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Lie algebra structures on Ω^1(M) and Ω^1(TM) for a Riemannian manifold

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RIASSUNTO: *Sia (M,g) una varietà Riemanniana. La metrica g e le parentesi di Lie dei campi vettoriali inducono varie strutture di algebre di Lie su $\Omega^1(M)$ ed $\Omega^1(TM)$. In questo articolo studiamo una struttura di algebra di Lie su $\Omega^1(M)$ "duale" dell'algebra di Lie $\mathcal{T}(M)$. Inoltre, definiamo due strutture di algebra di Lie su $\Omega^1(TM)$, mediante la forma simplettica indotta da g su TM e per mezzo del rilevamento orizzontale (completo) di g su TM. Infine, studiamo in dettaglio alcuni omomorfismi tra queste tre algebre di Lie.*

ABSTRACT: *Let (M,g) be a Riemannian manifold. The metric and the Lie bracket of vector fields yield several Lie algebra structures on $\Omega^1(M)$ and $\Omega^1(TM)$. In this paper a Lie algebra structure on $\Omega^1(M)$ "dual" to the Lie algebra $\mathcal{T}(M)$ is studied. Moreover two Lie algebra structures on $\Omega^1(TM)$ are defined by using the symplectic form on TM associated with g and the horizontal (complete) lift of g to TM, respectively. Some Lie algebra homomorphisms between these three Lie algebras are studied in details.*

KEY WORDS: *Riemannian manifold - Vertical lift - Complete lift - Horizontal lift - Lie Algebra.*

A.M.S. CLASSIFICATION: 53C20 - 53C15

Introduction

The Lie bracket of vector fields on a manifold M defines on $\mathcal{T}(M)$ a natural structure of a Lie algebra. Similar natural Lie algebra structure on $\Omega^1(M)$ does not exist. But the Lie algebra structure on $\mathcal{T}(M)$ can be transferred on $\Omega^1(M)$ by using an isomorphism between $\mathcal{T}(M)$ and $\Omega^1(M)$. This approach is frequently used for symplectic manifolds (M,ω), where the isomorphism between $\mathcal{T}(M)$ and $\Omega^1(M)$ is induced by the symplectic form ω and the obtained bracket on $\Omega^1(M)$ is the Poisson bracket of 1-forms, [4].

In this paper we consider M to be a Riemannian manifold with a metric g. The metric induces the canonical isomorphism $g^\flat : \mathcal{T}(M) \to \Omega^1(M)$ and its inverse $g^\sharp$. By using them we define the bracket on $\Omega^1(M)$ by $\{\alpha,\beta\}_g = g^\flat([g^\sharp(\alpha), g^\sharp(\beta)])$. $\Omega^1(M)$ with $\{\,,\}_g$ is a Lie algebra which is in some sense dual to the Lie algebra on $\mathcal{T}(M)$.

The same construction is used to define two Lie algebra structures on $\Omega^1(TM)$. A first structure of a Lie algebra on $\Omega^1(TM)$ is defined by using the natural lift of the metric to the canonical symplectic form on TM, which is usually defined as $\omega(g) = dd_v h$, where d_v is the vertical differentiation and $h(u) = \frac{1}{2}g(u,u)$ is the induced function on TM, [1]. The bracket $\{\,,\}_\omega$ on $\Omega^1(TM)$ defined by using $\omega(g)$ is the usual Poisson bracket of 1-forms on a symplectic manifold, [4].

A second structure of a Lie algebra on $\Omega^1(TM)$ is defined by using the horizontal lift $G(g)$ of the metric to a symmetric (0,2)-tensor field on TM, [3]. In the case of Riemannian manifolds, if the Levi-Civita connection is used, the horizontal lift coincides with the complete lift of the metric, [7].

Let us remark that to define a Lie algebra structure on $\Omega^1(TM)$ we could use other regular (0,2)-tensor fields on TM which are naturally induced from the metric, for instance the Sasaki metric.

We study relations between the Lie algebra structures on $\Omega^1(M)$ and $\Omega^1(TM)$. We use mappings from $\Omega^1(M)$ to $\Omega^1(TM)$ given by the classical vertical, complete and horizontal lifts of 1-forms. In Section 1 we recall the basic properties of these lifts, but for details we recommend to see [7]. We deduce that the complete lift of 1-forms is a Lie algebra homomorphism from the Lie algebra $(\Omega^1(M),\{\,,\}_g)$ to $(\Omega^1(TM),\{\,,\}_G)$ and that the lift $(2\alpha^H - \alpha^C)$ defines a Lie algebra homomorphism from the Lie algebra $(\Omega^1(M),\{\,,\}_g)$ to $(\Omega^1(TM),\{\,,\}_\omega)$. Moreover we deduce that the differences between $\{\alpha,\beta\}_g^H$, $\{\alpha^H,\beta^H\}_\omega$ and $\{\alpha^H,\beta^H\}_G$ depend on the curvature tensor of the Levi-Civita connection and the difference between $\{\alpha,\beta\}_g^C$ and $\{\alpha^C,\beta^C\}_\omega$ depends on the curvature tensor and the covariant differentials of $g^\sharp(\alpha)$ and $g^\sharp(\beta)$.

All manifolds and mappings are assumed to be infinitely differentiable.

1 – Lifts of functions, vector fields and 1-forms to TM

Let M be a manifold and $p_M : TM \to M$ be the tangent bundle. Let (x^i) be local coordinates on M and (x^i, u^i) the induced fibred coordinates on TM.

