

as it is easy to see.

This definition introduced by A.C. ZAAENEN [9] is equivalent to that due to A.S. BESICOVITCH [3, theorem 1, p. 95]. For the properties of B^q_{ap} spaces the reader is referred to [1,3,5,9].

Nevertheless, for reader's convenience, we report some properties we will use in the sequel.

Let $f \in B^q_{ap}$ and $P_n \in \mathcal{P}$, $\forall\, n \in \mathbb{N}$. Assume that $P_n \to f$ in B^q_{ap}, then there exists finite

$$\left(\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|^q dx\right)^{1/q} =: \|f\|_q$$

and the following relation holds true

$$\|f\|_q = \lim_{n\to\infty} \|P_n\|_q .$$

If in addition $g \in B^{q'}_{ap}$ with $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ and $Q_n \to g$ in $B^{q'}_{ap}$ then there exists finite

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}dx =: (f|g)$$

and it results

$$(f|g) = \lim_{n\to\infty} (P_n|Q_n)$$

$$|(f|g)| \le \|f\|_q \|g\|_{q'} .$$

For any complex Banach space B, let B^* denote the dual of B and let τ denote the natural mapping of B into B^{**}, i.e.

$$\tau(u)(u^*) = u^*(u) = \langle u, u^*\rangle \qquad \forall\, u \in B\,, u^* \in B^* .$$

We recall that a Banach space $B(\|\cdot\|)$ is called uniformly convex if given $\varepsilon > 0$ there is $\delta > 0$ such that whenever u and v are elements of B with $\|u\| \le 1$, $\|v\| \le 1$ and $\|u - v\| \ge \varepsilon$ we have $\left\|\dfrac{u+v}{2}\right\| \le 1 - \delta$.

3 – Preliminary results

In this section we prove two lemmata we will use in the sequel. To this aim we give preliminarily two elementary inequalities.

INEQUALITY 1.

$$\left|\frac{u}{|u|}-\frac{v}{|v|}\right|\leq 2\frac{|u-v|}{|u|+|v|},\quad \forall\, u,v\in \mathbb{C}\setminus\{0\}. \tag{3.1}$$

PROOF. Observe that (3.1) is equivalent to

$$4|u-v|^2-(|u|+|v|)^2\left|\frac{u}{|u|}-\frac{v}{|v|}\right|^2\geq 0,\quad \forall\, u,v\in \mathbb{C}\setminus\{0\}. \tag{3.2}$$

By easy calculations (3.2) follows, writing u and v in trigonometric form. □

INEQUALITY 2. *For any* $s\in[1,2]$ *one has*

$$\left|(\operatorname{sign} u)|u|^{s-1}-(\operatorname{sign} v)|v|^{s-1}\right|\leq (1+2^{2-s})|u-v|^{s-1},\quad \forall\, u,v\in\mathbb{C}. \tag{3.3}$$

PROOF. If $u=0$ or $v=0$, (3.3) is obvious. Let us assume now $u\neq 0$ and $v\neq 0$. We can write

$$(\operatorname{sign} u)|u|^{s-1}-(\operatorname{sign} v)|v|^{s-1}=(\operatorname{sign} u)(|u|^{s-1}-|v|^{s-1})+(\operatorname{sign} u-\operatorname{sign} v)|v|^{s-1}, \tag{3.4}$$

and

$$(\operatorname{sign} u)|u|^{s-1}-(\operatorname{sign} v)|v|^{s-1}=(\operatorname{sign} v)(|u|^{s-1}-|v|^{s-1})+|u|^{s-1}(\operatorname{sign} u-\operatorname{sign} v). \tag{3.5}$$

Adding (3.4) and (3.5), dividing by 2 we obtain, using $|\operatorname{sign} u+\operatorname{sign} v|\leq 2$,

$$\begin{aligned}&\left|(\operatorname{sign} u)|u|^{s-1}-(\operatorname{sign} v)|v|^{s-1}\right|\leq\\ &\leq\left||u|^{s-1}-|v|^{s-1}\right|+\frac{1}{2}|\operatorname{sign} u-\operatorname{sign} v|(|u|^{s-1}+|v|^{s-1}).\end{aligned} \tag{3.6}$$

