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On the $\overline{\partial}$-equation in domains between strictly pseudoconvex hypersurfaces

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RIASSUNTO: *Si costruisce un operatore integrale che risolve la $\overline{\partial}$-equazione in domini compresi tra due iperfuperficie pseudoconvesse e si riconosce che la soluzione è Lip (1/2). Si utilizzano la formula integrale di Leray-Koppelman e la costruzione di Henkin-Ramirez.*

ABSTRACT: *We construct integral solution operators for the $\overline{\partial}$-equation in domains between strictly pseudoconvex hypersurfaces and obtain the Lip(1/2)-estimate for these domains.*

KEY WORDS: *$\overline{\partial}$-equation, Strictly pseudoconvex hypersurfaces, Lip(1/2)-estimate, Integral solution operator.*

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1 – Introduction

This paper deals with integral solution operators for the $\overline{\partial}$-equation in domains between strictly pseudoconvex hypersurfaces and we obtain the Lip(1/2)- estimate for these domains.

Similar questions in strictly pseudoconvex domains have been extensively studied; see HENKIN-LEITERER [3] and RANGE [4]. More precisely we consider a domain $D \subset \mathbb{C}^n$ of the form $D = D_2 - \overline{D}_1$ where D_1 and D_2 are strictly pseudoconvex domains and $\overline{D}_1 \subset D_2$. In case $\overline{D}_1$ is holomorphically convex in D_2, we construct explicit integral operators

$$S_q : Z^1_{(0,q)}(\overline{D}) \to C^1_{(0,q-1)}(D)$$

($1 \leq q \leq n-2$) defined on the set $Z^1_{(0,q)}(\overline{D})$ of $(0,q)$-forms which are continuous on $\overline{D}$, C^1 in D and $\overline{\partial}$-closed in D; these operators solve the $\overline{\partial}$-equation in D and satisfy the Lip(1/2)-estimate. The main tools we use are the Leray - Koppelman integral formula and the Henkin - Ramirez construction.

In the general case, i.e., if $\overline{D}_1$ is not necessarily holomorphically convex in D_2, we prove that for each $f \in Z^1_{(o,q)}(\overline{D})$ there is a solution g of the equation $\overline{\partial} g = f$ which is Lip(1/2) in D. This follows from the first case considered combined with the fact that $H^{(0,q)}_{\overline{\partial}}(G-K) = 0$ for $1 \leq q \leq n-2$ if G is a pseudoconvex domain in $\mathbb{C}^n$ and $K(\subset G)$ is a Stein compactum.

Finally this technique can be applied to other situations too as we indicate in the last section.

2 – Preliminaries

2.1. THE LERAY-KOPPELMAN FORMULA. Let $D \subset \mathbb{C}^n$ be a bounded domain with C^1 boundary and $Y : (\partial D) \times D \to \mathbb{C}^n$ be a C^1-map so that

$$(1) \qquad \sum_{j=1}^{n} (\zeta_j - z_j) Y_j(\zeta, z) = 1 \quad \text{for every} \quad (\zeta, z) \in (\partial D) \times D\,.$$

Set $b_j(\zeta, z) = \dfrac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|}$ and $\eta^Y_j(\zeta, z, \lambda) = (1-\lambda) b_j(\zeta, z) + \lambda Y_j(\zeta, z)$ for $j = 1, \dots, n$ and $\lambda \in [0,1]$. Define $\omega'_q(\eta^Y)$ and $\omega'_q(Y)$ as follows

$$\omega'_q(\eta^Y)(\zeta, z, \lambda) = c(n,q) \det \left[\eta^Y_j, \overbrace{\overline{\partial}_z \eta^Y_j}^{q}, \overbrace{(\overline{\partial}_\zeta + d_\lambda)\eta^Y_j}^{n-q-1} \right]$$

and

$$\omega'_q(Y)(\zeta, z) = c(n,q) \det \left[Y_j, \overbrace{\overline{\partial}_z Y_j}^{q}, \overbrace{\overline{\partial}_\zeta Y_j}^{n-q-1} \right]$$

where $c(n,q) = \frac{1}{(2\pi i)^n} \cdot \frac{(n-1)!}{q!(n-q-1)!}$. (In the above determinants j runs from $j=1$ to $j=n$ forming the n-rows of them).