Let $f \in C^\infty(M)$ be a function. The *vertical lift* of f is the function $f^V = f \circ p_M$, $f^V \in C^\infty(TM)$. The *complete lift* of f is the function $f^C \in C^\infty(TM)$ such that $f^C(u) = df_x(u)$, $x = p_M(u)$. The coordinate expression of f^C is

$$f^C(u) = \frac{\partial f(x)}{\partial x^i} u^i. \tag{1.1}$$

Let ξ be a vector field on M. The *vertical lift* of ξ is the vector field ξ^V on TM such that $\xi^V f^C = (\xi f)^V$, $\forall f \in C^\infty(M)$. In coordinates, if $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$, then

$$\xi^V = \xi^i(x)\frac{\partial}{\partial u^i}. \tag{1.2}$$

The *complete lift* of ξ is the vector field ξ^C on TM such that $\xi^C f^C = (\xi f)^C$, $f \in C^\infty(M)$. In coordinates

$$\xi^C = \xi^i(x)\frac{\partial}{\partial x^i} + \frac{\partial \xi^i(x)}{\partial x^j} u^j \frac{\partial}{\partial u^i}. \tag{1.3}$$

REMARK 1.1. The complete lift of a vector field is called also the natural lift or the tangent lift or the flow lift and it can be defined also by

$$\exp(t\xi^C) = T(\exp(t\xi))$$

or by

$$\xi^C = i_M \circ T\xi,$$

where i_M is the canonical involution of TTM and $T\xi : TM \to TTM$ is the tangent prolongation of $\xi : M \to TM$.

Let $\alpha \in \Omega^1(M)$. The *vertical lift* of α is the 1-form $\alpha^V \in \Omega^1(TM)$ such that $\alpha^V(\xi^C) = (\alpha(\xi))^V$, for any $\xi \in \mathcal{T}(M)$. In coordinates, if $\alpha = \alpha_i(x)\, dx^i$, then

$$\alpha^V = \alpha_i(x)\, dx^i. \tag{1.4}$$

The *complete lift* of α is the 1-form $\alpha^C \in \Omega^1(TM)$ such that $\alpha^C(\xi^C) = (\alpha(\xi))^C$, for any $\xi \in \mathcal{T}(M)$. In coordinates

$$\alpha^C = \frac{\partial \alpha_i(x)}{\partial x^j} u^j\, dx^i + \alpha_i(x)\, du^i. \tag{1.5}$$

REMARK 1.2. In fact α^V is the pull-back of α with respect to p_M, i.e. $\alpha^V = (p_M)^*\alpha$. The complete lift of a 1-form can be characterized by

$$\alpha^C = s_M \circ T\alpha,$$

where $s_M : TT^*M \to T^*TM$ is the canonical transformation, [6].

LEMMA 1.1. *We have*
(1) $(fh)^V = f^V h^V$, $(fh)^C = f^C h^V + f^V h^C$,
(2) $(f\xi)^V = f^V \xi^V$, $(f\xi)^C = f^C \xi^V + f^V \xi^C$,
(3) $(f\alpha)^V = f^V \alpha^V$, $(f\alpha)^C = f^C \alpha^V + f^V \alpha^C$,
(4) $\xi^V(f^V) = 0$, $\xi^C(f^V) = (\xi f)^V$, $\alpha^V(\xi^V) = 0$, $\alpha^C(\xi^V) = (\alpha(\xi))^V$
for any $\alpha \in \Omega^1(M)$, $f, h \in C^\infty(M)$ *and* $\xi \in \mathcal{T}(M)$.

PROOF. (1) It is obvious.
(2) It follows from (1.1), (1.2) and (1.3).
(3) It follows from (1.1), (1.4) and (1.5).
(4) It follows from (1.1) - (1.5). □

Now let M be a Riemannian manifold with a metric g. Let Γ be the Levi-Civita connection on M, i.e. its Christoffel symbols are given by

$$\Gamma^i_{jk}(x) = \frac{g^{im}(x)}{2}(g_{mj,k} + g_{mk,j} - g_{jk,m}), \tag{1.6}$$

where $g_{ij,k}$ stands for $\frac{\partial g_{ij}(x)}{\partial x^k}$. Then for any $u \in TM$ the tangent space $T_u TM$ splits with respect to Γ into the horizontal and the vertical subspaces, i.e.

$$T_u TM = H_u \oplus V_u.$$

The connection Γ can be interpreted as a mapping $\Gamma : TM \oplus TM \to TTM$ called the horizontal lift. We denote $\Gamma(u,X) = X_u^H \in H_u$. This mapping is an isomorphism between the vector spaces T_xM and H_u, $p_M(u) = x$. If ξ is a vector field on M, then $\Gamma(u,\xi)$ is the vector field on TM denoted ξ^H and called the *horizontal lift* of ξ. In coordinates, if $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$, then

$$\xi^H = \xi^i(x)\frac{\partial}{\partial x^i} - \Gamma^i_{am}(x)u^a\xi^m(x)\frac{\partial}{\partial u^i}. \tag{1.7}$$

We have the natural canonical equivalence $\nu : TM \oplus TM \to VTM$ defined by $\nu(u,X)(df(u)) = Xf$, $\forall f \in C^\infty(M)$. Then we denote $\nu(u,X) = X_u^V \in V_u$ and $\nu(u,\xi) = \xi^V$ for any vector field ξ on M, where ξ^V is the vertical lift defined above.

Obviously, each tangent vector $Z_u \in T_uTM$ can be written in the form $Z_u = Z_u^h + Z_u^v$, where $Z_u^h \in H_u$, $Z_u^v \in V_u$ are uniquely determined vectors. Then there are unique vectors $X \in T_xM$, $Y \in T_xM$ such that $X_u^H = Z_u^h$ and $Y_u^V = Z_u^v$.

The *horizontal lift* of a 1-form α on M is the 1-form α^H on TM such that $\alpha^H(\xi^H) = 0$ and $\alpha^H(\xi^V) = (\alpha(\xi))^V$ for any $\xi \in \mathcal{T}(M)$. In coordinates

$$\alpha^H = \Gamma^a_{ik}(x)u^k\alpha_a(x)\,dx^i + \alpha_i(x)\,du^i. \tag{1.8}$$

LEMMA 1.2. *We have*
(1) $(f\xi)^H = f^V\xi^H$, $(f\alpha)^H = f^V\alpha^H$,
(2) $\alpha^C(\xi^H) + \alpha^H(\xi^C) = \alpha^C(\xi^C)$,
for any $f \in C^\infty(M)$, $\alpha \in \Omega^1(M)$ *and* $\xi \in \mathcal{T}(M)$.

PROOF. (1) From the coordinate expressions (1.7) and (1.8) it is easy to see that the horizontal lifts of vector fields and of 1-forms are $C^\infty(M)$-linear.

