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Local Bertini theorems for geometric properties over a non perfect field

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RIASSUNTO: *Sia k un campo infinito di caratteristica arbitraria, (A, M, K) una k-algebra essenzialmente di tipo finito, con K/k separabile e $\mathbb{P}$ una proprietà locale. Diciamo che $LB_k(\mathbb{P})$ è verificato se: per il generico $\alpha = (\alpha_1, \dots, \alpha_n) \in K^n \implies \mathbb{P}(A/x_\alpha A) \supseteq \mathbb{P}(A) \cap V(x_\alpha) \cap U_{\mathbb{P}}$ (con $x_\alpha = \sum \alpha_i x_i$, $\langle x_1, \dots, x_n \rangle = M$, $U_{\mathbb{P}}$ aperto non vuoto di $\operatorname{Spec} A$ e $\mathbb{P}(A) = \{\mathfrak{P} \in \operatorname{Spec} A | A_{\mathfrak{P}}$ è $\mathbb{P}\}$). Proviamo che: $LB_k(\mathbb{P})$ vale per $\mathbb{P} \implies LB_k(\mathbb{GP})$ vale per la corrispondente proprietà geometrica (in particolare per $\mathbb{P}$ = regolare, normale, ridotto, R_s, $LB_k(\mathbb{P})$ vale). Come applicazione otteniamo il teorema di* BERTINI *per le sezioni ipersuperficiali di una varietà $X \subseteq \mathbf{P}_k^n$ relativamente alle proprietà geometriche.*

ABSTRACT: *Let k be an infinite field of arbitrary characteristic, (A, M, K) a k-algebra of essentially finite type, with K/k separable and $\mathbb{P}$ a local property. We say that $LB_k(\mathbb{P})$ holds if: for the general $\alpha = (\alpha_1, \dots, \alpha_n) \in K^n \implies \mathbb{P}(A/x_\alpha A) \supseteq \mathbb{P}(A) \cap V(x_\alpha) \cap U_{\mathbb{P}}$ ($x_\alpha = \sum \alpha_i x_i$, $\langle x_1, \dots, x_n \rangle = M$, $U_{\mathbb{P}}$ non empty open subset of $\operatorname{Spec} A$ and $\mathbb{P}(A) = \{\mathfrak{P} \in \operatorname{Spec} A | A_{\mathfrak{P}}$ is $\mathbb{P}\}$). We show that: $LB_k(\mathbb{P})$ holds $\implies LB_k(\mathbb{GP})$ holds for the corresponding geometric property (in particular for $\mathbb{P}$ = regular, normal, reduced, R_s, $LB_k(\mathbb{GP})$ holds). As an application we obtain a Bertini theorem for hypersurface sections of a variety $X \subseteq \mathbf{P}_k^n$ concerning the geometric properties.*

KEY WORDS: *Local ring - Local geometric property - Hypersurface sections.*

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1 – Introduction

BERTINI showed that, given a smooth projective variety X contained in $\mathbf{P}_k^n$ with $k = \mathbb{C}$, the general hypersurface section of X is smooth too (see [2], Chap. 10 n. 25; for a modern approach see e.g. [7] Th. 8.18 or [8] Th. 6.3).

There have been many generalisation of this theorem: we recall the recent algebraic studies on trasversality made by KLEIMAM in [9] and SPEISER in [11] where they introduced a fully modern point of view of schemes over an algebraically closed field of arbitrary characteristic.

Another approach to this problem has been proposed by FLENNER in [4] (following GROTHENDIECK, see [6]).

He shows that, given a field k of arbitrary characteristic and given a local k-algebra (A, M, K) with K/k separable, then, for the general $\alpha = (\alpha_1, \dots, \alpha_n) \in K^n \implies$

$$\mathbb{P}(A/x_\alpha A) \supseteq \mathbb{P}(A) \cap V(x_\alpha) \cap U_{\mathbb{P}} \tag{1}$$

where $x_\alpha = \sum \alpha_i x_i$, $\{x_1, \dots, x_n\}$ is a generator system of M and $U_{\mathbb{P}}$ is a non empty open subset of $\operatorname{Spec} A$ depending on $\mathbb{P}$, being $\mathbb{P}$ one of the following local properties: regular, normal, reduced, R_s and S_r.

