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A uniform estimate of the perimeter for minimizers of a free boundary problem

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RIASSUNTO: *Si studia un problema di superficie minima per un funzionale di area con un termine addizionale, del tipo introdotto da H.W. Alt e L.A. Caffarelli, che porta ad un problema di frontiera libera. Si stabilisce una stima uniforme del perimetro introducendo una misura di Radon e dimostrando la proprietà di minimo di un sottografo di un minimizzatore. Il punto cruciale nella prova di questa proprietà di minimo è quella di costruire una funzione campione appropriata per il funzionale.*

ABSTRACT: *We treat a minimal surface problem for the area functional with such an additional term causing the free boundary as introduced by H.W.Alt and L.A. Caffarelli. By introducing a Radon measure and showing the minimality of the subgraph of a minimizer, we establish a uniform estimate of the perimeter. The crucial step in the proof of the minimality is to construct a testing function appropriate to our functional.*

KEY WORDS: *Free boundary - Perimeter.*

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– Introduction

Let Ω be a bounded domain in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, with the Lipschitz boundary $\partial\Omega$ and S a subset of $\partial\Omega$ with a positive $(n-1)$-dimensional Hausdorff measure.

We consider the free boundary problem:

$$(P) \quad \begin{cases} \text{In the class } BV(\Omega), \text{ minimize} \\ J(w) = \int\limits_{\Omega} \sqrt{1+|Dw|^2} + \int\limits_{\Omega} Q^2 \chi_{w>0}\, d\mathcal{L}^n + \int\limits_{S} |w-u^0|\, d\mathcal{H}^{n-1}, \end{cases}$$

where Q is a bounded measurable function, u^0 a non-negative function in $BV(\Omega)$ and $\chi_{w>0}$ means the characteristic function of the set $\Omega(w > 0) := \{x \in \Omega \,|\, w(x) > 0\}$.

Let u be a minimizer of the problem (P) (see [6, Theorem 1.1] for the existence) and we denote by U the subgraph of u in $\Omega \times \mathbb{R}^1$ defined by $U = \{(x,t) \in \Omega \times \mathbb{R}^1 \,|\, t < u(x)\}$. Then our main result is that the following uniform estimate for $|D\chi_U|$, the perimeter of U holds:

THEOREM. *For any n-dimensional ball B_ρ compactly contained in $\Omega \times \mathbb{R}^1$, we have*

$$\int_{B_\rho} |D\chi_U| \leq \frac{1}{2}(1 + Q_{\max}^2)(n+1)\,\omega_{n+1}\rho^n,$$

where $Q_{\max} = \sup_\Omega |Q|$ and ω_{n+1} is the volume of the $(n+1)$-dimensional unit ball.

In case U is an area minimizing set in $\mathbb{R}^{n+1}$ (see [3, Theorem 1.20] for the definition) the following uniform estimate has been proved to hold ([3, (5.14)]): For any n-dimensional ball $B_\rho \subset \mathbb{R}^{n+1}$

$$\int_{B_\rho} |D\chi_U| \leq \frac{1}{2}(n+1)\,\omega_{n+1}\rho^n. \tag{1}$$

This is one of the estimates used in the proof of the smoothness of minimal surfaces (see [3, Section 8]).

In order to prove Theorem we show the minimality of U as in Main Lemma of Section 1. This is stated by making use of a Radon measure, constructed in Section 1 and denoted by $Q^2|\partial_t\chi_U|$, correspoding to the second term of $J(u)$. In fact, we arrive at the uniform estimate in Theorem by taking the same comparison set as in the proof [3, page 72] of (1) (see the proof of Theorem). The idea of such treatment of the second term of $J(w)$ is taken from the paper [4].

In the proof of the minimality of U, the most important step is to obtain the following inequality: For a measurable set E in $\Omega \times \mathbb{R}^1$ satisfying $E \supset \Omega \times \mathbb{R}^1_-$,

$$\int_\Omega Q^2\chi_{w_E>0}\, d\mathcal{L}^n \leq \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t\chi_E|,$$

where w_E on the left side is, roughly speaking, the function which gives for each $x \in \Omega$ the length of the set $\{(x,t) \in \mathbb{R}^{n+1} \mid (x,t) \in E \cap (\Omega \times \mathbb{R}^1_+)\}$. This inequality is proved to hold by making a device of constructing an original testing function given by (2.6), which is absolutely continuous only in the t-direction, takes the value zero in $\Omega\,(w_E = 0) \times \mathbb{R}^1$, and has zero L^1-trace on $\Omega \times \{0\}$.

In Appendix, we state, without the proof, a result which gives the representation of the Radon measure $Q^2|\partial_t \chi_E|$ in terms of the n-dimensional Hausdorff measure $\mathcal{H}^n$ when E is a set, the boundary ∂E being the n-dimensional C^1-surface and, in some sense, finite. Making use of the result, we can directly show the transformation equality (2.14) for the second term of J.