For $f \in C_{(o,q)}(D)$ and $z \in D$ let us set

$$T_q^Y f(z) = (-1)^q \int_{\zeta \in D} f(\zeta) \wedge \omega'_{q-1}(b)(\zeta,z) \wedge \omega(\zeta) - (-1)^q \int_{(\zeta,\lambda)\in(\partial D)\times[0,1]} f(\zeta)\wedge\omega'_{q-1}(\eta^Y)(\zeta,z,\lambda)\wedge\omega(\zeta)$$

and

$$L_q^Y f(z) = (-1)^q \int_{\zeta \in \partial D} f(\zeta)\wedge\omega'_q(Y)(\zeta,z)\wedge\omega(\zeta)$$

where $\omega(\zeta) = d\zeta_1 \wedge \ldots \wedge d\zeta_n$.

Then the following Leray-Koppelman formula is proved:

Every $f \in C^1_{(0,q)}(\overline{D})$ can be decomposed in $C_{(0,q)}(D)$ as follows:

$$f = \overline{\partial}(T_q^Y f) + T_{q+1}^Y(\overline{\partial} f) + L_q^Y f. \tag{2}$$

(see HENKIN-LEITERER [3]).

2.2. REMARK. Suppose $f \in C_{(0,q)}(\overline{D})$ so that $f \in C^1_{(0,q)}(D)$ and $\overline{\partial} f = 0$ in D. Then $T_q^Y f$ and $L_q^Y f$ can be defined in this case too and

$$f = \overline{\partial}(T_q^Y f) + L_q^Y f \quad \text{in} \quad D. \tag{3}$$

(The point here is that f is not assumed to be in $C^1(\overline{D})$). To prove (3) we must show that for $\varphi \in C^\infty_{(n,n-q)}(D)$ with compact support $K \subset D$ we have

$$\int_K f\wedge\varphi = (-1)^{q-1}\int_K T_q^Y f\wedge\overline{\partial}\varphi + \int_K L_q^Y f\wedge\varphi. \tag{4}$$

Let $U \supset \partial D$ be a sufficiently small neighborhood of ∂D and $\tilde{Y} : U \times K \to \mathbb{C}^n$ a C^1 extension of $Y : \partial D \times K \to \mathbb{C}^n$ so that (1) still holds. Let us approximate D from inside with domains $D_\varepsilon, \varepsilon > 0 (D_\varepsilon \subset\subset D)$

with smooth boundaries ∂D_ε. Then (2) applied to D_ε and $f \in C^1_{(0,q)}(\overline{D}_\varepsilon)$ gives

$$\int_K f \wedge \varphi = (-1)^{q-1} \int_K T_q^{\widetilde{Y},\varepsilon} f \wedge \overline{\partial} \varphi + \int_K L_q^{Y,\varepsilon} f \wedge \varphi$$

since $\overline{\partial} f = 0$ in D. Letting $\varepsilon \to 0$ we see that

$$\int_K T_q^{\widetilde{Y},\varepsilon} f \wedge \overline{\partial} \varphi \longrightarrow \int_K T_q^Y f \wedge \overline{\partial} \varphi \quad \text{and} \quad \int_K L_q^{Y,\varepsilon} f \wedge \varphi \longrightarrow \int_K L_q^Y f \wedge \varphi$$

and (4) follows.

2.3. THE HENKIN-RAMIREZ CONSTRUCTION. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2-boundary and ρ a C^2 strictly plurisubharmonic function in a neighborhood of $\overline{\Omega}$ such that $\Omega = \{\rho < 0\}$, $\partial\Omega = \{\rho = 0\}$ and $d\rho \neq 0$ on $\partial\Omega$. Then there exists a neighborhood $\widetilde{V}$ of $\partial\Omega$ and V of $\overline{\Omega}$, small constants $\varepsilon > 0$ and $c > 0$ and functions $g_j(\zeta, z)$, $j = 1, \dots, n$ which are C^1 in $(\zeta, z) \in \widetilde{V} \times V$, holomorphic in z (for each fixed ζ) and so that

$$G(\zeta, z) = \sum_{j=1}^n (\zeta_j - z_j) g_j(\zeta, z) = \begin{cases} \phi(\zeta, z) E_1(\zeta, z) & \text{for} \quad |\zeta - z| \leq \varepsilon \\ E_2(\zeta, z) & \text{for} \quad |\zeta - z| > \varepsilon \end{cases}$$