(2) It can be computed in coordinates by using (1.3), (1.5), (1.7) and (1.8). □

LEMMA 1.3. *We have*
(1) $[\xi^C,\eta^C]=[\xi,\eta]^C$, $[\xi^C,\eta^V]=[\xi,\eta]^V$, $[\xi^V,\eta^V]=0$,
(2) $[\xi^H,\eta^V]=(\nabla_\xi\eta)^V$, $[\xi^V,\eta^H]=-(\nabla_\eta\xi)^V$
for any $\xi,\eta\in\mathcal{T}(M)$, *where* ∇ *is the covariant differentiation with respect to the Levi-Civita connection.*

PROOF. It is easy to prove it by using the coordinate expressions. □

Now let $F=F^i_j(x)\frac{\partial}{\partial x^i}\otimes dx^j$ be a (1,1)-tensor field on M, i.e. F can be considered as a mapping $TM\to TM$ over M. We define the "lift" of F to a vertical vector field $F^\wedge$ on TM by

$$F^\wedge_u=(F(u))^V,$$

where $F(u)$ is the value of F in u. In coordinates

$$F^\wedge=F^i_j(x)u^j\frac{\partial}{\partial u^i}. \tag{1.9}$$

LEMMA 1.4. *Let* F,G *be two (1,1)-tensor fields on* M. *Then*

$$[F^\wedge,G^\wedge]=-[F,G]^\wedge,$$

where on the right hand side there is the usual commutator of (1,1)-tensor fields, i.e. $[F,G](u)=F(G(u))-G(F(u))$.

PROOF. It can be easily proved in coordinates by using (1.9). □

LEMMA 1.5. *We have*

$$\xi^C-\xi^H=(\nabla\xi)^\wedge,$$

for any $\xi\in\mathcal{T}(M)$.

PROOF. From (1.3) and (1.7) we have $\xi^C-\xi^H=(\frac{\partial\xi^i}{\partial x^j}+\Gamma^i_{jk}\xi^k)u^j\frac{\partial}{\partial u^i}=(\nabla\xi)^\wedge$. □

Let us remark that $(\nabla\xi)^\wedge_u = (\nabla_u\xi)^V$.

LEMMA 1.6. *We have*
(1) $[\xi^H, \eta^H] = [\xi, \eta]^H - (R(\xi, \eta))^\wedge$,
(2) $[\xi^H, (\nabla\eta)^\wedge] + [(\nabla\xi)^\wedge, \eta^H] = (R(\xi, \eta))^\wedge + [\nabla\xi, \nabla\eta]^\wedge + (\nabla[\xi, \eta])^\wedge$,
(3) $[\xi^V, (\nabla\eta)^\wedge] + [(\nabla\xi)^\wedge, \eta^V] = [\xi, \eta]^V$
for any $\xi, \eta \in \mathcal{T}(M)$, where R is the curvature tensor of the Levi-Civita connection.

PROOF. (1) It can be easily proved by using the coordinate expressions.

(2) From Lemma 1.3 (1) and Lemma 1.5 we have

$$[\xi^H + (\nabla\xi)^\wedge, \eta^H + (\nabla\eta)^\wedge] = [\xi, \eta]^H + (\nabla[\xi, \eta])^\wedge$$
$$[\xi^H, \eta^H] + [\xi^H, (\nabla\eta)^\wedge] + [(\nabla\xi)^\wedge, \eta^H] + [(\nabla\xi)^\wedge, (\nabla\eta)^\wedge] = [\xi, \eta]^H + (\nabla[\xi, \eta])^\wedge$$

and (2) now follows from Lemma 1.4 and (1).

(3) From Lemma 1.3 (1), Lemma 1.3 (2) and Lemma 1.5 we have

$$\begin{aligned} &[\xi^V, \eta^C - \eta^H] + [\xi^C - \xi^H, \eta^V] = \\ &= [\xi^V, \eta^C] + [\xi^C, \eta^V] - [\xi^V, \eta^H] - [\xi^H, \eta^V] = \\ &= 2[\xi, \eta]^V + (\nabla_\eta\xi)^V - (\nabla_\xi\eta)^V = [\xi, \eta]^V \end{aligned}$$

which is a consequence of the zero torsion of the Levi-Civita connection, i.e. $\nabla_\eta\xi - \nabla_\xi\eta = [\eta, \xi]$. □

COROLLARY 1.1. *We have*
(1) $[\xi^C, (\nabla\eta)^\wedge] + [(\nabla\xi)^\wedge, \eta^C] = R(\xi, \eta)^\wedge + 3[\nabla\xi, \nabla\eta]^\wedge + (\nabla[\xi, \eta])^\wedge$,
(2) $[\xi^H, \eta^C] + [\xi^C, \eta^H] = [\xi, \eta]^C + [\xi, \eta]^H - R(\xi, \eta)^\wedge + [\nabla\xi, \nabla\eta]^\wedge$
for any $\xi, \eta \in \mathcal{T}(M)$.

PROOF. (1) It follows from Lemma 1.4, Lemma 1.5 and Lemma 1.6 (2).

(2) It follows from Lemma 1.4, Lemma 1.5, Lemma 1.6 (1) and Lemma 1.6 (2). □

Let $\kappa = \kappa_{ij}(x)\, dx^i \otimes dx^j$ be a (0,2)-tensor field on M. We define the "lift" of κ to a 1-form $\kappa^\wedge$ on TM by

$$\kappa^\wedge_u = (\kappa(\cdot, u))^V.$$

In coordinates

$$\kappa^\wedge = \kappa_{ij}(x) u^j dx^i. \tag{1.10}$$

LEMMA 1.7. *We have*

$$\alpha^C - \alpha^H = (\nabla\alpha)^\wedge,$$

for any $\alpha \in \Omega^1(M)$.

PROOF. In coordinates we have

$$\alpha^C - \alpha^H = (\alpha_{i,k} - \Gamma^a_{ik}\alpha_a)u^k\, dx^i = (\nabla\alpha)^\wedge,$$

where $\alpha_{i,k} = \frac{\partial\alpha_i}{\partial x^k}$. □

Let us remark that $(\nabla\alpha)^\wedge_u = (\nabla_u\alpha)^V$.