We here sum up notations used in this paper.

Let G be an open set of $\mathbb{R}^l$, $l \geq 1$. The function spaces $C_0(G)$, $C^1(G)$, $C_0^1(G)$, $C^\infty(G)$, $C_0^\infty(G)$, $L^1(G)$, $L^1_{\text{loc}}(G)$ and $L^\infty(G)$ are as in [2], and $BV(G)$ is as in [3]. We denote by $\mathbb{R}^1_+$ and $\mathbb{R}^1_-$ the set of positive and negative numbers respectively:

$$\mathbb{R}^1_+ = \{t \in \mathbb{R}^1 \mid t > 0\},$$
$$\mathbb{R}^1_- = \{t \in \mathbb{R}^1 \mid t < 0\}.$$

Let w be a non-negative function defined in Ω. Then we write $\Omega\,(w > 0) = \{x \in \Omega \mid w(x) > 0\}$, $\Omega\,(w = 0) = \{x \in \Omega \mid w(x) = 0\}$ and

$$\chi_{w>0}(x) = \begin{cases} 1 & x \in \Omega\,(w > 0), \\ 0 & x \in \Omega\,(w = 0). \end{cases}$$

For $w \in BV(\Omega)$ we use the notation

$$\int_\Omega \sqrt{1 + |Dw|^2} := \sup_{\substack{\zeta \in C_0^1(\Omega, \mathbb{R}^{n+1}) \\ |\zeta| \leq 1}} \int_\Omega \left(\zeta^{n+1} + w \operatorname{div} \widehat{\zeta}\right) d\mathcal{L}^n,$$

where ζ^{n+1} is the $(n+1)$-th component of $\zeta = (\zeta^1, \cdots, \zeta^n, \zeta^{n+1})$, and $\widehat{\zeta} = (\zeta^1, \cdots, \zeta^n)$. For a function f defined in $\Omega \times \mathbb{R}^1$, we use the notation

$$\operatorname{spt} f = \overline{\{x \in \Omega \times \mathbb{R}^1 \mid f(x) \neq 0\}},$$

where $\overline{A}$ means the closure of the set $A \subset \Omega \times \mathbb{R}^1$ with respect to the $(n+1)$-dimensional Euclidean topology.

1 – Construction of a Radon measure and the proof of Theorem

In this and later sections, we extend to $\Omega \times \mathbb{R}^1$ the domain of the definition of Q as follows:

$$(1.1) \qquad Q(x,t) = Q(x) \qquad \text{for } (x,t) \in \Omega \times \mathbb{R}^1.$$

Let w be a function in $BV(\Omega)$ and W be the subgraph of w defined by $W = \{(x,t) \in \Omega \times \mathbb{R}^1 \,|\, t < w(x)\}$, and χ_W be the characteristic function of W in $\Omega \times \mathbb{R}^1$:

$$\chi_W(x,t) = \begin{cases} 1 & (x,t) \in W, \\ 0 & (x,t) \in (\Omega \times \mathbb{R}^1) \setminus W. \end{cases}$$

Then, χ_W belongs to $L^1_{\text{loc}}(\Omega \times \mathbb{R}^1)$ and has the derivative $D\chi_W$ which is the $(n+1)$-dimensional vector valued Radon measure with the finite total variation

$$(1.2) \qquad \int_{\Omega\times\mathbb{R}^1} |D\chi_W| = \sup_{\substack{\zeta \in C_0^1(\Omega\times\mathbb{R}^1, \mathbb{R}^{n+1}) \\ |\zeta| \le 1}} \int_{\Omega\times\mathbb{R}^1} \chi_W \operatorname{div} \zeta \, d\mathcal{L}^{n+1}$$

(see [3, Theorem 14.6]).

On the other hand, we shall introduce a Radon measure defined in $\Omega \times \mathbb{R}^1$ and denoted by $Q^2|\partial_t \chi_W|$. For this purpose, we treat an arbitrary function f belonging to $L^1_{\text{loc}}(\Omega \times \mathbb{R}^1)$ whose derivative has the finite total variation in $\Omega \times \mathbb{R}^1$. We here remark that, for any open set $G \subset \Omega \times \mathbb{R}^1$, the total variation of Df in G:

$$(1.3) \qquad \int_G |Df| = \sup_{\substack{\zeta \in C_0^1(G) \\ |\zeta| \le 1}} \int_G f \operatorname{div} \zeta \, d\mathcal{L}^{n+1}$$

is finite. Then we have

LEMMA 1.1.. *Let f be as above. Then, for any open set $G \subset \Omega \times \mathbb{R}^1$ we have*

$$\sup_{\substack{\zeta \in C_0^1(G) \\ |\zeta| \leq 1}} \int_G f Q^2 \partial_t \zeta \, d\mathcal{L}^{n+1} \leq Q_{\max}^2 \int_G |Df|, \tag{1.4}$$

where Q is the extended function defined by (1.1) *and* $Q_{\max} = \sup_\Omega |Q|$.