where $\phi(\zeta, z)$ is the modified Levi polynominal of ρ, i.e.,

$$\phi(\zeta, z) = \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{j,k=1}^n \varphi_{jk}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k)$$

where φ_{jk} are C^∞ functions which are sufficiently close to the continuous functions $\dfrac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}$; also $E_1(\zeta, z) \neq 0$ for $|\zeta - z| \leq \varepsilon$ and $E_2(\zeta, z) \neq 0$ for $|\zeta - z| > \varepsilon$; moreover $2 \operatorname{Re} \phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2$ for $(\zeta, z) \in \widetilde{V} \times V$ and $|\zeta - z| \leq \varepsilon$. (For the proof see HENKIN-LEITERER [3]).

3 – Domains between strictly pseudoconvex hypersurfaces

3.1. CHOOSING "COMMON" DEFINING FUNCTION. Let us consider a domain D of the form $D = D_2 - \overline{D}_1$ where $D_1, D_2 \subset \mathbb{C}^n$ are bounded pseudoconvex domains with C^2 boundaries and $\overline{D}_1 \subset D_2$. Let $\rho_1 \in C^2(\overline{D}_1)$ be a strictly plurisubharmonic function in a neighborhood U of $\overline{D}_1$ so that $D_1 = \{\rho_1 < 0\}$, $\partial D_1 = \{\rho_1 = 0\}$ and $d\rho_1 \neq 0$ on ∂D_1. Let $\rho_2 \in C^2(\overline{D}_2)$ be a strictly plurisubharmonic function in a neighborhood of $\overline{D}_2$ so that $D_2 = \{\rho_2 < 1\}$, $\partial D_2 = \{\rho_2 = 1\}$ and $d\rho_2 \neq 0$ on ∂D_2. Let us assume that

$$D_1 \subset\subset \left\{\rho_2 < \frac{1}{6}\right\} \subset \left\{\rho_2 < \frac{2}{3}\right\} \subset\subset U.$$

Set $\rho = \psi\rho_1 + \chi \circ \rho_2$ where ψ is a C^∞ function on $\mathbb{C}^n$ with compact support in $\left\{\rho_2 < \frac{2}{3}\right\}$ and $\psi \equiv 1$ on the set $\left\{\rho_2 \leq \frac{1}{3}\right\}$; χ is a C^∞ convex increasing function on $(-\infty, 2]$, strictly increasing on $\left[\frac{2}{3}, 2\right]$, which is to be chosen later. On the set $\left\{\frac{2}{3} < \rho_2 < 2\right\}$, $\psi = 0$ and therefore $\rho = \chi \circ \rho_2$; hence ρ is strictly plurisubharmonic there.

On the other hand

$$\sum_{j,k=1}^{n} \frac{\partial^2(\chi \circ \rho_2)}{\partial z_j \partial \overline{z}_k} t_j \overline{t}_k = (\chi'' \circ \rho_2). \left|\sum_{j=1}^{n} \frac{\partial \rho_2}{\partial z_j} t_j\right|^2 + (\chi' \circ \rho_2). \sum_{j,k=1}^{n} \frac{\partial^2 \rho_2}{\partial z_j \partial \overline{z}_k} t_j \overline{t}_k$$

and $\sum_{j,k=1}^{n} \frac{\partial^2 \rho_2}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k \geq a > 0$ for $|t| = 1$, $\rho_2(z) \leq \frac{2}{3}$.

Let $M =: \sup_{\substack{\rho_2(z) \leq \frac{2}{3} \\ |t|=1}} \left|\sum_{j,k=1}^{n} \frac{\partial^2(\psi\rho_1)}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k\right| < \infty.$

Since $(\chi'' \circ \rho_2).|\sum_{j=1}^{n} \frac{\partial \rho_2}{\partial z_j} t_j| \geq 0$, we will have $\sum_{j,k=1}^{n} \frac{\partial^2 \rho_2}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k > 0$ for $z \in \left\{\frac{1}{3} < \rho_2 < \frac{2}{3}\right\}$, $|t| = 1$ provided that $\chi' > \frac{M}{a}$ on $\left[\frac{1}{3}, \frac{2}{3}\right]$.