These results, applied to the local ring of the vertex of the affine cone corresponding to a projective variety X, imply, by standard techniques, the corresponding global BERTINI theorem for the variety X.

In this work we want to show that every time we have a result like (1) for a property $\mathbb{P}$ we have the same result for the corresponding geometric $\mathbb{GP}$ and that the corresponding global results hold (there are known only for geometrically regular, see [8] Chap. I ¶ 6).

In section 3 we introduce some topological remarks that we use in next section: we show that if k is a subfield of K, (k infinite), every non empty open set of K^n can be contracted to a non empty open set of k^n.

In section 4, that is the main section, we give a local BERTINI theorem for the properties $\mathbb{GP}$ in an axiomatic form and we show that there are properties $\mathbb{GP}$ (for example $\mathbb{GP} = S_r$, geom. R_s, geom. regular, geom. normal, geom. reduced) to which we can apply the theorem. In these cases we show that the $\mathbb{GP}$-locus is open.

In the last section 5 we deduce a global BERTINI's theorem for the hypersurface sections of a variety X in a projective space over a field of arbitrary characteristic and for the above cited $\mathbb{GP}$ (we extend for many geometrical properties Th. 6.3 in [8] concerning the only geometrical regular property).

2 – Preliminaries and notation

In this section we fix the standard notation to be used in the following.

The rings considered are always commutative with an identity element.

If A is a ring, $\Omega(A)$ is the set of maximal ideals of A.

We recall here the definition of essentially finite type algebra and some properties of this algebra that we shall have to use in section 4.

DEFINITION 2.1 ([5] Chap. IV 1.3.8). *Let T be a ring. A T-algebra S is of essentially finite type (e.f.t. for short) if S is a T-isomorphic to $R^{-1}C$ where C is a T-algebra of finite type and R is a multiplicatively closed subset of C.*

PROPERTIES 2.2 ([5] Chap. IV 1.3.9 (ii); [10] 34.A).

(i) *If S is a T-algebra of e.f.t. and T' is a T-algebra then $S' = S \otimes_T T'$ is a T'-algebra of e.f.t.*

(ii) *If S is a T-algebra of e.f.t. and T is a excellent ring then S is an excellent ring.*

In the following all topological spaces are considered with their ZARISKY topology. If A is a ring we put $V(x_1, \dots, x_n)$ the closed subset of Spec A corresponding to the ideal generated by the elements $x_1, \dots, x_n$ of A.

Let $F[\underline{T}] = F[T_1, \dots, T_n]$ be a polynomial ring with coefficients in the field F. We identify $F^n = \{(\alpha_1, \dots, \alpha_n) | \alpha_i \in F\}$ with the topological subspace $S = \{(T_1 - \alpha_1, \dots, T_n - \alpha_n) | \alpha_1 \in F\}$ of $\max \operatorname{Spec} F[\underline{T}]$. (We observe that $\overline{F}^{\,n} = \max \operatorname{Spec}[\underline{T}]$ where $\overline{F}$ denotes the algebraic closure of the field F).

The expression "x general in X", where X is topological space, means that x is in a dense open subset of X.

We recall here the definition of *geometric property*.

DEFINITION 2.3. *Let $\mathbb{P}$ be a local property and A a local ring containing a field k. We say that A is geometrically $\mathbb{P}$ over k* (**GP** *for short*) *if $A \otimes_k \bar{k}$ is $\mathbb{P}$.*

(See also [5] Chap. IV 6.7.7 for equivalent definitions).

Finally we put $\mathbb{P}(A) = \{\mathfrak{P}\,\mathrm{Spec}\,A | A_{\mathfrak{P}}$ verifies the local property $\mathbb{P}\}$.