PROOF. Let G be an open set in $\Omega \times \mathbb{R}^1$ and let ζ be an arbitrary function belonging to $C_0^1(G)$ and satisfying that $|\zeta| \leq 1$ in G. Let $\{Q_j\}_{j=1}^\infty \subset C^\infty(\Omega)$ be such that $\sup_\Omega |Q_j| \leq Q_{\max}^2$ for all j and Q_j converges almost everywhere in Ω as $j \to \infty$ to Q (refer to [2, Lemma 7.2]). We extend to $\Omega \times \mathbb{R}^1$ the domain of the definition of Q_j in the same way as (1.1). Then for each Q_j, we have from (1.3)

$$\int_G f Q_j^2 \partial_t \zeta \, d\mathcal{L}^{n+1} = \int_G f \partial_t (Q_j^2 \zeta) \, d\mathcal{L}^{n+1} \leq Q_{\max}^2 \int_G |Df|. \tag{1.5}$$

Since Q_j converges almost everywhere in $\Omega \times \mathbb{R}^1$ as $j \to \infty$ to Q, we obtain by letting $j \to \infty$ in (1.5) that

$$\int_G f Q^2 \partial_t \zeta \, d\mathcal{L}^{n+1} \leq Q_{\max}^2 \int_G |Df|.$$

Taking supremum over all such ζ, we arrive at (1.4). □

For a function f as in Lemma 1.1, we define $\mathcal{L}_f$ as a linear functional on $C_0^1(\Omega \times \mathbb{R}^1)$ which gives for each $\zeta \in C_0^1(\Omega \times \mathbb{R}^1)$ the value

$$\mathcal{L}_f \zeta = \int_{\Omega \times \mathbb{R}^1} f Q^2 \partial_t \zeta \, d\mathcal{L}^{n+1}.$$

By virtue of Lemma 1.1, the functional $\mathcal{L}_f$ is continuous on $C_0^1(\Omega \times \mathbb{R}^1)$ and hence $\mathcal{L}_f$ is uniquely extended to a continuous linear functional $\tilde{\mathcal{L}}_f$ defined on $C_0(\Omega \times \mathbb{R}^1)$. Here, let us apply the Riesz representation theorem to $\tilde{\mathcal{L}}_f$ (see [5, Theorem 4.1]): There exists a unique finite Radon measure μ_f defined in $\Omega \times \mathbb{R}^1$ and a unique μ_f-measurable function ν_f

defined in $\Omega \times \mathbb{R}^1$ satisfying that $|\nu_f| = 1$ almost everywhere, with respect to the measure μ_f, in $\Omega \times \mathbb{R}^1$ and satisfying, for any $\zeta \in C_0(\Omega \times \mathbb{R}^1)$,

$$\widetilde{\mathcal{L}}_f \zeta = \int_{\Omega \times \mathbb{R}^1} \zeta \nu_f \, d\mu_f.$$

In the sequel, we shall use the notation $Q^2|\partial_t f|$ instead of μ_f. Recalling the method of the construction of the Radon measure $Q^2|\partial_t f|$ (see the proof of [5, Theorem 4.1]), we have

$$\int_G Q^2|\partial_t f| = \sup_{\substack{\zeta \in C_0^1(G) \\ |\zeta| \leq 1}} \int_G f Q^2 \partial_t \zeta \, d\mathcal{L}^{n+1}, \tag{1.6}$$

where G is an open set in $\Omega \times \mathbb{R}^1$, and moreover, combining (1.6) with (1.4), we obtain

$$\int_G Q^2|\partial_t f| \leq Q^2_{\max} \int_G |Df|.$$

Furthermore, owing to the outer regularity of the Radon measure ([5, Theorem 1.3]),

$$\int_A Q^2|\partial_t f| \leq Q^2_{\max} \int_A |Df| \tag{1.7}$$

for any subset A of $\Omega \times \mathbb{R}^1$.