Therefore if we choose χ so that $\chi = 0$ on $\left(-\infty, \frac{1}{6}\right]$, convex, increasing on $(-\infty, 2]$ and $\chi' > \frac{M}{a}$ on $\left[\frac{1}{3}, 2\right]$ then ρ is a C^2 strictly plurisubharmonic function on a neighborhood of $\overline{D}_2$, $\rho = \rho_1$ on a neighborhood of $\overline{D}_1$ and $\rho = \chi \circ \rho_2$ on a neighborhood of ∂D_2.

Modifying the above construction (essentially by rescaling the parameters) we can easily prove the following lemma

3.2. LEMMA. *Let* $D = D_2 - \overline{D}_1$ *where* D_1, D_2 *are bounded strictly pseudoconvex domains in* $\mathbb{C}^n$ *with* C^2*-boundaries,* $\overline{D}_1 \subset D_2$ *and such that there exists a* C^2 *strictly plurisubharmonic function on* D_2 *which is a defining function for* D_1*. Then there exists* $\rho \in C^2(\overline{D}_2)$ *strictly plurisubharmonic in a neighborhood of* $\overline{D}_2$ *such that* $\partial D_2 = \{\rho = 1\}$, $\partial D_1 = \{\rho = 0\}, D_1 = \{\rho < 0\}$, $D_2 = \{\rho < 1\}$ *and* $d\rho \neq 0$ *on* $\partial D_1 \cup \partial D_2$.

3.3. A FUNCTION Y FOR $\{-a < \rho < a\}$. Let ρ be as in § 2.3. Then if $a > 0$ is sufficiently small then the set $A =: \{-a < \rho < a\} \subset \widetilde{V}$ and the boundary ∂A consists of two hypersurfaces $\Gamma_1 = \{\rho = a\}$ and $\Gamma_2 = \{\rho = -a\}$; by Sard's theorem a can be chosen so that Γ_1 and Γ_2 are smooth, i.e., $d\rho \neq 0$ on ∂A.

Now for $\zeta \in \Gamma_1$ and $z \in A$ let us define $Y_j(\zeta, z) = [G(\zeta, z)]^{-1} g_j(\zeta, z)$ (with the notation of § 2.3); for $\zeta \in \Gamma_2$ and $z \in A$ let us define $Y_j(\zeta, z) = [G(z, \zeta)]^{-1} g_j(z, \zeta)$.

Since $2 \operatorname{Re} \phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2$ for $(\zeta, z) \in \widetilde{V} \times \widetilde{V}$, $|\zeta - z| \leq \varepsilon$, it follows that $Y_j(\zeta, z)$, $j = 1, \dots, n$, are well-defined for $(\zeta, z) \in (\partial A) \times A$ and that $\sum_{j=1}^{n} (\zeta_j - z_j) Y_j(\zeta, z) = 1$. Moreover $Y_j(\zeta, z)$ is holomorphic in z for each $\zeta \in \Gamma_1$ and for each $z \in A$, $Y_j(\zeta, z)$ extends holomorphically in ζ in a neighborhood of Γ_2. Therefore $\omega'_q(Y)(\zeta, z) = 0$ for $\zeta \in \partial A$, $z \in A$ and $1 \leq q \leq n-2$. It follows that if we set $S_q^A f =: T_q^Y f$ for $f \in C_{(0,q)}(\overline{A})$ then $f = \overline{\partial}[S_q^A f]$ in A provided that $f \in C^1(A)$ and $\overline{\partial} f = 0$ (by remark 2.2).

4 – Integral operators

In this section we prove first the following

4.1. THEOREM 1. *Let* D *be as in lemma 3.2. Then for* $1 \leq q \leq n-2$ *there are linear operators*

$$S_q : Z^1_{(0,q)}(\overline{D}) \to C^1_{(0,q-1)}(D)$$

where $Z^1_{(0,q)}(\overline{D}) = \{f \in C_{(0,q)}(\overline{D}) \cap C^1_{(0,q)}(D) : \overline{\partial} f = 0$ *in* $D\}$ *such that*

(i) *$\bar\partial(S_q f) = f$ in D for $f \in Z^1_{(0,q)}(\overline{D})$*
(ii) *there is a constant $C > 0$ so that $|S_q f|_{1/2,D} \le C|f|_{\overline{D}}$ for $f \in Z^1_{(0,q)}(\overline{D})$ and*
(iii) *the estimate in* (ii) *is stable for small C^2 perturbations of ∂D.*

(Here $|.|_{1/2,D}$ denotes the Lipschitz norm of order $1/2$ in D and $|.|_{\overline{D}}$ the supremum norm in $\overline{D}$).