3 – Some topological remarks

For our aim we have to prove that, given an infinite field k, if K/k is a field extension and $\mathcal{L}$ is an open dense subset of K^n then $\mathcal{L} \cap k^n$ is an open dense subset of k^n (Prop. 3.3). We prove this fact in two steps (the first one for the "open" property, the second one for the "dense" property).

We consider the following commutative diagram:

$$\begin{array}{ccc} K^n & \xrightarrow[i]{} & \mathrm{Spec}\,K[T_1,\dots,T_n] \\ {\scriptstyle j}\uparrow & & \downarrow{\scriptstyle f} \\ k^n & \xrightarrow[h]{} & \mathrm{Spec}\ k[T_1,\dots,T_n] \end{array}$$

where i, h are the inclusions of canonical maps and, as well known, K^n (resp. k^n) is a topological subspace of $\mathrm{Spec}\,K[\underline{T}]$ (resp. $\mathrm{Spec}\,k[\underline{T}]$).

LEMMA 3.1. *Let K/k be a field extension, then k^n is a subspace of K^n.*

PROOF CASE 1. *K/k algebraic extension.*

$\mathcal{L}$ closed in $K^n \implies$ exists C closed in $\mathrm{Spec}\,K[\underline{T}]$ such that $\mathcal{L} = K^n \cap C$.

By Theorem 5.E in [10], the going-up theorem holds for $f^* : k[\underline{T}] \to K[\underline{T}]$ and then, by Theorem 6.J of [10], f is a closed map. So we have that $f(C)$ is closed in $\mathrm{Spec}\,k[\underline{T}]$. Taking the induced closed subset $f(C) \cap k^n$ of k^n we have to prove that $f(C) \cap k^n = \mathcal{L} \cap k^n$. The inclusion $\mathcal{L} \cap k^n \subset f(C) \cap k^n$ is trivial. The other one is easy if we remark that:

$$f^{-1}(h(k^n)) = i(j(k^n))\,.$$

CASE 2. *K/k purely trascendental extension.*

Let $\mathcal{L} = \{(x_1, \dots, x_n) \in K^n | g(x_1, \dots, x_n) = 0$ with $g \in K[\underline{T}]\}$ be a fundamental closed set of K^n. Among the coefficients of g there are only a finite number t of elements of K trascendental over k and so we can reduce to the trascendental extension of finite type. Using induction on t we can consider that there is only one trascendental element Z (i.e. $t = 1$).

So $g(T_1, \dots, T_n) = a_{i_1 \dots i_n}(Z)T^{i_1} \dots T^{i_n}$ with $a_{i_1 \dots i_n}(Z) \in k(Z)$.

$(k_1, \dots, k_n) \in k^n \cap \mathcal{L} \iff g(k_1, \dots, k_n) = 0 \implies b_{i_1 \dots i_n}(Z)k^{i_1} \dots \dots k^{i_n} = 0$ with $b_{i_1 \dots i_n}(Z) \in k[Z]$ (obtained by clearing denominators and simplifying) $\iff g_r(k_1, \dots, k_n)Z^r + \dots + g_0(k_1, \dots, k_n) = 0$ (obtained ordering $b_{i_1 \dots i_n}(Z)k^{i_1} \dots k^{i_n}$ like a polynomial in Z) where $g_i(T_1, \dots, T_n) \in k[\underline{T}]$.

But Z is trascendental over k and so $(k_1, \dots, k_n) \in k^n \cap \mathcal{L} \iff g_i(k_1, \dots, k_n) = 0\ \forall i\ 0 \le i \le r$. Then we have $\mathcal{L} \cap k^n = V(g_1, \dots, g_r)$.

General case. It is well known that every field extension can be written as $k \subseteq K' \subseteq K$ with K'/k purely trascendental and K/K' algebraic. So we can apply subsequently Case 2 and Case 1. □

Lemma 3.2. *Let k be an infinite field, then k^n is irreducible.*

Proof. We want to show that the intersection of two non empty open sets is still non empty.