Taking into account (3.1) and the inequalities

$$|a^r - b^r| \leq |a-b|^r\,, \ \frac{a^r+b^r}{|a+b|^r} \leq 2^{1-r}\,, \quad \forall\, a,b \in \mathbb{R}^+\,, \forall\, r \in [0,1] \tag{3.7}$$

with $r = s-1$, using $2-s \geq 0$ from (3.6) we get

$$\begin{aligned}
&\Big|(\operatorname{sign} u)|u|^{s-1} - (\operatorname{sign} v)|v|^{s-1}\Big| \leq \\
\leq &|u-v|^{s-1} + \frac{|u|^{s-1}+|v|^{s-1}}{|u|+|v|}|u-v|^{2-s}|u-v|^{s-1} \leq \\
\leq &|u-v|^{s-1} + \frac{|u|^{s-1}+|v|^{s-1}}{|u|+|v|}\big||u|+|v|\big|^{2-s}|u-v|^{s-1} \leq \\
\leq &|u-v|^{s-1} + \ 2^{2-s}|u-v|^{s-1} = \\
= &(1+2^{2-s})|u-v|^{s-1}\,.
\end{aligned}$$

□

LEMMA 3.1. *If g is uniformly almost periodic then for any $q \in]1,+\infty[$ the function*

$$f(x) = \Big(\operatorname{sign} g(x)\Big)|g(x)|^{q-1} \tag{3.8}$$

is u.a.p. too.

PROOF. The claim follows from the continuity of the function $h(z) = (\operatorname{sign} z)|z|^{q-1}$, for $q \in]1,+\infty[$, and a classical theorem on u.a.p. functions (cf [1], VII p. 6). □

LEMMA 3.2. *Let $q' \in]1,+\infty[$. If $(g_n)_{n\in\mathbb{N}} \in (C^0_{ap})^{\mathbb{N}}$ is a CAUCHY sequence in $B^{q'}_{ap}$ then $(\operatorname{sign} g_n(x)|g_n(x)|^{q'-1})_{n\in\mathbb{N}} \in (C^0_{ap})^{\mathbb{N}}$ is a CAUCHY sequence in B^q_{ap}, where obviously $\dfrac{1}{q} = 1 - \dfrac{1}{q'}$.*

PROOF. By Lemma 3.1, $g_n \in C^0_{ap}$ implies $\operatorname{sign} g_n |g_n|^{q'-1} \in C^0_{ap}$ for any $n \in \mathbb{N}$ and for any $q' \in]1, +\infty[$.

Preliminarly we will consider the case $q' \in]1, 2]$.

Using (3.3) and the first inequality of (3.7) we can write for any $x \in \mathbb{R}$

$$\begin{aligned}
&\Big|(\operatorname{sign} g_n(x))|g_n(x)|^{q'-1} - (\operatorname{sign} g_m(x))|g_m(x)|^{q'-1}\Big|^q \le \\
&\le (1 + 2^{2-q'})\Big||g_n(x)| - |g_m(x)|\Big|^{(q'-1)q} \le \qquad (3.9)\\
&\le (1 + 2^{2-q'})|g_n(x) - g_m(x)|^{q'},
\end{aligned}$$

since $(q'-1)q = q'$. By (3.9), taking into account that by hypothesis $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B^{q'}_{ap}$ we derive

$$\begin{aligned}
0 &\le \|(\operatorname{sign} g_n)|g_n|^{q'-1} - (\operatorname{sign} g_m)|g_m|^{q'-1}\|^q_q \le \\
&\le (1 + 2^{2-q'})\|g_n - g_m\|^{q'}_{q'} \xrightarrow[n,m \to +\infty]{} 0 .
\end{aligned}$$

Assume now $q' \in]2, +\infty[$. Since $q' - 2 > 0$, for any $n \in \mathbb{N}$ we can write

$$(\operatorname{sign} g_n(x))|g_n(x)|^{q'-1} = g_n(x)|g_n(x)|^{q'-2}, \forall\, x \in \mathbb{R} .$$