4.2. PROOF IN THE CASE $D = A = \{-a < \rho < a\}$. With notation as in § 3.3 we claim that the operator S^A_q proves Theorem 1 in this case. Indeed (i) was already proved in § 3.3. Furthermore the integral which defines $S^A_q f$ involves three integrals: one over D, one over $\Gamma_1 \times [0,1]$ and another one over $\Gamma_2 \times [0,1]$. The first two of them can be estimated exactly as in the case of strictly pseudoconvex domains and to be proved that they satisfy estimate (ii). As far as the third term is concerned we only have to go through the proof of the classical case of strictly pseudoconvex domains (see RANGE [4]) and see that with minor modifications of the proof we can prove that this term also satisfies estimate (ii).

Finally (iii) is proved analogously to the case of strictly pseudoconvex domains.

4.3. LEMMA. *Let ρ be as in lemma 3.2 and $A = \{a_1 < \rho < a_3\}$ and $B = \{a_2 < \rho < a_4\}$ where $0 \le a_1 < a_2 < a_3 < a_4 \le 1$. Let us fix a $q \ge 2$ and suppose that Theorem 1 holds for A and B for this q and for $A \cap B$ and $q-1$. Then the theorem holds for $A \cup B$ and q.*

PROOF. The hypotheses mean that there are operators S^A_q, S^B_q and $S^{A\cap B}_{q-1}$ with properties analogous to these of S_q.

Let $f \in Z^1_{(0,q)}(\overline{A \cup B})$ and let $\chi_A \in C^\infty_0(A)$ (compact support in A), $\chi_B \in C^\infty_0(B)$ so that $\chi_A + \chi_B = 1$ on $A \cup B$. Define

$$S^{A\cup B}_q f =: \begin{cases} S^A_q f - \bar\partial[\chi_B S^{A\cap B}_{q-1}(S^A_q f - S^B_q f)] & \text{on } A \\ S^B_q f + \bar\partial[\chi_A S^{A\cap B}_{q-1}(S^A_q f - S^B_q f)] & \text{on } B. \end{cases}$$

We claim that $S^{A\cup B}_q f$ is well-defined. To see this let us first observe that $\bar\partial(S^A_q f) = f$ on A and that $S^A_q f$ (which is C^1 in A) has continuous extension to $\overline{A}$ by estimate (ii) (which we assume that it holds in A) and similarly for $S^B_q f$; therefore $S^A_q f - S^B_q f \in C^1_{(0,q-1)}(A \cap B) \cap C_{(0,q-1)}(\overline{A \cap B})$

and $\overline{\partial}(S_q^A f - S_q^B f) = 0$ in $A \cap B$; hence $S_{q-1}^{A\cap B}(S_q^A f - S_q^B f)$ is well-defined and $\chi_B S_{q-1}^{A\cap B}(S_q^A f - S_q^B f) \in C^1_{(0,q-1)}(A)$.

Similarly we see that the part of $S_q^{A\cup B} f$ on B is well-defined too. Finally on $A \cap B$ we have (since $\chi_A + \chi_B = 1$)

$$\{S_q^A f - \overline{\partial}[\chi_B S_{q-1}^{A\cap B}(S_q^A f - S_q^B f)]\} - \{S_q^B f + \overline{\partial}[\chi_A S_{q-1}^{A\cap B}(S_q^A f - S_q^B f)]\} =$$
$$= (S_q^A f - S_q^B f) - \overline{\partial} S_{q-1}^{A\cap B}(S_q^A f - S_q^B f) = 0$$

where we have used that $S_{q-1}^{A\cap B}$ satisfies (i) in $A \cap B$. It is now easy to check that $S_q^{A\cup B}$ satisfies (i), (ii) and (iii) in $A \cup B$; this completes the proof of the lemma.