For this is clearly sufficient to show that if $f, g \in k[T_1, \dots, T_n]$ and $V(f) \neq k^n$, $V(g) \neq k^n$ then $V(fg) \neq k^n$. We use induction on n.

If $n = 1$ we consider the polynomial: $fg = (f_0 + \dots + f_i T^i)(g_0 + \dots + g_h T^h)$.

$fg = 0$ has the most $i + h$ solutions in $\bar{k}$ (and so in k) and this proves that $V(fg) \neq k$ because k is infinite.

Suppose now that the conclusion is true for any number of variables smaller than n.

We have $fg = (f_0 + f_1 T_n \dots + f_i T_n^i)(g_0 + g_1 T_n \dots + g_h T_n^h) = f_0 g_0 + \dots + f_i g_h T^{i+h}$ with $f_j, g_l \in k[T_1, \dots, T_{n-1}]$ for $0 \le j \le i$ and $0 \le l \le h$. Observe that $f_i g_h$ is a polynomial in $n-1$ variables $\implies$ by the inductions hypothesis, there exists an element $w = (k_1, \dots, k_{n-1}) \in k^{n-1}$ such that $f_i(k_1, \dots, k_{n-1}) g_h(k_1, \dots, k_{n-1}) \neq 0$. For this w we can find an element $a \in k$ such that $f(k_1, \dots, k_{n-1}, a) g(k_1, \dots, k_{n-1}, a) \neq 0$ because the polynomial in a single variable $f(k_1, \dots, k_{n-1}, T_n) g(k_1, \dots, k_{n-1}, T_n)$ has at most $i + h$ solutions in k and k is infinite.

Then there exists k_n such that $y = (k_1, \dots, k_{n-1}, k_n) \in V(fg)$. □

From the above lemmas we get:

PROPOSITION 3.3. *Let K be an extension of the infinite field k.*
If $\mathcal{L}$ is an open dense subset of K^n then $\mathcal{L} \cap k^n$ is an open dense subset of k^n.

PROOF. By Lemma 3.1 we know that $\mathcal{L} \cap k^n$ is open in k^n. By Lemma 3.2 it is enough to show that $\mathcal{L} \cap k^n$ is non empty. It is sufficient to prove this fact for $\mathcal{L} = K^n - V(f)$ with $f \in K[T_1, \dots, T_n]$, by induction on n.

If $n = 1$, $f(T) = K_0 + \dots + K_r T^r$ has at most r solutions in K and so in k.

Suppose that it is true for any integer $m < n$.

Put $f(T_1, \dots, T_n) = f_0 + f_1 T_n \dots + f_i T_n^i$ where $f_j \in K[T_1, \dots, T_{n-1}]$ for $0 \leq j \leq i$.

By induction hypothesis there exists $(k_1, \dots, k_{n-1}) \in k^{n-1}$ such that $f_i(k_1, \dots, k_{n-1}) \neq 0$. Considering $f(k_1, \dots, k_{n-1}, T_n)$ we observe that f has at most i solutions in k. Let $a \in k$ be a non solution for $f(k_1, .., k_{n-1}, T^n)$, then $(k_1, \dots, k_{n-1}, a) \in \mathcal{L} \cap k^n$. □

4 – Main result

The main purpose of this paragraph is to give a local BERTINI theorem for the geometric properties. Let's state some conventions for the following.

Let (A, M, K) and k a local ring and a field, respectively. When we say that S is a k-algebra we mean that A is a noetherian k-algebra, k is an infinite field and K is separable over k.