As $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B^{q'}_{ap}$, there exists $H > 0$ such that

$$\|g_n\|_{q'} \le H \quad , \forall n \in \mathbb{N} . \qquad (3.10)$$

We are going now to show that

$$\|g_n|g_n|^{q'-2} - g_m|g_m|^{q'-2}\|_q \xrightarrow[n,m \to +\infty]{} 0 . \qquad (3.11)$$

Adding and substracting $g_n(x)|g_m(x)|^{q'-2}$ and using $(a+b)^q \le 2^{q-1}(a^q + b^q)$ $\forall a, b \in \mathbb{R}^+ \cup \{0\}$, we can write

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T} \Big|g_n(x)|g_n(x)|^{q'-2} - g_m(x)|g_m(x)|^{q'-2}\Big|^q dx \le \\
&\le \frac{2^{q-1}}{2T}\int_{-T}^{T} |g_n(x)|^q \,\Big|\, |g_n(x)|^{q'-2} - |g_m(x)|^{q'-2}\Big|^q dx + \qquad (3.12)\\
&+ \frac{2^{q-1}}{2T}\int_{-T}^{T} |g_m(x)|^{q(q'-2)}\Big|g_n(x) - g_m(x)\Big|^q dx .
\end{aligned}$$

Now we apply Hölder inequality to the second addend of the last term of inequality (3.12) with $p = \dfrac{q'-1}{q'-2}$ and $p' = q'-1$ respectively; since $q(q'-2)p = q'$ and $qp' = q'$, by taking the limit as $T \to +\infty$, we get

$$
\begin{aligned}
&2^{q-1} \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big|g_m(x)\Big|^{q(q'-2)} \Big|g_n(x) - g_m(x)\Big|^q dx \leq \\
&\leq 2^{q-1} \Big\|g_m\Big\|_{q'}^{q(q'-2)} \Big\|g_n - g_m\Big\|_{q'}^{q} \xrightarrow[n,m\to+\infty]{} 0,
\end{aligned} \tag{3.13}
$$

taking into account relation (3.10) and the hypothesis that $(g_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $B_{ap}^{q'}$.

Consider now the first addend of the last term of inequality (3.12). We apply Hölder inequality with $p = q'-1$ and $p' = \dfrac{q'-1}{q'-2}$ and, since $pq = q'$ and $qp' = \dfrac{q'}{q'-2}$, we obtain

$$
\begin{aligned}
&\frac{1}{2T} \int_{-T}^{T} |g_n(x)|^q \Big||g_n(x)|^{q'-2} - |g_m(x)|^{q'-2}\Big|^q dx \leq \\
&\leq \Big(\frac{1}{2T} \int_{-T}^{T} |g_n(x)|^{q'} dx\Big)^{\frac{1}{q'-1}} \cdot \\
&\cdot \Big(\frac{1}{2T} \int_{-T}^{T} \Big||g_n(x)|^{q'-2} - |g_m(x)|^{q'-2}\Big|^{\frac{q'}{q'-2}} dx\Big)^{\frac{q'-2}{q'-1}}.
\end{aligned} \tag{3.14}
$$

Therefore to reach our goal we are going now to establish a suitable inequality for the second factor of the last term of inequality (3.14).

For $q'-2 = 1$ we have for any $T > 0$

$$
\frac{1}{2T} \int_{-T}^{T} \Big||g_n(x)| - |g_m(x)|\Big|^{q'} dx \leq \frac{1}{2T} \int_{-T}^{T} |g_n(x) - g_m(x)|^{q'} dx \tag{3.15}
$$

and we are done. Hence we shall consider only the two cases A) $q'-2 < 1$, B) $q'-2 > 1$.