Let w be a function in $BV(\Omega)$. Then, as stated above, the function χ_W, the characteristic function of the subgraph of w, belongs to $L^1_{\text{loc}}(\Omega \times \mathbb{R}^1)$ and has the finite total variation in $\Omega \times \mathbb{R}^1$. Hence, the Radon measure $Q^2|\partial_t \chi_W|$ is defined as above. Then there holds the following lemma which plays an important role in the proof of Theorem and will be proved in Section 3:

MAIN LEMMA. *Let u be a minimizer of the problem* (P) *and let U be the subgraph of u in $\Omega \times \mathbb{R}^1$. Then for every measurable set F in $\Omega \times \mathbb{R}^1$ such that* spt $(\chi_F - \chi_U)$ *is compactly contained in $\Omega \times \mathbb{R}^1$, we have*

$$\int_{\Omega \times \mathbb{R}^1} |D\chi_U| + \int_{\Omega \times \mathbb{R}^1_+} Q^2|\partial_t \chi_U| \leq \int_{\Omega \times \mathbb{R}^1} |D\chi_F| + \int_{\Omega \times \mathbb{R}^1_+} Q^2|\partial_t \chi_F|. \tag{1.8}$$

By making use of Main Lemma, we can demonstrate Theorem in the following way:

THE PROOF OF THEOREM. Take $U\backslash\overline{B}_r$, $0<r<\rho$, as a comparison set F in Main Lemma: Since spt $(\chi_U-\chi_{U\backslash\overline{B}_r})\subset B_\rho$,

$$\int_{B_\rho}|D\chi_U|+\int_{B_\rho\cap(\Omega\times\mathbf{R}^1_+)}Q^2|\partial_t\chi_U|\leq\int_{B_\rho}|D\chi_{U\backslash\overline{B}_r}|+\int_{B_\rho\cap(\Omega\times\mathbf{R}^1_+)}Q^2|\partial_t\chi_{U\backslash\overline{B}_r}|.$$

Then from (1.7) and [3, Remark 2.13] we have for almost all $r<\rho$

$$\begin{aligned}\int_{B_\rho}|D\chi_U|&\leq(1+Q^2_{\max})\int_{B_\rho}|D\chi_{U\backslash\overline{B}_r}|=\\ &=(1+Q^2_{\max})\left\{\int_{B_\rho\backslash\overline{B}_r}|D\chi_U|+\mathcal{H}^n(\partial B_r\cap U)\right\}.\end{aligned}\tag{1.9}$$

We next take $U\cup B_r$ as a comparison set. Then, by the similar calculation, we have for almost all $r<\rho$ that

$$\int_{B_\rho}|D\chi_U|\leq(1+Q^2_{\max})\left\{\int_{B_\rho\backslash\overline{B}_r}|D\chi_U|+\mathcal{H}^n(\partial B_r\backslash U)\right\}.\tag{1.10}$$

From (1.9) and (1.10) we obtain for almost all $r<\rho$

$$\begin{aligned}\int_{B_\rho}|D\chi_U|&\leq(1+Q^2_{\max})\left\{\int_{B_\rho\backslash\overline{B}_r}|D\chi_U|+\max\Big(\mathcal{H}^n(\partial B_r\setminus U),\mathcal{H}^n(\partial B_r\cap U)\Big)\right\}\leq\\ &\leq(1+Q^2_{\max})\left\{\int_{B_\rho\backslash\overline{B}_r}|D\chi_U|+\frac{1}{2}(n+1)\,\omega_{n+1}r^n\right\}.\end{aligned}\tag{1.11}$$

By letting $r\nearrow\rho$, the integration in the last term of (1.11) vanishes. This completes the proof of Theorem. □

2 – A result connecting $J(w)$ with the parametric form

In this section we investigate the relations between the first two terms of $J(w)$ and the parametric form, which will be used in the proof of Main Lemma. Let E be a measurable set in $\Omega \times \mathbb{R}^1$ satisfying

$$(2.1) \qquad \begin{cases} (2.1-\text{a}) & E \supset \Omega \times \mathbb{R}^1_-; \\ (2.1-\text{b}) & \mathcal{L}^{n+1}(E \cap (\Omega \times \mathbb{R}^1_+)) < \infty. \end{cases}$$

We define for $k > 0$ the function w_k in Ω as follows:

$$(2.2) \qquad w_k(x) = \int_0^k \chi_E(x,\tau)\, d\mathcal{L}^1(\tau) \qquad \text{for } x \in \Omega.$$

We here remark that (2.1-b) implies that w_k converges in $L^1(\Omega)$ as $k \to \infty$ to a function which is denoted by w_E:

$$w_E = \lim_{k\to\infty} w_k \qquad \text{in } L^1(\Omega).$$

Now we state the main result of this section:

PROPOSITION 2.1.. *Let E be a measurable set in $\Omega \times \mathbb{R}^1$ satisfying* (2.1-a,b). *Then*

$$\int_\Omega \sqrt{1+|Dw_E|^2} + \int_\Omega Q^2 \chi_{w_E>0} d\mathcal{L}^n \le \int_{\Omega\times\mathbb{R}^1} |D\chi_E| + \int_{\Omega\times\mathbb{R}^1_+} Q^2 |\partial_t \chi_E|.$$