4.4. LEMMA. *With notation as in § 3.3 let $A = \{-a < \rho < a\}$ for $a > 0$ small. Suppose that ∂A is smooth, i.e., $d\rho \neq 0$ on ∂A. Then every holomorphic function f on A which is continuous on $\overline{A}$ extends to a holomorphic function on $\{\rho < a\}$.*

PROOF. If $Y(\zeta, z)$ $(\zeta \in \Gamma_1 \cup \Gamma_2, z \in A)$ is as in § 3.3 then by the Cauchy-Fantappiè formula for the domain A we obtain

$$f(z) = \int\limits_{\zeta\in\Gamma_1} f(\zeta)\omega_0'(Y)(\zeta,z)\wedge\omega(\zeta) - \int\limits_{\zeta\in\Gamma_2} f(\zeta)\omega_0'(Y)(\zeta,z)\wedge\omega(\zeta) \text{ for } z \in A\,.$$

Since $Y(\zeta, z)$ extends holomorphically in ζ in a neighborhood of Γ_2 for each $z \in A$, it follows that the integral over Γ_2 is zero. Therefore

$$f(z) = \int\limits_{\zeta\in\Gamma_1} f(\zeta)\omega_0'(Y)(\zeta,z)\wedge\omega(\zeta) \quad \text{for} \quad z \in A\,.$$

But the above integral is defined for all $z \in \{\rho < a\}$ and is holomorphic there; hence it gives us the required extension of f.

4.5. LEMMA. *Let A and B be as in lemma 4.3. If Theorem 1 holds for A and B when $q = 1$ then the theorem holds for $A \cup B$ and $q = 1$ too.*

PROOF. By hypotheses we have operators S_1^A and S_1^B for A and B. Let $f \in Z^1_{(0,1)}(\overline{A \cup B})$. Then $S_1^A f$ and $S_1^B f$ are C^1 functions in A and B and $\overline{\partial}(S_1^A f - S_1^B f) = 0$ in $A \cap B$. Therefore $S_1^A f - S_1^B f$ is holomorphic in $A \cap B$ and by Lemma 4.4 there exists $F^A f$ a holomorphic function in the set $\{\rho < a_3\}$ (which contains A) which is the holomorphic extension of $S_1^A f - S_1^B f$.

Define $S_1^{A \cup B} f$ as follows:

$$S_1^{A\cup B} f = \begin{cases} S_1^A f - F^A f & \text{on} \quad A \\ S_1^B f & \text{on} \quad B\,. \end{cases}$$

It is now easy to check that $S_1^{A \cup B}$ satisfies (i), (ii) and (iii) of Theorem 1 for $A \cup B$ and $q = 1$; this completes the proof of the lemma.

4.6. PROOF OF THEOREM 1. We choose ρ as in lemma 3.2. Then for each $\beta \in [0, 1]$ we consider the strictly pseudoconvex domain $\Omega_\beta = \{\rho < \beta\}$ and the corresponding Henkin-Ramirez construction as described in § 2.3. Then by compactness of $\overline{D}$ we can cover it by a finite number of sets of the form $A = \{a_1 < \rho < a_2\}$ such that the theorem holds for each of these A's (also in view of § 4.2). Applying now lemmas 4.3 and 4.5 several times we can easily prove that the theorem holds for D.

Another version of Theorem 1 is the following

4.7. THEOREM 2. *Let $D_1, D_2 \subset C^n$ be bounded strictly pseudoconvex domains with C^2 boundary, $\overline{D}_1 \subset D_2$ so that $\overline{D}_1$ is holomorphically convex in D_2. Then for each $1 \leq q \leq n-2$ there is a linear operator $S_q : Z^1_{(0,q)}(\overline{D}) \to C^1_{(0,q-1)}(D)$ which satisfies (i), (ii) and (iii) of Theorem 1.*