CASE A. Assume $q' - 2 < 1$. Using the following inequality

$$|a^r - b^r| \leq |a - b|^r \quad \forall a, b \in \mathbb{R}^+ \cup \{0\}$$

with $r = q' - 2 < 1$, we can write for any $n, m \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\Big||g_n(x)|^r - |g_m(x)|^r\Big|^{\frac{q'}{r}} \leq \Big||g_n(x)| - |g_m(x)|\Big|^{q'} \leq$$
$$\leq |g_n(x) - g_m(x)|^{q'}.$$

Hence we have for any $T > 0$ and for any $n, m \in \mathbb{N}$

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} \Big||g_n(x)|^{q'-2} - |g_m(x)|^{q'-2}\Big|^{\frac{q'}{q'-2}} dx \leq \\ \leq \frac{1}{2T}\int_{-T}^{T} |g_n(x) - g_m(x)|^{q'} dx. \end{aligned} \tag{3.16}$$

CASE B. Let us assume now $q' - 2 > 1$. Using the inequality

$$|a - b|^r \leq |a^r - b^r| \quad \forall a, b \in \mathbb{R}^+ \cup \{0\}$$

with $r = \dfrac{q'}{q'-2} > 1$, we get for any $T > 0$ and $n, m \in \mathbb{N}$

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} \Big||g_n(x)|^{q'-2} - |g_m(x)|^{q'-2}\Big|^{\frac{q'}{q'-2}} dx \leq \\ \leq \frac{1}{2T}\int_{-T}^{T} \Big||g_n(x)|^{q'} - |g_m(x)|^{q'}\Big| dx. \end{aligned} \tag{3.17}$$

For any $T > 0$, let us define

$$T_{n,m} = \left\{x \in [-T, T] \colon |g_n(x)|^{q'} \geq |g_m(x)|^{q'}\right\}.$$

Using the inequalities

$$rb^{r-1}(a-b) \leq a^r - b^r \leq ra^{r-1}(a-b) \quad \forall a,b \in \mathbb{R}^+ \cup \{0\}$$

with $q' = r > 1, a = |g_n(x)|^{q'}$ and $b = |g_m(x)|^{q'}$, from (3.17) we derive, for any $T > 0$ and for any $n, m \in \mathbb{N}$

(3.18)

$$\frac{1}{2T}\int_{-T}^{T} \Big||g_n(x)|^{q'-2} - |g_m(x)|^{q'-2}\Big|^{\frac{q'}{q'-2}} dx \leq$$

$$\leq \frac{1}{2T}\Big[\int_{T_{n,m}} |g_n(x)|^{q'} - |g_m(x)|^{q'} dx - \int_{T_{m,n}} |g_n(x)|^{q'} - |g_m(x)|^{q'} dx\Big] \leq$$

$$\leq \frac{1}{2T}\Big[\int_{T_{n,m}} r|g_n(x)|^{q'-1}\Big(|g_n(x)| - |g_m(x)|\Big)dx - \int_{T_{m,n}} r|g_m(x)|^{q'-1}\Big(|g_n(x)| +$$

$$- |g_m(x)|\Big)dx\Big] =$$

$$= \frac{r}{2T}\Big[\int_{T_{n,m}} |g_n(x)|^{q'-1}\Big(|g_n(x)| - |g_m(x)|\Big)dx + \int_{T_{m,n}} |g_m(x)|^{q'-1}\Big(|g_m(x)| +$$

$$- |g_n(x)|\Big)dx\Big] =$$

$$= \frac{r}{2T}\Big[\int_{T_{n,m}} |g_n(x)|^{q'-1}\Big||g_n(x)| - |g_m(x)|\Big|dx + \int_{T_{m,n}} |g_m(x)|^{q'-1}\Big||g_m(x)| +$$

$$- |g_n(x)|\Big|dx\Big] \leq$$

$$\leq \frac{r}{2T}\Big[\int_{-T}^{T} |g_n(x)|^{q'-1}\Big|g_n(x) - g_m(x)\Big|dx + \int_{-T}^{T} |g_m(x)|^{q'-1}\Big|g_m(x) - g_n(x)\Big|dx\Big].$$