PROOF. It is proved in [3, Theorem 14.8] that

$$\int_\Omega \sqrt{1+|Dw_E|^2} \le \int_{\Omega\times\mathbb{R}^1} |D\chi_E|.$$

Hence, in order to establish the conclusion it is sufficient to show

$$(2.3) \qquad \int_\Omega Q^2 \chi_{w_E>0} d\mathcal{L}^n \le \int_{\Omega\times\mathbb{R}^1_+} Q^2 |\partial_t \chi_E|.$$

We first show (2.3) under a bounded condition for E: Let E be satisfy

$$E \subset \Omega \times (-\infty, K) \tag{2.4}$$

for some positive number K. In the following, we assume, for simplicity, that $K = 1$: $E \subset \Omega \times (-\infty, 1)$. Then we remark that

$$w_E(x) = \int_0^1 \chi_E(x,\tau)\, d\mathcal{L}^1(\tau) \qquad \text{for } x \in \Omega. \tag{2.5}$$

To show (2.3) we introduce the function η in $\Omega \times \mathbb{R}^1_+$ defined by

$$\eta(x,t) = \begin{cases} \dfrac{1}{w_E(x)} \displaystyle\int_0^t \chi_E(x,\tau) d\mathcal{L}^1(\tau) & \text{for } (x,t) \in \Omega\,(w_E > 0) \times (0,1), \\ 2 - t & \text{for } (x,t) \in \Omega\,(w_E > 0) \times [1,2], \\ 0 & \text{otherwise in } \Omega \times \mathbb{R}^1. \end{cases} \tag{2.6}$$

Then it holds that:

(i) η is $\mathcal{L}^{n+1}$*-measurable in* $\Omega \times \mathbb{R}^1$;

(ii) $0 \le \eta \le 1$ *in* $\Omega \times \mathbb{R}^1_+$;

(iii) $\eta = 0$ *on* $\Omega \times \{0\}$ *and on* $\Omega \times \{2\}$;

(iv) *For each* $x_0 \in \Omega$, *the 1-dimensional function* $\eta(x_0, t)$, $t > 0$, *is absolute continuous and* $\partial_t \eta \in L^1(\Omega \times \mathbb{R}^1_+)$.

The assertion (i) holds, since η is the pointwise limit of the sequence $\{\eta_j\}_{j=1}^{\infty}$ defined by

$$\eta_j(x,t) = \begin{cases} \dfrac{1}{w_E(x)} \displaystyle\int_0^{k/2^j} \chi_E(x,\tau) d\mathcal{L}^1(\tau) & \text{for } (x,t) \in \Omega(w_E > 0) \times \left(\dfrac{k}{2^j}, \dfrac{k+1}{2^j}\right), \\ & k = 0, 1, \ldots, 2^j - 1, \\ 2 - t & \text{for } (x,t) \in \Omega(w_E > 0) \times [1,2], \\ 0 & \text{otherwise in } \Omega \times \mathbb{R}^1, \end{cases}$$

which is $\mathcal{L}^{n+1}$-measurable in $\Omega \times \mathbb{R}^1$. The assertion (ii), (iii) and the first one in (iv) are directly proved in view of (2.6). We shall finally make sure that $\partial_t \eta \in L^1(\Omega \times \mathbb{R}^1_+)$. Since it holds that

$$\partial_t \eta(x,t) = \begin{cases} \dfrac{\chi_E(x,t)}{w_E(x)} & \text{for almost all } (x,t) \in \Omega\,(w_E > 0) \times (0,1), \\ -1 & \text{for all } (x,t) \in \Omega\,(w_E > 0) \times (1,2), \\ 0 & \text{otherwise in } \Omega \times \mathbb{R}^1, \end{cases} \tag{2.7}$$

we infer the integrability of $\partial_t \eta$ in the following way:

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^1_+} |\partial_t \eta| d\mathcal{L}^{n+1} &= \int_{\Omega\,(w_E>0)} d\mathcal{L}^n(x) \Big(\int_0^1 \frac{\chi_E(x,\tau)}{w_E(x)} d\mathcal{L}^1(\tau) + \int_1^2 1 d\mathcal{L}^1(\tau) \Big) \\ &= 2\mathcal{L}^n(\Omega\,(w_E > 0)) < \infty, \end{aligned}$$

(2.5) being used in the last equality.