2 – A Lie algebra structure on $\Omega^1(M)$

Let (M, g) be a Riemannian manifold and (x^i) be a local coordinate chart on M. Then $g = g_{ij}(x) dx^i \odot dx^j$, $\det(g_{ij}) \neq 0$, $g_{ij} = g_{ji}$. The metric g defines the $C^\infty(M)$-linear isomorphism

$$g^\flat : \mathcal{T}(M) \to \Omega^1(M), \quad g^\flat(\xi) = g(\cdot, \xi)$$

and its inverse

$$g^\sharp : \Omega^1(M) \to \mathcal{T}(M), \quad g^\sharp(\alpha) = \tilde{g}(\alpha, \cdot),$$

where $\tilde{g}$ is the dual metric. In coordinates we have, if $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$, $\alpha = \alpha_i(x)\, dx^i$,

$$g^\flat(\xi) = g_{im}\xi^m\, dx^i \tag{2.1}$$

and

$$g^\sharp(\alpha) = g^{im}\alpha_m \frac{\partial}{\partial x^i}, \tag{2.2}$$

where $g^{im}g_{mj} = \delta^i_j$. In what follows we shall write $\xi^\flat$ instead of $g^\flat(\xi)$ and $\alpha^\sharp$ instead of $g^\sharp(\alpha)$.

LEMMA 2.1. *We have*
(1) $(\nabla\alpha^\sharp)^\wedge_u = ((\nabla_u\alpha)^\sharp)^V$,
(2) $(\alpha^\sharp)^C - (\alpha^\sharp)^H = ((\nabla_u\alpha)^\sharp)^V$
for any $\alpha \in \Omega^1(M)$.

PROOF. (1) It follows from (1.9) and (2.2) by using the Christoffel symbols (1.6).

(2) It follows from (1) and Lemma 1.5. □

DEFINITION 2.1. *We define an* $\mathbb{R}$*-bilinear mapping*

$$\{\,,\}_g : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$$

by

$$\{\alpha, \beta\}_g =: [\alpha^\sharp, \beta^\sharp]^\flat,$$

where on the right hand side there is the Lie bracket on $\mathcal{T}(M)$.

Hence the bracket $\{\,,\}_g$ is defined by the following commutative diagram

$$\begin{array}{ccc} \Omega^1(M) \times \Omega^1(M) & \xrightarrow{\{,\}_g} & \Omega^1(M) \\ {\scriptstyle g^\flat \times g^\flat}\uparrow & & \uparrow{\scriptstyle g^\flat} \\ \mathcal{T}(M) \times \mathcal{T}(M) & \xrightarrow{[,]} & \mathcal{T}(M) \end{array}$$

REMARK 2.1. Since the Levi-Civita connection has the zero torsion we get

$$\{\alpha, \beta\}_g = \nabla_{\alpha^\sharp}\beta - \nabla_{\beta^\sharp}\alpha.$$

The coordinate expression of $\{\alpha, \beta\}_g$ is the following

$$\begin{aligned} \{\alpha, \beta\}_g = [&g_{iq,p}(g^{qm}g^{pk} - g^{qk}g^{pm})\alpha_m\beta_k + \\ &+ g^{pm}(\alpha_m\beta_{i,p} - \beta_m\alpha_{i,p})]\, dx^i, \end{aligned} \tag{2.3}$$

where $\alpha = \alpha_i(x)\, dx^i, \beta = \beta_i(x)\, dx^i$.

THEOREM 2.1. *We have*

$$\{\alpha, \beta\}_g = -\{\beta, \alpha\}_g,$$
$$\{\{\alpha, \beta\}_g, \gamma\}_g + \{\{\beta, \gamma\}_g, \alpha\}_g + \{\{\gamma, \alpha\}_g, \beta\}_g = 0,$$
$$\{f\alpha, h\beta\}_g = f(\alpha^\sharp h)\beta - h(\beta^\sharp f)\alpha + fh\{\alpha, \beta\}_g$$

for any $\alpha, \beta, \gamma \in \Omega^1(M)$ *and any* $f, h \in C^\infty(M)$.

PROOF. By Definition 2.1 we have $\{\alpha, \beta\}_g = [\alpha^\sharp, \beta^\sharp]^\flat = -[\beta^\sharp, \alpha^\sharp]^\flat = -\{\beta, \alpha\}_g$.

Further we have $\{\{\alpha, \beta\}_g, \gamma\}_g = [\{\alpha, \beta\}_g^\sharp, \gamma^\sharp]^\flat = [(([\alpha^\sharp, \beta^\sharp]^\flat)^\sharp, \gamma^\sharp]^\flat = [[\alpha^\sharp, \beta^\sharp], \gamma^\sharp]^\flat$ and the second identity now follows from the Jacobi identity of the Lie bracket and linearity of $g^\flat$.

Finally $\{f\alpha, h\beta\}_g = [(f\alpha)^\sharp, (h\beta)^\sharp]^\flat = [f(\alpha^\sharp), h(\beta^\sharp)]^\flat = f(\alpha^\sharp h)(\beta^\sharp)^\flat - h(\beta^\sharp f)(\alpha^\sharp)^\flat + fh[\alpha^\sharp, \beta^\sharp]^\flat = f(\alpha^\sharp h)\beta - h(\beta^\sharp f)\alpha + fh\{\alpha, \beta\}_g$. □

COROLLARY 2.1. *The bracket* $\{\,,\}_g$ *defines on* $\Omega^1(M)$ *a Lie algebra structure.* □

3 – Lie algebra structures on $\Omega^1(TM)$

In this Section we shall define two Lie algebra structures on $\Omega^1(TM)$.

We construct from the metric g a 2-form ω on TM as follows

$$\omega_u(g)(X^H, Y^H) = 0, \qquad \omega_u(g)(X^H, Y^V) = -g_x(X, Y),$$
$$\omega_u(g)(X^V, Y^H) = g_x(Y, X), \qquad \omega_u(g)(X^V, Y^V) = 0$$

for any $X, Y \in T_xM$, $p_M(u) = x$. The matrix expression of $\omega(g)$ is

$$\omega(g) = \begin{bmatrix} (g_{mj}\Gamma^m_{ai} - g_{mi}\Gamma^m_{aj})u^a & -g_{ij} \\ g_{ij} & 0 \end{bmatrix}$$

which can be rewritten by using the Christoffel symbols (1.6) as

$$\omega(g) = \begin{bmatrix} (g_{mj,i} - g_{mi,j})u^m & -g_{ij} \\ g_{ij} & 0 \end{bmatrix}. \tag{3.1}$$

REMARK 3.1. The 2-form $\omega(g)$ is in fact the canonical symplectic form on TM derived from the metric g. Usually this form is defined by $\omega(g) = dd_v h$, where $h(u) = \frac{1}{2}g(u,u)$ and d_v is the vertical differential, [1].