Using Hölder inequality to the first addend and second addend of the last term of inequality (3.18) with $p = q$ and $p' = q'$, since $(q'-1)q = q'$ we obtain

$$\frac{1}{2T}\int_{-T}^{T}\left||g_n(x)|^{q'}-|g_m(x)|^{q'}\right|dx\leq$$

$$(3.19)\qquad \leq r\left(\frac{1}{2T}\int_{-T}^{T}|g_n(x)|^{q'}dx\right)^{\frac{1}{q}}\left(\frac{1}{2T}\int_{-T}^{T}|g_n(x)-g_m(x)|^{q'}dx\right)^{\frac{1}{q'}}+$$

$$+r\left(\frac{1}{2T}\int_{-T}^{T}|g_m(x)|^{q'}dx\right)^{\frac{1}{q}}\left(\frac{1}{2T}\int_{-T}^{T}|g_n(x)-g_m(x)|^{q'}dx\right)^{\frac{1}{q'}}.$$

Now we can pass to the limit in (3.12) for $T\longrightarrow+\infty$, and we get that relation (3.11) holds true, by using inequality (3.10) and inequalities (3.15) or (3.16) or (3.19) according to $q'=3$ or $q'<3$ or $q'>3$ respectively. The proof of Lemma 3.2 is complete. □

4 – Reflexivity of the B^q_{ap} spaces

This section is devoted to establish some properties of B^q_{ap} spaces and to obtain that B^q_{ap} spaces are reflexive for any $q\in]1,+\infty[$.

This study is based on arguments used in the literature on the theory of L^q spaces and carries out a strong analogy between B^q_{ap} spaces and the L^q spaces.

The following Proposition gives the CLARKSON inequalities for the Besicovitch spaces. The reader is referred, for these inequalities in L^q, e.g. to [9, p. 129-130].

PROPOSITION 4.1. CLARKSON INEQUALITIES. *Let $q\in[2,+\infty[$, then*

$$(4.1)\qquad \left\|\frac{f+g}{2}\right\|_q^q+\left\|\frac{f-g}{2}\right\|_q^q\leq\frac{1}{2}\left(\left\|f\right\|_q^q+\left\|g\right\|_q^q\right),\forall\, f,g\in B^q_{ap}.$$

If $q\in]1,2]$ and $\frac{1}{q'}=1-\frac{1}{q}$, then

$$(4.2)\qquad \left\|\frac{f+g}{2}\right\|_q^{q'}+\left\|\frac{f-g}{2}\right\|_q^{q'}\leq\left(\frac{1}{2}\left\|f\right\|_q^q+\frac{1}{2}\left\|g\right\|_q^q\right)^{\frac{1}{q-1}},\forall\, f,g\in B^q_{ap}.$$

PROOF. Let us consider two sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ of trigonometric polynomials converging in B^q_{ap} to f and g respectively. Since for any $T>0$ we have that $P_n, Q_n \in L^q[-T,T]$, $\forall\, n\in\mathbb{N}$, we can apply CLARKSON inequalities in $L^q[-T,T]$ to any P_n and Q_n. Dividing by $2T$ if $q\in[2,+\infty[$ and by $(2T)^{\frac{1}{q-1}}$ if $q\in]1,2]$, passing to the limit as $T\to+\infty$, we get for any $n\in\mathbb{N}$

$$\left\|\frac{P_n+Q_n}{2}\right\|_q^q+\left\|\frac{P_n-Q_n}{2}\right\|_q^q\le\frac{1}{2}\left(\left\|P_n\right\|_q^q+\left\|Q_n\right\|_q^q\right)\quad\text{for}\quad q\in[2,+\infty[$$

and, since $q'=\dfrac{q}{q-1}$,

$$\left\|\frac{P_n+Q_n}{2}\right\|_q^{q'}+\left\|\frac{P_n-Q_n}{2}\right\|_q^{q'}\le\left(\frac{1}{2}\left\|P_n\right\|_q^q+\frac{1}{2}\left\|Q_n\right\|_q^q\right)^{\frac{1}{q-1}}\quad\text{for}\quad q\in]1,2]\,.$$

By taking the limit as $n\to+\infty$ and observing that

$$P_n\to f\,, Q_n\to g\,, (P_n+Q_n)\frac{1}{2}\to\frac{1}{2}(f+g)\,, (P_n-Q_n)\frac{1}{2}\to(f-g)\frac{1}{2}$$

in B^q_{ap} we get the thesis. □

Now we are going to use this proposition to prove that the space B^q_{ap} is uniformly convex for $q\in]1,+\infty[$. This fact implies by MIL'MAN's Theorem [cf. LARSEN, p. 217] that they are reflexive.