DEFINITION 4.1. *Let $\mathbb{P}$ be a local property of commutative rings. We say that $\mathbb{P}$ is a local* BERTINI *property if, for every local k-algebra (A, M, K) of e.f.t. and every set of generators $\langle x_1, \dots, x_n \rangle$ of M, the following condition holds:*

$LB_k(\mathbb{P})$ *for the general $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n$ we have*

$$\mathbb{P}(A/x_\alpha A) \supseteq \mathbb{P}(A) \cap V(x_\alpha) \cap U_{\mathbb{P}}$$

where $x_\alpha = \sum \alpha_i x_i$, and $U_{\mathbb{P}}$ is either Spec A *or* Spec $A - \{M\}$, *depending only on $\mathbb{P}$. We say briefly that $LB_k(\mathbb{P})$ holds.*

REMARK 4.2 We observe that $LB_k(\mathbb{P})$ holds for $\mathbb{P}$ =regular and $U_{\mathbb{P}} = \operatorname{Spec} A$, for $\mathbb{P}$ =normal, reduced, SERRE's properties R_s and S_r and $U_{\mathbb{P}} = \operatorname{Spec} A - \{M\}$ (in fact more general statements hold: see [4] Theorem 4.1 and Corollaries 4.2 and 4.3).

We want to prove that if A is a k-algebra of e.f.t. and $LB_k(\mathbb{P})$ holds for some property $\mathbb{P}$ then $LB_k(\mathbb{GP})$ holds too for the corresponding geometric property.

We need some lemmas.

LEMMA 4.3. *Let (A, M, K) be a k-algebra of e.f.t. and $B = A \otimes_k \bar{k}$. Then, for every $\mathcal{M} \in \Omega(B)$, $(B_{\mathcal{M}}, \mathcal{M}B_{\mathcal{M}}, K_{\mathcal{M}})$ is a $\bar{k}$-algebra of e.f.t.*

PROOF. Recall that $\varphi : A \to B$ is a flat homomorphism.

STEP 1. B is a semilocal $\bar{k}$-algebra and $MB_{\mathcal{M}} = \mathcal{M}B_{\mathcal{M}}$ $\forall \mathcal{M} \in \Omega(B)$.

Clearly B is a $\bar{k}$-algebra of e.f.t. and, being integral over A, we have $MB \subseteq \operatorname{Rad}(B)$.

$B/MB = K \otimes_A (A \otimes_k \bar{k}) = K \otimes_k \bar{k}$ and $\dim K \otimes_k \bar{k} = 0$. In fact $K \otimes_k \bar{k}$ is noetherian[(1)] (because B is a $\bar{k}$-algebra of e.f.t. by prop. 2.2 (i) and so it is noetherian) and integral over K and we can apply Theorem 20 in [10]. So $K \otimes_k \bar{k}$ is an artinian ring (Theorem 8.5 in [1]) and this proves that B is semilocal.

$K \otimes_k \bar{k}$ is also reduced (because K/k is separable and we can apply (27.1) Lemma 1 in [10]) and $\dim(K \otimes_k \bar{k})_{\mathcal{M}} = \dim(B/MB)_{\mathcal{M}} = 0$. This proves that $(B/MB)_{\mathcal{M}} = B_{\mathcal{M}}/MB_{\mathcal{M}}$ is a field, that is $MB_{\mathcal{M}} = \mathcal{M}B_{\mathcal{M}}$.

STEP 2. $K_{\mathcal{M}}$ is separable over $\bar{k}$ for every $\mathcal{M} \in \Omega(B)$ because every extension of an algebrically closed field is separable. □

LEMMA 4.4. *Let (A, M, K) be a k-algebra of e.f.t., $\{x_1, \ldots, x_n\}$ a generator system of M and $B = A \otimes_k \bar{k}$. If $LB_k(\mathbb{P})$ holds then:*

a) *for the general $\alpha = (\alpha_1, \ldots, \alpha_n) \in \bar{k}^n$: $\mathbb{P}(B/x_\alpha B) \supseteq \mathbb{P}(B) \cap V(x_\alpha B) \cap \mathbb{U}_{\mathbb{P}}$,*

b) *for the general $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$: $\mathbb{P}(B/x_\alpha B) \supseteq \mathbb{P}(B) \cap V(x_\alpha B) \cap \mathbb{U}_{\mathbb{P}}$*

where $x_\alpha = \sum \alpha_i x_i$ and $\mathbb{U}_{\mathbb{P}} = \operatorname{Spec} B$ (resp. $\operatorname{Spec} B - \Omega(B)$) if $\mathbb{U}_{\mathbb{P}} = \operatorname{Spec} A$ (resp. $\operatorname{Spec} A - \{M\}$).