Now let us show (2.3). Let φ be a function in $C^1_0(\Omega)$ such that $|\varphi| \leq 1$ in Ω. From (i), $|\varphi\eta| \leq 1$ in $\Omega \times \mathbb{R}^1_+$. Furthermore, by (ii) and (iii), we can substitute $\varphi\eta$ to ζ in (1.6) with $f = \chi_E$ and $G = \Omega \times \mathbb{R}^1_+$:

$$\begin{aligned} \int_{\Omega \times R^1_+} Q^2 |\partial_t \chi_E| &\geq \int_{\Omega(w_E>0)} Q^2(x)\varphi(x) d\mathcal{L}^n(x) \int_0^1 \Big(\chi_E(x,t)\partial_t\eta(x,t)\Big) d\mathcal{L}^1(t) \\ &= \int_{\Omega(w_E>0)} Q^2(x)\varphi(x) d\mathcal{L}^n(x) \frac{1}{w_E(x)} \int_0^1 \chi_E(x,t) d\mathcal{L}^1(t) \\ &= \int_\Omega \chi_{w_E>0} Q^2 \varphi d\mathcal{L}^n, \end{aligned}$$

(2.4) with $K = 1$, (2.5), (2.7), and the equality $\chi_E = \chi_E^2$ being used. Letting $\varphi \nearrow \chi_\Omega$, we arrive at (2.3).

We next prove (2.3) without the assumption (2.4). For a positive number k, let $E_k = E \cap \{\Omega \times (-\infty, k)\}$. Then E_k satisfies (2.4) with $K = k$, and therefore we have

$$\int_\Omega Q^2 \chi_{w_k>0}\, d\mathcal{L}^n \leq \int_{\Omega \times \mathbb{R}^1_+} Q^2 |\partial_t \chi_{E_k}|. \tag{2.8}$$

Let us calculate the limit of the right side of (2.8) as $k \to \infty$. Because $E_k \subset \Omega \times (0,k]$ and $E_k = E$ in $\Omega \times (0,k)$, we infer

$$\int_{\Omega\times\mathbf{R}^1_+} Q^2|\partial_t\chi_{E_k}| = \int_{\Omega\times(0,k)} Q^2|\partial_t\chi_E| + \int_{\Omega\times\{k\}} Q^2|\partial_t\chi_{E_k}|. \tag{2.9}$$

From (1.7), we have for $\mathcal{L}^1$-almost all $k > 0$ that

$$\int_{\Omega\times\{k\}} Q^2|\partial_t\chi_{E_k}| \le Q^2_{\max} \int_{\Omega\times\{k\}} |D\chi_{E_k}| = Q^2_{\max}\mathcal{L}^n(E\cap(\Omega\times\{k\})) \tag{2.10}$$

(refer to [3, Remark 2.14] for the last equality). By (2.1-b), $\lim_{k\to\infty}\mathcal{L}^n(E\cap(\Omega\times\{k\}))= 0$, and hence from (2.10)

$$\lim_{k\to\infty}\int_{\Omega\times\{k\}} Q^2|\partial_t\chi_{E_k}| = 0. \tag{2.11}$$

Letting $k \to \infty$ in (2.9), we deduce from (2.11)

$$\lim_{k\to\infty}\int_{\Omega\times\mathbf{R}^1_+} Q^2|\partial_t\chi_{E_k}| = \int_{\Omega\times\mathbf{R}^1_+} Q^2|\partial_t\chi_E|. \tag{2.12}$$

Next, since w_k converges in $L^1(\Omega)$ to w_E, in the same way as in the proof of the lower-semicontinuity of the functional J (see the proof of [1, Theorem 1.3]) we can choose a sequence $\{k_j\}_{j=1}^{\infty}$ such that $k_j \to \infty$ as $j \to \infty$ and

$$\liminf_{j\to\infty}\int_\Omega Q^2\chi_{w_{k_j}>0}\, d\mathcal{L}^n \ge \int_\Omega Q^2\chi_{w_E>0}\, d\mathcal{L}^n. \tag{2.13}$$

Thus, by passing $j \to \infty$ in (2.8) with k replaced by k_j, $j = 1,\cdots,$ (2.3) follows from (2.12) and (2.13). This completes the proof of Proposition 2.1. □

In case that E, in Proposition 2.1, is the subgraph of a non-negative $BV(\Omega)$-function, we can assert the stronger result:

COROLLARY 2.1. *Let w be a non-negative function belonging to $BV(\Omega)$. Then*

$$\int_{\Omega} \sqrt{1+|Dw|^2} + \int_{\Omega} Q^2 \chi_{w>0}\, d\mathcal{L}^n = \int_{\Omega\times\mathbf{R}^1} |D\chi_W| + \int_{\Omega\times\mathbf{R}^1_+} Q^2 |\partial_t \chi_W|.$$

PROOF. Since it is proved in [3, Lemma 14.6] that

$$\int_{\Omega} \sqrt{1+|Dw|^2} = \int_{\Omega\times\mathbf{R}^1} |D\chi_W|,$$

in order to derive the conclusion we have only to show

$$\int_{\Omega} Q^2 \chi_{w>0} d\mathcal{L}^n = \int_{\Omega\times\mathbf{R}^1_+} Q^2 |\partial_t \chi_W|. \tag{2.14}$$