THEOREM 4.1. *For any $q\in]1,+\infty[$, the space B^q_{ap} is uniformly convex and hence it is reflexive.*

PROOF. Let assume $\varepsilon>0$ and $f,g\in B^q_{ap}$ such that

$$\|f\|_q\le 1\,, \|g\|_q\le 1\quad\text{and}\quad \|f-g\|_q\ge\varepsilon\,.$$

If $q\in[2,+\infty[$, from (4.1) we get

$$\left\|\frac{f+g}{2}\right\|_q^q\le 1-\left(\frac{\varepsilon}{2}\right)^q$$

and hence

$$\left\|\frac{f+g}{2}\right\|_q \leq 1-\delta\,, \quad \text{where} \quad \delta = 1-\left[1-\left(\frac{\varepsilon}{2}\right)^q\right]^{\frac{1}{q}}.$$

For $q \in]1,2[$, by (4.2) we obtain

$$\left\|\frac{f+g}{2}\right\|_q^{\frac{q}{q-1}} \leq 1-\left(\frac{\varepsilon}{2}\right)^{\frac{q}{q-1}}$$

from which it follows immediately that

$$\left\|\frac{f+g}{2}\right\|_q \leq 1-\delta\,, \quad \text{where} \quad \delta = 1-\left[1-\left(\frac{\varepsilon}{2}\right)^{\frac{q}{q-1}}\right]^{\frac{q-1}{q}}.$$

□

REMARK 4.1. *The space B^1_{ap} is not uniformly convex.*

Setting

$$f(x)=\begin{cases}\pi\sin x & ,0\leq x\leq\pi\\ 0 & ,-\pi\leq x<0\end{cases} \quad \text{and} \quad g(x)=\begin{cases}0 & ,0\leq x\leq\pi\\ -\pi\sin x & ,-\pi\leq x<0\end{cases}$$

we can consider this function extended to the whole real axis by periodicity. Since f and g are continuous periodic functions, they are also u.a.p.. Then $f, g \in B^1_{ap} \subset C^0_{ap}$. It is easy to verify that $\|f\|_1 = \|g\|_1 = 1$ and $\|f-g\|_1 = 2$, whereas $\|f+g\|_1 = 2$.

The space $B^\infty_{ap} = C^0_{ap}$ is not uniformly convex; indeed for $f(x) = \sin^2 x$ and $g(x) = \sin x$ we have

$$\|f\|_\infty = \|g\|_\infty = 1 \quad \text{and} \quad \|f+g\|_\infty = \|f-g\|_\infty = 2\,.$$

5 – Riesz representation theorem for B^q_{ap} spaces

In [2], the authors have proved that assuming $q \in]1, +\infty[$ and $\frac{1}{q} + \frac{1}{q'} = 1$, each function g in $B^{q'}_{ap}$ defines a bounded linear functional G on B^q_{ap} by

$$(5.1) \qquad G(f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\, dx\,, \quad \forall\, f \in B^q_{ap}$$

verifying the condition

$$(5.2) \qquad \|G\| = \sup_{\|f\|_q = 1} \left| G(f) \right| = \|g\|_{q'}\,;$$

hence $B^{q'}_{ap}$ is isometrically embedded in the dual of B^q_{ap}.

The proof of (5.2) is based on Hölder inequality and Lemmata 3.1, 3.2 in this paper [see Theorem 5.1 and Theorem 5.2].