Let $S_\omega : \mathcal{T}(TM) \to \Omega^1(TM)$ be the induced $C^\infty(TM)$-linear isomorphism given by $S_\omega(\Xi) = -i_\Xi\omega$, $\Xi \in \mathcal{T}(TM)$.

DEFINITION 3.1. *We define an $\mathbb{R}$-bilinear mapping*

$$\{\,,\}_\omega : \Omega^1(TM) \times \Omega^1(TM) \to \Omega^1(TM)$$

by

$$\{\Phi, \Psi\}_\omega =: S_\omega([X_\Phi, X_\Psi]),$$

where X_Φ and X_Ψ are uniquely determined vector fields on TM such that $S_\omega(X_\Phi) = \Phi$ and $S_\omega(X_\Psi) = \Psi$, respectively.

Hence the bracket $\{\,,\}_\omega$ is the usual Poisson bracket of 1-forms on a symplectic manifold, [4], and is defined by the following commutative diagram

$$\begin{array}{ccc} \Omega^1(TM) \times \Omega^1(TM) & \xrightarrow{\{,\}_\omega} & \Omega^1(TM) \\ {\scriptstyle S_\omega \times S_\omega}\uparrow & & \uparrow{\scriptstyle S_\omega} \\ \mathcal{T}(TM) \times \mathcal{T}(TM) & \xrightarrow{[,]} & \mathcal{T}(TM) \end{array}$$

THEOREM 3.1. *We have*

$$\{\Phi, \Psi\}_\omega = -\{\Psi, \Phi\}_\omega,$$
$$\{\{\Phi, \Psi\}_\omega, \Theta\}_\omega + \{\{\Psi, \Theta\}_\omega, \Phi\}_\omega + \{\{\Theta, \Phi\}_\omega, \Psi\}_\omega = 0,$$
$$\{F\Phi, H\Psi\}_\omega = F(X_\Phi H)\Psi - H(X_\Psi F)\Phi + FH\{\Phi, \Psi\}_\omega$$

for any $\Phi, \Psi, \Theta \in \Omega^1(TM)$ and any $F, H \in C^\infty(TM)$.

PROOF. By Definition 3.1 we have

$$\{\Phi,\Psi\}_\omega = S_\omega([X_\Phi,X_\Psi]) = -S_\omega([X_\Psi,X_\Phi]) = -\{\Psi,\Phi\}_\omega.$$

Further we have $X_{\{\Phi,\Psi\}_\omega} = [X_\Phi,X_\Psi]$ and

$$\{\{\Phi,\Psi\}_\omega,\Theta\}_\omega = S_\omega([X_{\{\Phi,\Psi\}_\omega},X_\Theta]) = S_\omega([[X_\Phi,X_\Psi],X_\Theta])$$

and the second identity follows from the Jacobi identity of the Lie bracket and from the linearity of S_ω.

Finally

$$\begin{aligned}\{F\Phi,H\Psi\}_\omega &= S_\omega([X_{F\Phi},X_{H\Psi}]) = S_\omega([FX_\Phi,HX_\Psi]) = \\ &= S_\omega(F(X_\Phi H)X_\Psi - H(X_\Psi F)X_\Phi + FH[X_\Phi,X_\Psi]) = \\ &= F(X_\Phi H)\Psi - H(X_\Psi F)\Phi + FH\{\Phi,\Psi\}_\omega.\end{aligned}$$ □

COROLLARY 3.1. *The bracket $\{\,,\}_\omega$ defines on $\Omega^1(TM)$ a Lie algebra structure.* □

Now, we construct from the metric g a symmetric (0,2)-tensor field $G(g)$ on TM as follows

$$G_u(g)(X^H,Y^H)=0, \qquad G_u(g)(X^H,Y^V)=g_x(X,Y),$$
$$G_u(g)(X^V,Y^H)=g_x(Y,X), \qquad G_u(g)(X^V,Y^V)=0$$

for all $X,Y\in T_xM$, $p_M(u)=x$. The matrix expression of $G(g)$ is

$$G(g) = \begin{bmatrix} (g_{mj}\Gamma^m_{ai} + g_{mi}\Gamma^m_{aj})u^a & g_{ij} \\ g_{ij} & 0 \end{bmatrix}$$

which can be rewritten by using the Christoffel symbols (1.6) as

$$G(g) = \begin{bmatrix} g_{ij,m}u^m & g_{ij} \\ g_{ij} & 0 \end{bmatrix}. \tag{3.2}$$

Let $S_G : \mathcal{T}(TM) \to \Omega^1(TM)$ be the induced $C^\infty(TM)$-linear isomorphism given by $S_G(\Xi) = G(\cdot,\Xi)$, $\Xi\in\mathcal{T}(TM)$.

REMARK 3.2. The (0,2)-tensor field $G(g)$ is the horizontal lift of the metric g to the space of metrics on TM and it is a pseudo-Riemannian metric of signature (n, n) on TM, [3]. In the case of Riemannian manifolds, if the Levi-Civita connection is used, the horizontal lift coincides with the complete lift of the metric, [7]. Let us note that the induced transformation $TTM \to T^*TM$ coincides with $s_M \circ Tg^\flat \circ i_M$, where i_M is the canonical involution on TTM, $Tg^\flat$ is the tangent mapping of $g^\flat : TM \to T^*M$ and $s_M : TT^*M \to T^*TM$ is the canonical transformation, [6].

DEFINITION 3.2. *We define an $\mathbb{R}$-bilinear mapping*

$$\{\,,\}_G : \Omega^1(TM) \times \Omega^1(TM) \to \Omega^1(TM)$$

by

$$\{\Phi, \Psi\}_G =: S_G([\widetilde{X}_\Phi, \widetilde{X}_\Psi]),$$

where $\widetilde{X}_\Phi$ and $\widetilde{X}_\Psi$ are uniquely determined vector fields on TM such that $S_G(\widetilde{X}_\Phi) = \Phi$ and $S_G(\widetilde{X}_\Psi) = \Psi$, respectively.