PROOF. Let $\rho_1 \in C^2(\overline{D}_1)$, $\rho_2 \in C^2(\overline{D}_2)$ be strictly plurisubharmonic defining functions for the domains D_1 and D_2. Let $W \supset \overline{D}_1$ be the open set where ρ_1 is defined. Since $\overline{D}_1$ is holomorphically convex in D_2 there exists $\rho \in C^2(D_2)$, strictly plurisubharmonic so that $\rho < 0$ on $\overline{D}_1$ and $\rho > 0$ on $D_2 - W$ (we may assume $\overline{W} \subset D_2$). Let $\tilde{D}_1 =: \{\rho < 0\}$ and $\tilde{D}_2 =: \{\rho < \varepsilon\}$ for some small $\varepsilon > 0$. By Sard's theorem we may assume that $d\rho \neq 0$ on $\partial\tilde{D}_1$ and $\partial\tilde{D}_2$; also we assume that ε is small enough so that $\tilde{D}_2 \subset\subset W$. Notice that $D_1 \subset\subset \tilde{D}_1 \subset\subset \tilde{D}_2 \subset\subset D_2$. Now Theorem 1 gives operators S_q (with the properties (i), (ii), and (iii)) for the open sets $\tilde{D}_2 - \overline{D}_1$ and $D_2 - \overline{\tilde{D}}_1$ as well as for their intersection $(\tilde{D}_2 - \overline{D}_1) \cap (D_2 - \overline{\tilde{D}}_1) = \tilde{D}_2 - \overline{\tilde{D}}_1$ $(1 \leq q \leq n-2)$. Hence the conclusion of Theorem 1 holds also for $(\tilde{D}_2 - \overline{D}_1) \cup (D_2 - \overline{\tilde{D}}_1) = D_2 - \overline{D}_1$ (by the proof of lemmas 4.3 and 4.5). This completes the proof of the theorem.

4.8. REMARK. Let D_2 be as in Theorem 2 and $\Omega_1, \dots, \Omega_N \subset\subset D_2$ be strictly convex domains with C^2 boundaries so that the sets $\overline{\Omega}_1, \dots, \overline{\Omega}_N$ are pairwise disjoint. Let $\lambda_k \in C^2(\mathbb{C}^n)$ be strictly convex function which is a defining function for Ω_k (for example the distance to the boundary function). Now for $z \in D_2 - \overline{\Omega}_k$ and $\zeta \in \partial\Omega_k$ let us define

$$Y_j^k(\zeta, z) =: \frac{\frac{\partial \lambda_k}{\partial z_j}(z)}{\sum_{j=1}^n \frac{\partial \lambda_k}{\partial z_j}(z)(\zeta_j - z_j)}, j = 1, \dots, n, \; k = 1, \dots, N.$$

Also for $z \in D_2$ and $\zeta \in \partial D_2$ let $Y_j^0(\zeta, z), j = 1, \dots, n$ be the functions associated to D_2 by the Henkin-Ramirez construction. Using the maps $\{Y^0, Y^1, \dots, Y^N\}$ we can easily construct operators S_q for the domain $D = D_2 - \bigcup_{k=1}^N \overline{\Omega}_k$, $1 \leq q \leq n-2$, with properties (i), (ii) and (iii) of Theorem 1. Combined this with Theorem 2 gives the following

4.9. THEOREM 3. *Let $D_2, \Omega_1, \dots, \Omega_N$ be as in § 4.8 and $\Omega'_k (\subset\subset \Omega_k)$ be strictly pseudoconvex domains with C^2-boundaries so that $\overline{\Omega}'_k$ is holomorphically convex in Ω_k. Set $D =: D_2 - \bigcup_{k=1}^N \overline{\Omega}'_k$. Then there are operators $S_q : Z^1_{(0,q)}(\overline{D}) \to C^1_{(0,q-1)}(D)$, $1 \leq q \leq n-2$ which satisfy* (i), (ii) *and* (iii) *of Theorem 1.*

PROOF. Let $\Omega''_k \subset\subset \Omega_k$ be strictly convex domains with C^2 boundaries so that $\Omega'_k \subset\subset \Omega''_k$.

By the previous remark the conclusion of Theorem 1 holds for the domain $D_2 - \bigcup_{k=1}^N \overline{\Omega}''_k$; by Theorem 2 it also holds for $\bigcup_{k=1}^N (\Omega_k - \overline{\Omega}'_k)$. Since their intersection (which is $\bigcup_{k=1}^N (\Omega_k - \overline{\Omega}''_k)$) also satisfies that conclusion, the theorem follows from the proofs of Lemmas 4.3 and 4.5.