THEOREM 5.1. *The mapping A: $B^{q'}_{ap} \to (B^q_{ap})^*$ defined by $Ag = G$, where G is given by (5.1), i.e.*

$$(5.3) \quad \langle f, Ag \rangle = G(f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\, dx = (f, g)\,, \quad \forall\, f \in B^q_{ap}$$

is a surjective antilinear isometry from $B^{q'}_{ap}$ to $(B^q_{ap})^*$.

PROOF. The antilinearity of A is evident; moreover we know now that A is an isometry. We are going to prove that A is onto. Let us consider $E = A(B^{q'}_{ap})$. From the continuity of A, since $B^{q'}_{ap}$ is a Banach space, it follows that E is a closed linear subspace of $(B^q_{ap})^*$. If we show that E is dense in $(B^q_{ap})^*$, i.e. $\overline{E} = (B^q_{ap})^*$ we reach our goal.

For this purpose we will prove that if φ is a linear continuous functional on $(B^q_{ap})^*$ such that

$$\varphi = 0 \quad \text{on} \quad E \quad \text{i.e.} \quad \varphi(Ag) = 0\,, \quad \forall\, g \in B^{q'}_{ap}\,,$$

then φ is identically zero on $(B^q_{ap})^*$ [8, p. 90, Corollary 4.2.8].

Let $\varphi \in (B^q_{ap})^{**}$ be such that $\varphi(Ag) = 0$, $\forall\, g \in B^{q'}_{ap}$. Since B^q_{ap} is reflexive, there exists some $f \in B^q_{ap}$ such that $\tau(f) = \varphi$, i.e.

$$\varphi(h^*) = \tau(f)(h^*) = \langle f, h^* \rangle = h^*(f), \quad \forall\, h^* \in (B^q_{ap})^* .$$

On the other hand $\varphi(Ag) = 0\ \forall\, g \in B^{q'}_{ap}$ means

$$\begin{aligned} 0 &= \varphi(Ag) = \tau(f)(Ag) = \langle f, Ag \rangle = \Big(Ag(f) = G(f)\Big) = \\ &= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\, dx\,, \quad \forall\, g \in B^{q'}_{ap} . \end{aligned} \tag{5.4}$$

Since $f \in B^q_{ap}$ there exists $(P_n)_{n\in\mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ such that $P_n \to f$ in B^q_{ap} as $n \to +\infty$; hence by Lemma 3.1 and Lemma 3.2.

$$f_n(x) = \operatorname{sign} P_n(x) \Big| P_n(x) \Big|^{q-1}$$

is u.a.p. for any $n \in \mathbb{N}$ and f_n converges to some $\tilde{f}$ in $B^{q'}_{ap}$. Setting $g = \tilde{f}$ in (5.4) we can write

$$\begin{aligned} 0 = \varphi(A\tilde{f}) &= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{\tilde{f}(x)}\, dx = \\ &= \lim_{n\to\infty} \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} P_n(x)\overline{\operatorname{sign} P_n(x)} \Big| P_n(x) \Big|^{q-1} dx = \\ &= \lim_{n\to\infty} \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big| P_n(x) \Big|^{q} dx = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big| f(x) \Big|^{q} dx = \|f\|_q^q . \end{aligned}$$

Therefore $\|f\|_q = 0$ and so $f = 0$ in B^q_{ap} and the proof is complete. $\square$

By this result we are now in a position to give the following characterization of the spaces $(B^q_{ap})^*$ for $q \in]1, +\infty[$:

THEOREM 5.2. RIESZ REPRESENTATION THEOREM. *Let G be a bounded linear functional on B^q_{ap}, for $q \in]1, +\infty[$. Then there exists one and only one function $g \in B^{q'}_{ap}$ such that*

$$G(f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\, dx\,, \quad \forall\, f \in B^q_{ap}\,.$$

We have also $\|G\| = \|g\|_{q'}$.

Observe that Theorem 5.2 claims that there is a natural representation of the bounded linear functionals on B^q_{ap} by elements of $B^{q'}_{ap}$.

The case $q = 1$ and $q = \infty$ are open problems.

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