Hence the bracket $\{\,,\}_G$ is defined by the following commutative diagram

$$\begin{array}{ccc} \Omega^1(TM) \times \Omega^1(TM) & \xrightarrow{\{,\}_G} & \Omega^1(TM) \\ {\scriptstyle S_G \times S_G}\uparrow & & \uparrow{\scriptstyle S_G} \\ \mathcal{T}(TM) \times \mathcal{T}(TM) & \xrightarrow{[,]} & \mathcal{T}(TM) \end{array}$$

THEOREM 3.2. *We have*

$$\begin{gathered} \{\Phi, \Psi\}_G = -\{\Psi, \Phi\}_G, \\ \{\{\Phi, \Psi\}_G, \Theta\}_G + \{\{\Psi, \Theta\}_G, \Phi\}_G + \{\{\Theta, \Phi\}_G, \Psi\}_G = 0, \\ \{F\Phi, H\Psi\}_G = F(\widetilde{X}_\Phi H)\Psi - H(\widetilde{X}_\Psi F)\Phi + FH\{\Phi, \Psi\}_G \end{gathered}$$

for any $\Phi, \Psi, \Theta \in \Omega^1(TM)$ and any $F, H \in C^\infty(TM)$.

PROOF. The proof is the same as for Theorem 3.1. □

COROLLARY 3.2. *The bracket $\{\,,\}_G$ defines on $\Omega^1(TM)$ a Lie algebra structure.* □

4 – Lie algebra homomorphisms

In this section we shall study homomorphisms (given by the vertical, complete and horizontal lifts of 1-forms) between the Lie algebras $(\Omega^1(M), \{\,,\}_g)$, $(\Omega^1(TM), \{\,,\}_\omega)$ and $(\Omega^1(TM), \{\,,\}_G)$.

LEMMA 4.1. *We have*
(1) $(\alpha^\sharp)^V = -X_{\alpha^V}$,
(2) $(\alpha^\sharp)^H = X_{\alpha^H}$,
(3) $X_{(\nabla\alpha)^\wedge} = -(\nabla\alpha^\sharp)^\wedge$,
(4) $X_{\alpha^C} = (\alpha^\sharp)^C - 2(\nabla\alpha^\sharp)^\wedge$
for any $\alpha \in \Omega^1(M)$.

PROOF. (1) By (2.2), (3.1) and (1.4) we have

$$(\alpha^\sharp)^V = g^{ia}\alpha_a \frac{\partial}{\partial u^i} = -X_{\alpha^V}\,.$$

(2) By (2.2), (3.1) and (1.9) we have

$$(\alpha^\sharp)^H = g^{ia}\alpha_a \frac{\partial}{\partial x^i} - \Gamma^i_{jk} g^{ja}\alpha_a u^k \frac{\partial}{\partial u^i} = X_{\alpha^H}\,.$$

(3) From (1) and Lemma 2.1 (1) we have

$$(\nabla\alpha^\sharp)^\wedge = ((\nabla_u\alpha)^\sharp)^V = -X_{(\nabla_u\alpha)^V} = -X_{(\nabla\alpha)^\wedge}\,.$$

(4) From (2), (3), Lemma 1.7 and Lemma 2.1 (2) we have

$$X_{\alpha^C} = X_{\alpha^H} + X_{(\nabla\alpha)^\wedge} = (\alpha^\sharp)^H - (\nabla\alpha^\sharp)^\wedge = (\alpha^\sharp)^C - 2(\nabla\alpha^\sharp)^\wedge\,. \qquad \square$$

LEMMA 4.2. *We have*

$$S_\omega(F^\wedge) = -(g^\flat F)^\wedge,$$

where $g^\flat F$ *is the (0,2)-tensor field obtained by the result of the mapping* $(g^\flat \otimes id) : TM \otimes T^*M \to T^*M \otimes T^*M$ *on* F.

PROOF. It can be easily computed in coordinates. □

COROLLARY 4.1. *We have*
(1) $\{\alpha^V, \beta^H\}_\omega + \{\alpha^H, \beta^V\}_\omega = \{\alpha, \beta\}_g^V$,
(2) $\{\alpha^V, (\nabla\beta)^\wedge\}_\omega + \{(\nabla\alpha)^\wedge, \beta^V\}_\omega = -\{\alpha, \beta\}_g^V$,
(3) $\{\alpha^V, \beta^C\}_\omega + \{\alpha^C, \beta^V\}_\omega = 0$
for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. (1) It follows from Definition 3.1, Lemma 1.3 (2) and Lemma 4.1 (1).
(2) It follows from Lemma 1.6 (3), Lemma 4.1 (1) and Lemma 4.1 (3).
(3) It follows from (1), (2), and Lemma 1.7. □

THEOREM 4.1. *We have*

$$\{\alpha^H, \beta^H\}_\omega = \{\alpha, \beta\}_g^H + (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge$$

for any $\alpha, \beta \in \Omega^1(M)$, *where* R *is the curvature tensor of the Levi-Civita connection.*

PROOF. We have

$$\begin{aligned}\{\alpha^H, \beta^H\}_\omega &= S_\omega([X_{\alpha^H}, X_{\beta^H}]) = S_\omega([(\alpha^\sharp)^H, (\beta^\sharp)^H]) = \\ &= S_\omega([\alpha^\sharp, \beta^\sharp]^H - R(\alpha^\sharp, \beta^\sharp)^\wedge) = \\ &= S_\omega((\{\alpha, \beta\}_g^\sharp)^H) + (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge = \\ &= \{\alpha, \beta\}_g^H + (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge.\end{aligned}$$ □

LEMMA 4.3. *We have*
(1) $\{(\nabla\alpha)^\wedge, (\nabla\beta)^\wedge\}_\omega = (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge$,
(2) $\{\alpha^H, (\nabla\beta)^\wedge\}_\omega + \{(\nabla\alpha)^\wedge, \beta^H\}_\omega =$
$= (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge + (\nabla\{\alpha, \beta\}_g)^\wedge$
for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. (1) We have $\{(\nabla\alpha)^\wedge, (\nabla\beta)^\wedge\}_\omega = S_\omega([X_{(\nabla\alpha)^\wedge}, X_{(\nabla\beta)^\wedge}]) = = S_\omega([(\nabla\alpha^\sharp)^\wedge, (\nabla\beta^\sharp)^\wedge]) = -S_\omega([\nabla\alpha^\sharp, \nabla\beta^\sharp]^\wedge) = (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge$.