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(1) $\dim K \otimes k = 0$ also by Corollaries 5.8 and 5.9 in [1] which don't require the noetherian property.

PROOF. a) In fact the condition $LB_{\bar{k}}(\mathbb{P})$ holds for $(B_{\mathcal{M}}, MB_{\mathcal{M}}, K_{\mathcal{M}})$ $\forall \mathcal{M} \in \Omega(B)$ by Lemma 4.3. So we can find an open dense subset $\mathcal{L}_{\mathcal{M}}$ of $\bar{k}^n$ such that $\forall \alpha \in \mathcal{L}_{\mathcal{M}}$ $\mathbb{P}(B_{\mathcal{M}}/x_\alpha B_{\mathcal{M}}) \supseteq \mathbb{P}(B_{\mathcal{M}}) \cap V(x_\alpha B_{\mathcal{M}}) \cap U_{\mathbb{P}}$. But B is semilocal by 4.3 so it has a finite number of maximal ideals: $\mathcal{M}_1, \dots, \mathcal{M}_d$.

Putting $\mathcal{L} = \mathcal{L}_{\mathcal{M}_1} \cap \dots \cap \mathcal{L}_{\mathcal{M}_d}$, this is an open dense subset of $\bar{k}^n$ (by Lemma 3.2), independent from M_i and so $\forall \alpha \in \mathcal{L}$ we have $\mathbb{P}(B/x_\alpha B) \supseteq \mathbb{P}(B) \cap V(x_\alpha B) \cap \mathbb{U}_{\mathbb{P}}$.

b) Use a) and Proposition 3.3.

THEOREM 4.5. *If $LB_k(\mathbb{P})$ holds for some local property $\mathbb{P}$ then $LB_k(\mathbb{GP})$ holds for the corresponding geometric property $\mathbb{GP}$ with $U_{\mathbb{GP}} = U_{\mathbb{P}}$.*

PROOF. If (A, M, K) is a k-algebra of e.f.t. and $\mathfrak{P} \in \mathbb{GP}(A) \cap V(x_\alpha) \cap U_{\mathbb{GP}}$ we have to prove that $\mathfrak{P} \in \mathbb{GP}(A/x_\alpha A)$.

Clearly we have: $\mathfrak{P} \in \mathbb{GP}(A/x_\alpha A) \iff (A_{\mathfrak{P}}/x_\alpha A_{\mathfrak{P}}) \otimes_k \bar{k}$ is $\mathbb{P} \iff (A/x_\alpha A) \otimes_A (A_{\mathfrak{P}} \otimes_k \bar{k})$ is $\mathbb{P}$.

Considering $\varphi : A \to B = A \otimes_k \bar{k}$ and $S = A - \mathfrak{P} \implies A_{\mathfrak{P}} \otimes_k \bar{k} \simeq S^{-1}B$ by Prop. 3.5 in [1]. If $\mathfrak{Q} \in \operatorname{Spec}(A_{\mathfrak{P}} \otimes_k \bar{k})$, let Q its image in $S^{-1}B$.

Then, $\forall \mathfrak{Q} \in \operatorname{Spec}(A_{\mathfrak{P}} \otimes_k \bar{k})$, $(A_{\mathfrak{P}} \otimes_k \bar{k})_{\mathfrak{Q}} \simeq B_Q$ is $\mathbb{P}$, i.e. $Q \in \mathbb{P}(B)$. It is also $Q \supset (x_\alpha)^e$ and $Q \in \mathbb{U}_{\mathbb{P}}$ according to the notations of Lemma 4.4 (note that if $\mathfrak{P} \neq M$ then $Q \notin \Omega(B)$). Applying Lemma 4.4 to B we have: $(B_Q)/(x_\alpha)B_Q \simeq (A_{\mathfrak{P}} \otimes_k \bar{k})_{\mathfrak{Q}}/(x_\alpha)(A_{\mathfrak{P}} \otimes_k \bar{k})_{\mathfrak{Q}}$ is $\mathbb{P}$ $\forall \mathfrak{Q} \in \operatorname{Spec}(A_{\mathfrak{P}} \otimes_k \bar{k}) \implies (A/x_\alpha A) \otimes_A (A_{\mathfrak{P}} \otimes_k \bar{k})$ is $\mathbb{P} \implies (A_{\mathfrak{P}}/x_\alpha A_{\mathfrak{P}}) \otimes_k \bar{k}$ is $\mathbb{P} \implies \mathfrak{P} \in \mathbb{GP}(A/x_\alpha A)$. □