5 – The general case

In this section we will prove the following

5.1. THEOREM 4. *Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded strictly pseudoconvex domains with C^2 boundaries, $\overline{D}_1 \subset D_2$ and set $D =: D_2 - \overline{D}_1$. Then for every $f \in Z^1_{(0,q)}(\overline{D})$, $1 \leq q \leq n-2$, there is a solution $g \in C^1_{(0,q)}(D)$ of the equation $\overline{\partial} g = f$ in D which is Lip(1/2) in D.*

For the proof of this theorem we will use the following

5.2. REMARK. Let $G \subset \mathbb{C}^n$ be a pseudoconvex domain and $K(\subset G)$ be a Stein compactum. Then $H^{(0,q)}_{\overline{\partial}}(G-K) = 0$ for $1 \leq q \leq n-2$. This is well-known but for completeness we outline a proof of it.

By Serre duality theorem

$$ {}^{\sigma}H^q(G-K, O^p) \approx [{}^{\sigma}H^{n-q}_c(G-K, O^{n-p})]^* $$

and $H^q(G-K, O^p)$ is separated if and only if $H^{n-q+1}_c(G-K, O^{n-p})$ is. (Here ${}^{\sigma}H$ denotes separated cohomology, H_c is cohomology with compact supports and O^p is the sheaf of holomorphic p-forms; see [1] and [6]). Therefore it suffices to show that $H^r_c(G-K, O^{n-p}) = 0$ for $2 \leq r \leq n-1$ and $H^n_c(G-K, O^{n-p})$ is separated. But since G is pseudoconvex, $H^r_c(G, O^{n-p}) = 0$ for $r \neq n$ and $H^n_c(G, O^{n-p})$ is separated, and the last assertion follows from the cohomology sequence with compact supports associated to a closed subset (see [2]).

5.3. PROOF OF THEOREM 4. Let $\tilde{D}_1$ and $\tilde{D}_2$ be C^2 strictly pseudoconvex domains so that $D_1 \subset\subset \tilde{D}_1 \subset\subset \tilde{D}_2 \subset\subset D_2$ and with boundary $\partial\tilde{D}_1$ sufficiently close to ∂D_1 and the boundary $\partial\tilde{D}_2$ sufficiently close to ∂D_2 in such a way the conclusion of Theorem 1 holds for $\tilde{D}_1 - \overline{D}_1$ and $D_2 - \overline{\tilde{D}}_2$ (This follows from § 2.3 and § 4.2). On the other hand it follows from remark 5.2 that the equation $\overline{\partial} g = f$ has a C^∞ solution in an appropriate neighborhood W of the compact set $\overline{\tilde{D}}_2 - \tilde{D}_1$ ($W \subset\subset D_2 - \overline{D}_1$).

Furthermore the conclusion of Theorem 1 holds also for the (intersections) $W - \overline{\tilde{D}}_2$ and $W \cap \tilde{D}_1$ (provided that W is appropriately chosen). Hence Theorem 4 follows from the proofs of lemmas 4.3 and 4.5.

5.4. COMMENTS. (i) One can consider more general situations than the ones considered in Theorem 1. For example D_1, D_2 could have non-smooth boundary in the sense of [3] (chapter 3); or piece-wise smooth strictly pseudoconvex boundaries in the sense of [5]. Also one could give examples of D_1, D_2 which are weakly pseudoconvex for which this method works and gives Lip(a)-estimates for some $a > 0$. Also one could replace $\mathbb{C}^n$ by a Stein manifold.

(ii) It is easy to see that if $D \subset \mathbb{C}^n$ is open and $K(\subset D)$ is a non-empty compact subset then $H_{\overline{\partial}}^{(0,n-1)}(D-K) \neq 0$.

Indeed assuming that $0 \in K$ we have that the $(0, n-1)$-form

$$f(z) = \left(\sum_{j=1}^{n} |z_j|^2\right)^{-n} \sum_{j=1}^{n} (-1)^j \overline{z}_j d\overline{z}_1 \wedge \ldots (j) \ldots \wedge d\overline{z}_n$$

has C^∞ coefficients in $D-K$ and $\overline{\partial} f = 0$ but the equation $\overline{\partial} g = f$ has no solution g in $D-K$ since $\int_{\partial\Omega} f(z) \wedge \ldots \wedge dz_n \neq 0$ (by the Bochner-Martinelli formula, where Ω is open with $K \subset \Omega \subset\subset D$ and $\partial\Omega$ is smooth) while $\int_{\partial\Omega} \overline{\partial} g(z) \wedge dz_1 \wedge \ldots \wedge dz_n = 0$ (by Stokes' formula). Hence the restriction of q to $1 \leq q \leq n-2$ is necessary.

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