(2) We have from Lemma 1.6 (2), Lemma 4.1 (2) and Lemma 4.1 (3)

$$\begin{aligned}\{\alpha^H, (\nabla\beta)^\wedge\}_\omega + \{(\nabla\alpha)^\wedge, \beta^H\}_\omega &= S_\omega([X_{\alpha^H}, X_{(\nabla\beta)^\wedge}] + [X_{(\nabla\alpha)^\wedge}, X_{\beta^H}]) = \\ &= -S_\omega([(\alpha^\sharp)^H, (\nabla\beta^\sharp)^\wedge] + [(\nabla\alpha^\sharp)^\wedge, (\beta^\sharp)^H]) = \\ &= -S_\omega((R(\alpha^\sharp, \beta^\sharp))^\wedge + [\nabla\alpha^\sharp, \nabla\beta^\sharp]^\wedge + (\nabla[\alpha^\sharp, \beta^\sharp])^\wedge) = \\ &= (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge + (\nabla\{\alpha, \beta\}_g)^\wedge. \qquad \square\end{aligned}$$

THEOREM 4.2. *We have*

$$\{\alpha^C, \beta^C\}_\omega = \{\alpha, \beta\}_g^C + 2(g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + 2(g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge$$

for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. By Lemma 1.7, Lemma 4.3 (1), Lemma 4.3 (2) and Theorem 4.1 we have

$$\begin{aligned}\{\alpha^C, \beta^C\}_\omega &= \{\alpha^H, \beta^H\}_\omega + \{\alpha^H, (\nabla\beta)^\wedge\}_\omega + \{(\nabla\alpha)^\wedge, \beta^H\}_\omega + \\ &+ \{(\nabla\alpha)^\wedge, (\nabla\beta)^\wedge\}_\omega = \{\alpha, \beta\}_g^H + 2(g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + \\ &+ 2(g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge + (\nabla\{\alpha, \beta\})^\wedge = \\ &= \{\alpha, \beta\}_g^C + 2(g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + 2(g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge. \qquad \square\end{aligned}$$

COROLLARY 4.2. *We have*

$$\begin{aligned}&\{\alpha^C, \beta^H\}_\omega + \{\alpha^H, \beta^C\}_\omega = \\ &= \{\alpha, \beta\}_g^C + \{\alpha, \beta\}_g^H + 3(g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge\end{aligned}$$

for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. It follows from Lemma 1.7, Theorem 4.1 and Theorem 4.2. $\square$

LEMMA 4.4. *We have*

(1) $(\alpha^\sharp)^V = \widetilde{X}_{\alpha^V}$,

(2) $(\alpha^\sharp)^H = \widetilde{X}_{\alpha^H}$,

(3) $\widetilde{X}_{(\nabla\alpha)^\wedge} = (\nabla\alpha^\sharp)^\wedge$,

(4) $\widetilde{X}_{\alpha^C} = (\alpha^\sharp)^C$

for any $\alpha \in \Omega^1(M)$.

PROOF. (1) From (1.2) and (3.2) we have $(\alpha^\sharp)^V = g^{ia}\alpha_a \frac{\partial}{\partial u^i} = \widetilde{X}_{\alpha^V}$.

(2) From (1.7) and (3.2) we have $(\alpha^\sharp)^H = g^{ia}\alpha_a \frac{\partial}{\partial x^i} - \Gamma^i_{jk} g^{ja}\alpha_a u^k \frac{\partial}{\partial u^i} = \widetilde{X}_{\alpha^H}$.

(3) It follows from (1) and Lemma 2.1.

(4) By (2) and (3) we have $\widetilde{X}_{\alpha^C} = \widetilde{X}_{\alpha^H} + \widetilde{X}_{(\nabla\alpha)^\wedge} = (\alpha^\sharp)^H + (\nabla\alpha^\sharp)^\wedge = (\alpha^\sharp)^C$. □

COROLLARY 4.3. *We have*
(1) $\{\alpha^V, \beta^H\}_G + \{\alpha^H, \beta^V\}_G = \{\alpha, \beta\}_g^V$,
(2) $\{\alpha^V, (\nabla\beta)^\wedge\}_G + \{(\nabla\alpha)^\wedge, \beta^V\}_G = \{\alpha, \beta\}_g^V$,
(3) $\{\alpha^V, \beta^C\}_G + \{\alpha^C, \beta^V\}_G = 2\{\alpha, \beta\}_g^V$
for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. (1) It follows from Definition 3.2, Lemma 1.3 (2) and Lemma 4.4 (1).

(2) It follows from Lemma 1.6 (3), Lemma 4.4 (1) and Lemma 4.1 (1).

(3) It follows from (1), (2) and Lemma 1.7. □

THEOREM 4.3. *We have*

$$\{\alpha^C, \beta^C\}_G = \{\alpha, \beta\}_g^C$$

for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. It follows from Definition 3.2, Lemma 1.3 (1) and Lemma 4.4 (4). □

COROLLARY 4.4. *The complete lift of 1-forms is a Lie algebra homomorphism from* $(\Omega^1(M), \{\,,\}_g)$ *to* $(\Omega^1(TM), \{\,,\}_G)$. □

LEMMA 4.5. *We have*

$$S_G(F^\wedge) = (g^\flat F)^\wedge.$$

PROOF. It can be easily proved in coordinates. □

THEOREM 4.4. *We have*

$$\{\alpha^H, \beta^H\}_G = \{\alpha, \beta\}_g^H - (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge$$

for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. It follows from Definition 3.2, Lemma 1.6 (1), Lemma 4.4 (2) and Lemma 4.5. □