Because W satisfies (2.1-a,b), there holds (2.3) with χ_E, w_E replaced by χ_W, w respectively. Now we show the reverse inequality. Let $\zeta \in C^1_0(\Omega \times \mathbb{R}^1_+)$ be such that $|\zeta| \le 1$ in $\Omega \times \mathbb{R}^1$. Then we have

$$\begin{aligned}\int_{\Omega\times\mathbf{R}^1_+} Q^2 \chi_W\ \partial_t \zeta d\mathcal{L}^{n+1} &= \int_{\Omega(w>0)} Q^2(x) d\mathcal{L}^n(x) \int_0^{w(x)} \partial_t \zeta(x,t) d\mathcal{L}^1(t)\\ &= \int_{\Omega(w>0)} Q^2(x) \zeta(x, w(x))\ d\mathcal{L}^n(x)\\ &\le \int_{\Omega} Q^2 \chi_{w>0} d\mathcal{L}^n.\end{aligned}$$

Taking supremum over all such ζ, we obtain the reverse inequality. □

3 – The proof of Main Lemma

From the results of Section 2 we prove Main Lemma, being stated in Section 1, in the following way:

THE PROOF OF MAIN LEMMA. Let F be as in Main Lemma.

Assume initially that $F \supset \Omega \times \mathbb{R}^1_-$. Then, since F satisfies (2.1-a,b), we can apply Proposition 2.1 to F:

$$(3.1)\quad \int_\Omega \sqrt{1+|Dw_F|^2} + \int_\Omega Q^2 \chi_{w_F>0} d\mathcal{L}^n \leq \int_{\Omega\times\mathbb{R}^1} |D\chi_F| + \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t \chi_F|.$$

Since spt $(\chi_F - \chi_U)$ is compactly contained in $\Omega \times \mathbb{R}^1$, $w_F = u$ on S. Hence, from the minimality of u,

$$(3.2)\quad \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega Q^2 \chi_{u>0} d\mathcal{L}^n \leq \int_\Omega \sqrt{1+|Dw_F|^2} + \int_\Omega Q^2 \chi_{w_F>0} d\mathcal{L}^n$$

and owing to Corollary 2.1,

$$(3.3)\quad \int_{\Omega\times\mathbb{R}^1} |D\chi_U| + \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t \chi_U| = \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega Q^2 \chi_{u>0} d\mathcal{L}^n.$$

Gathering (3.1), (3.2) and (3.3), we establish (1.8).

Next we show (1.8) for F which does not necessarily satisfy $F \supset \Omega \times \mathbb{R}^1_-$. Suppose that Main Lemma is not true. Then there exists a measurable set F such that spt $(\chi_F - \chi_U)$ is compactly contained in $\Omega \times \mathbb{R}^1$ and such that

$$(3.4)\quad \int_{\Omega\times\mathbb{R}^1} |D\chi_U| + \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t \chi_U| > \int_{\Omega\times\mathbb{R}^1} |D\chi_F| + \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t \chi_F|.$$

Let $H = \Omega \times \mathbb{R}^1_-$. Then, since $F = F \cup H$ in $\Omega \times \mathbb{R}^1_+$,

$$(3.5)\quad \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t \chi_F| = \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t \chi_{F\cup H}|$$

and furthermore, from Lemma 3.1 below,

$$\int_{\Omega\times\mathbf{R}^1} |D\chi_F| \geq \int_{\Omega\times\mathbf{R}^1} |D\chi_{F\cup H}|. \tag{3.6}$$

Substituting (3.5) and (3.6) to the right side of (3.4), we obtain

$$\int_{\Omega\times\mathbf{R}^1} |D\chi_U| + \int_{\Omega\times\mathbf{R}^1_+} Q^2|\partial_t\chi_U| > \int_{\Omega\times\mathbf{R}^1} |D\chi_{F\cup H}| + \int_{\Omega\times\mathbf{R}^1_+} Q^2|\partial_t\chi_{F\cup H}|. \tag{3.7}$$

From the non-negativity of minimizers ([6, Theorem 1.2]), $u \geq 0$ in Ω and so $U \supset \Omega \times \mathbb{R}^1_-$. Hence, we have

$$\text{spt }(\chi_{F\cup H} - \chi_U) \subset \text{spt }(\chi_F - \chi_U).$$

Therefore spt $(\chi_{F\cup H} - \chi_U)$ is compactly contained in $\Omega \times \mathbb{R}^1$. Moreover, since $F\cup H \supset \Omega\times\mathbb{R}^1_-$, $F\cup H$ satisfies the initial restriction. Hence, (3.7) is a contradiction. We thus accomplish the proof of Main Lemma. □