COROLLARY 4.6. *$LB_k(\mathbb{GP})$ holds for k-algebra of e.f.t. (A, M, K) if:*

i) $\mathbb{GP}$ =geom. regular and $U_{\mathbb{GP}}(A) = \operatorname{Spec} A$;

ii) $\mathbb{GP} = S_r$, geom. SERRE*'s property R_s, geom. normal, geom. reduced and $U_{\mathbb{GP}}(A) = \operatorname{Spec} A - \{M\}$*

(with the notation given in Def. 4.1).

PROOF. By Remark 4.2 and Theorem 4.5. □

In connection with Theorem 4.5 it is important know that the $\mathbb{GP}$-locus of an e.f.t. k-algebra is open, at least for the properties $\mathbb{P}$ cited above. This will be shown in Theorem 4.7 below.

THEOREM 4.7. *Let A be a k-algebra of finite type, then $\mathbb{GP}(A)$ is an open subset of* $\operatorname{Spec} A$ *for $\mathbb{GP} = S_r$, geom.* SERRE*'s property R_s, geom. regular, geom. normal, geom. reduced.*

PROOF. We may assume that A is a k-algebra of finite type.

Indeed if A is a k algebra of e.f.t. then (Def. 2.1) $A = S^{-1}C$ where C is a k-algebra of finite type and S is a multiplicatively closed subset of C. If U is an open subset of $\operatorname{Spec} C$ and if we call φ the continuous map defined from $\operatorname{Spec}(S^{-1}C)$ to $\operatorname{Spec} C$ induced by the canonical homomorphism $\varphi^* : C \to S^{-1}C$, then $\varphi^{-1}(U)$ is an open subset of $\operatorname{Spec}(S^{-1}C) = \operatorname{Spec} A$. Moreover the properties $\mathbb{GP}$ are preserved by localisation.

(a) Case $\mathbb{GP}$ = geom. regular, geom. normal, geom. R_n.

We use a proof that looks like ZARISKY's theorem in [5] Chap. IV, 6.12.5..

We consider $A \otimes_k k'$, where $k' = k^{p^{-\infty}}$.

The morphism $\operatorname{Spec}(k') \to \operatorname{Spec}(k)$ is a universal homeomorphism and so the morphism $\operatorname{Spec}(A \otimes_k k') \to \operatorname{Spec} A$ is a homeomorphism.

Then the projection of $\mathbb{P}(A \otimes_k k')$ in $\operatorname{Spec} A$ is just the set $\mathbb{GP}(A)$ (by Theorem 6.7.7. Chap. IV [5]).

We have only to show that $\mathbb{P}(A \otimes_k k')$ is open in $\operatorname{Spec}(A \otimes_k k')$. But this is true:

i) for $\mathbb{P}$ =regular by [5] Chap. IV 6.12.5;

ii) for $\mathbb{P} = R_n$ by i) and [5] Chap. IV 6.12.9;

iii) for $\mathbb{P}$ =normal by i) and [5] Chap. IV 6.13.5.

(b) Case $\mathbb{GP} = S_n$ and geom. reduced.