LEMMA 4.6. *We have*

$$\{\alpha^H, (\nabla\beta)^\wedge\}_G + \{(\nabla\alpha)^\wedge, \beta^H\}_G = \\ = \{\alpha, \beta\}_g^C - \{\alpha, \beta\}_g^H + (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge$$

for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. We get from Lemma 1.6 (2), Lemma 4.4 (2), Lemma 4.4 (3) and Lemma 4.5

$$\begin{aligned}\{\alpha^H, (\nabla\beta)^\wedge\}_G + \{(\nabla\alpha)^\wedge, \beta^H\}_G &= S_G([X_{\alpha^H}, X_{(\nabla\beta)^\wedge}] + [X_{(\nabla\alpha)^\wedge}, X_{\beta^H}]) = \\ &= S_G([(\alpha^\sharp)^H, (\nabla\beta^\sharp)^\wedge] + [(\nabla\alpha^\sharp)^\wedge, (\beta^\sharp)^H]) = \\ &= S_G((R(\alpha^\sharp, \beta^\sharp))^\wedge + [\nabla\alpha^\sharp, \nabla\beta^\sharp]^\wedge + (\nabla[\alpha^\sharp, \beta^\sharp])^\wedge) = \\ &= (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge + (\nabla\{\alpha, \beta\})^\wedge.\end{aligned}$$

□

COROLLARY 4.5. *We have*

$$\{\alpha^C, \beta^H\}_G + \{\alpha^H, \beta^C\}_G = \\ = \{\alpha, \beta\}_g^C + \{\alpha, \beta\}_g^H - (g^\flat R(\alpha^\sharp, \beta^\sharp))^\wedge + (g^\flat[\nabla\alpha^\sharp, \nabla\beta^\sharp])^\wedge$$

for any $\alpha, \beta \in \Omega^1(M)$.

PROOF. It follows from Lemma 1.7, Theorem 4.3 and Theorem 4.4. □

From Lemma 1.1 (3) and Lemma 1.2 (1) we see that the vertical, complete and horizontal lifts of 1-forms are $\mathbb{R}$-linear mappings from $\Omega^1(M)$ to $\Omega^1(TM)$. Hence we can define the linear combinations of these lifts by $\alpha^{(k_1V+k_2C+k_3H)} = k_1\alpha^V + k_2\alpha^C + k_3\alpha^H$, $k_i \in \mathbb{R}$.

THEOREM 4.5. *The morphism from $\Omega^1(M)$ to $\Omega^1(TM)$ which transforms any $\alpha \in \Omega^1(M)$ to $\alpha^{(2H-C)} \in \Omega^1(TM)$ is a Lie algebra homomorphism from the Lie algebra $(\Omega^1(M), \{\,,\}_g)$ to the Lie algebra $(\Omega^1(TM), \{\,,\}_\omega)$.*

PROOF. Suppose the commutative diagram

$$\begin{array}{ccc}
\Omega^1(TM)\times\Omega^1(TM) & \xrightarrow{\{,\}_\omega} & \Omega^1(TM) \\
{\scriptstyle S_\omega\times S_\omega}\uparrow & & \uparrow{\scriptstyle S_\omega} \\
\mathcal{T}(TM)\times\mathcal{T}(TM) & \xrightarrow{[,]} & \mathcal{T}(TM) \\
{\scriptstyle S_G\times S_G}\downarrow & & \downarrow{\scriptstyle S_G} \\
\Omega^1(TM)\times\Omega^1(TM) & \xrightarrow{\{,\}_G} & \Omega^1(TM)
\end{array}$$

It is easy to see that

$$(S_\omega \circ S_G^{-1})(\{\Phi,\Psi\}_G) = \{(S_\omega \circ S_G^{-1})(\Phi), (S_\omega \circ S_G^{-1})(\Psi)\}_\omega,$$

i.e. $S_\omega \circ S_G^{-1}$ is a Lie algebra isomorphism from the Lie algebra $(\Omega^1(TM), \{\,,\}_G)$ to the Lie algebra $(\Omega^1(TM), \{\,,\}_\omega)$. From Theorem 4.3 we see that the complete lift of 1-forms is a Lie algebra homomorphism from the Lie algebra $(\Omega^1(M),\{\,,\}_g)$ to the Lie algebra $(\Omega^1(TM),\{\,,\}_G)$. Combining these two Lie algebra homomorphisms, we get a Lie algebra homomorphism from the Lie algebra $(\Omega^1(M), \{\,,\}_g)$ to the Lie algebra $(\Omega^1(TM), \{\,,\}_\omega)$, such that the diagram

$$\begin{array}{ccc}
\Omega^1(TM) & \xrightarrow{S_\omega\circ S_G^{-1}} & \Omega^1(TM) \\
{\scriptstyle C}\uparrow & & \uparrow \\
\Omega^1(M) & = & \Omega^1(M)
\end{array}$$

commutes. In coordinates, if $\alpha = \alpha_i(x)\, dx^i \in \Omega^1(M)$, then

$$(S_\omega \circ S_G^{-1})(\alpha^C) = (-\alpha_{i,m}u^m + 2\Gamma^a_{im}u^m\alpha_a)\, dx^i + \alpha_i\, du^i.$$

It is easy to see that

$$(S_\omega \circ S_G^{-1})(\alpha^C) = 2\alpha^H - \alpha^C = \alpha^{(2H-C)}. \qquad \square$$

THEOREM 4.6. *$k_1V + k_2C + k_3H$ is a Lie algebra morphism from the Lie algebra $(\Omega^1(M), \{\,,\}_g)$ to the Lie algebra $(\Omega^1(TM), \{\,,\}_G)$ if and only if $k_1 = 0$, $k_2 = 1$, $k_3 = 0$.*

PROOF. It follows immediately from Theorem 4.3, Theorem 4.4, Corollary 4.3 (1), Corollary 4.3 (3) and Corollary 4.5. $\square$

THEOREM 4.7. *$k_1V + k_2C + k_3H$ is a Lie algebra morphism from the Lie algebra $(\Omega^1(M), \{\,,\}_g)$ to the Lie algebra $(\Omega^1(TM), \{\,,\}_\omega)$ if and only if $k_1 = 0$, $k_2 = -1$, $k_3 = 2$.*

PROOF. It follows immediately from Theorem 4.1, Theorem 4.2, Corollary 4.1 (1), Corollary 4.1 (3) and Corollary 4.2. $\square$

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