A is a k-algebra of finite type and so it is excellent by prop. 2.2 (ii). So we can apply consideration 7.9.7 Chap. IV [5] for $\mathbb{P} = S_n$ and prop. 4.6.13 Chap. IV [5] for $\mathbb{P}$ =reduced. □

Using Theorem 4.7 we have:

COROLLARY 4.8. *If (A, M, K) is a k-algebra of e.f.t. then $\mathbb{GP}(A)$ is an open subset of* $\operatorname{Spec} A$ *for $\mathbb{GP} = S_r$, geom.* SERRE*'s property R_s, geom. regular, geom. normal, geom. reduced.*

5 – Application to global Bertini theorems

We want now to deduce from Theorem 4.5 a global BERTINI theorem for geometric properties of hypersurface sections of a projective variety over an arbitrary field.

For this we use a standard technique involving the vertex of the affine cone (see also [4] ¶5).

We give some *notation*: let k be a field, $X \subseteq \mathbf{P}_k^n$ a projective variety over the field k and $Y \subseteq X$ a closed subset of X. Let $Y^+ \subset X^+ \subseteq \mathbf{A}_k^{n+1}$ be the corresponding affine cones; put $A = O_{X^+,v}$ (where v is the vertex) and let I be the ideal of Y^+ in A. Let $X(\bar{k}), Y(\bar{k})$ be the varieties obtained from X and Y by making the base extension field $k \to \bar{k}$.

PROPOSITION 5.1. *Let $\mathbb{P}$ be a local property which is preserved by polynomials and fractions and which descends by faithful flatness. With the notation given above, the following are equivalent:*

(i) $X - Y$ *is* $\mathbb{GP}$ *over* k;

(ii) $X^+ - Y^+$ *is* $\mathbb{GP}$ *over* k;

(iii) $\operatorname{Spec} A - V(I)$ *is* $\mathbb{GP}$ *over* k.

PROOF. $X - Y$ is $\mathbb{GP}$ over $k \iff X(\bar{x}) - Y(\bar{k})$ is $\mathbb{P} \underset{(1)}{\iff} X^+(\bar{k}) - Y^+(\bar{k})$ is $\mathbb{P}$ ($\iff X^+ - Y^+$ is $\mathbb{GP}$ over k) $\underset{(2)}{\iff} \operatorname{Spec} A(\bar{k}) - V(I(\bar{k}))$ is $\mathbb{P} \iff \operatorname{Spec} A - V(I)$ is $\mathbb{GP}$ over k; where the equivalences (1) and (2) are due to proposition 2.1 in [3]. □

In the following let $S = \oplus S_d$ be a graded k-algebra of finite type so that $S_0 \simeq k$ and $S = k[S_1]$.

THEOREM 5.2. *$S = k[S_1]$ a graded k-algebra, k a field with infinitely many elements and $\{f_0, \dots, f_{n(q)}\}$ a generator system of S_q as a k-vector space. Let $\mathbb{P}$ be as in 5.1.*

If $LB_k(\mathbb{GP})$ holds for some geometrical property $\mathbb{GP}$ then, for the general $\alpha = (\alpha_0, \dots, \alpha_{n(q)}) \in k^{n(q)+1}$ we have that:

$$\mathbb{GP}(\operatorname{Proj}(S/f_\alpha S)) \supseteq \mathbb{GP}(\operatorname{Proj}(S) \cap V^+(f_\alpha))$$

where $f_\alpha = \sum \alpha_i f_i$.

PROOF. For $q = 1$ we can apply Prop. 5.1 and Th. 4.5 (observe that K, the residue field of A, coincides with k and so it is separable over k). For $q > 1$ we can reduce to the hyperplane case using the VERONESE map of degree q. □

COROLLARY 5.3. *With the hypothesis and notation as in Theorem 5.2 we have* $\mathbb{GP}(\mathrm{Proj}(S/x_\alpha S)) \supseteq \mathbb{GP}(\mathrm{Proj}(S)) \cap V^+(x_\alpha)$ *for* $\mathbb{GP} = S_r$, *geom.* SERRE*'s property* R_s, *geom. regular, geom. normal, geom. reduced.*

PROOF. Apply Theorem 5.2 and Corollary 4.6. □

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