LEMMA 3.1. *Let F be a set as in Main Lemma and let $H = \Omega\times\mathbb{R}^1_-$. Then*

$$\int_{\Omega\times\mathbf{R}^1} |D\chi_{F\cup H}| \leq \int_{\Omega\times\mathbf{R}^1} |D\chi_F|. \tag{3.8}$$

PROOF. From [3, Lemma 15.1],

$$\int_{\Omega\times\mathbf{R}^1} |D\chi_{F\cap H}| + \int_{\Omega\times\mathbf{R}^1} |D\chi_{F\cup H}| \leq \int_{\Omega\times\mathbf{R}^1} |D\chi_F| + \int_{\Omega\times\mathbf{R}^1} |D\chi_H|. \tag{3.9}$$

By the assumption for F, there exists a positive number K such that $F \supset \Omega\times(-\infty, -K)$ and hence we can apply [3, Theorem 14.8] to $F\cap H$:

$$\int_{\Omega\times\mathbf{R}^1} |D\chi_{F\cap H}| \geq \int_{\Omega} \sqrt{1+|Dw_{F\cap H}|^2} \geq \mathcal{L}^n(\Omega) = \int_{\Omega\times\mathbf{R}^1} |D\chi_H|, \tag{3.10}$$

where $w_{F\cap H}$ is defined for $F\cap H$ as in the beginning of Section 2. From (3.9) and (3.10) we obtain (3.8). □

– Appendix

To begin with, we define a finite C^1-set: Let E be a subset of $\Omega \times \mathbb{R}^1$ satisfying the following (i) and (ii):

(i) The boundary ∂E is the n-dimensional C^1-surface.

(ii) $\mathcal{H}^n(\partial E \cap (\Omega \times \mathbb{R}^1)) < \infty$.

Then we denote by $\nu(\xi)$ the unit normal to ∂E at $\xi \in \partial E$ and put $\nu_t(\xi) = \langle \nu(\xi), e_t \rangle$, where $e_t = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$. From (i), ν_t is continuous on ∂E and hence the set $\partial E(\nu_t > 0) = \{\xi \in \partial E \mid \nu_t(\xi) > 0\}$ is decomposed into relatively open connected sets. Since each connected set has a positive $\mathcal{H}^n$-measure, then (ii) yields that the number of them is at most countable. Hence, we can write

$$\partial E(\nu_t > 0) = \sum_{k=1}^{\infty} \partial E_k^+ .$$

Set $\Omega_k^+ = \text{proj } \partial E_k^+$, the orthogonal projection of ∂E_k^+ onto Ω. Then each Ω_k^+ is an open set because for each $\xi \in \partial E_k^+$ it holds that $\nu_t(\xi) > 0$ and, from (i), ∂E_k^+ is n-dimensional C^1-surface in a neighbourhood of ξ. By replacing the inequality $\nu_t > 0$ with $\nu_t < 0$ in the above argument, we can also define the family of open sets $\{\Omega_k^-\}_{k=1}^{\infty}$.

DEFINITION A.1. *Let E be a subset of $\Omega \times \mathbb{R}^1$. Then, E is called a finite C^1-set if E satisfies* (i), (ii) *above mentioned and*

(iii) *For each $x \in \Omega$, there exists a neighbourhood N_x of x such that the sets $\{$positive integer $k \mid N_x \cap \Omega_k^+ \neq \phi\}$ and $\{$positive integer $k \mid N_x \cap \Omega_k^- \neq \phi\}$ are finite.*

In particular, from (i) and (ii) there holds that the characteristic function χ_E of E has the derivative with the finite total variation $\Omega \times \mathbb{R}^1$ for any finite C^1-set E (see [3, Example 1.4]). Therefore as in Section 1, the Radon measure $Q^2|\partial_t \chi_E|$ is defined and then there holds the following:

THEOREM A.1. *Let E be a finite C^1-set. Then for any open set $G \subset \Omega \times \mathbb{R}^1$ we have*

$$\int_G Q^2 |\partial_t \chi_E| = \int_{G \cap \partial E} Q^2 |\nu_t| \, d\mathcal{H}^n,$$

where ν is the unit normal to ∂E and ν_t is the t- component of ν.

COROLLARY A.1. *Let w be a function belonging to $C^1(\Omega)\cap BV(\Omega)$. Then*

$$\int_\Omega Q^2\chi_{w>0}d\mathcal{L}^n = \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t\chi_W|.$$

PROOF. The subgraph W of w in $\Omega\times\mathbb{R}^1$ satisfies (i)-(iii). In fact, (i) and (ii) directly follow from the condition $w\in C^1(\Omega)\cap BV(\Omega)$. (iii) is verified to hold by taking Ω as N_x for each $x\in\Omega$, because $\Omega_k^+=\Omega$ for $k=1$ and $\Omega_k^+=\phi$ for $k\geq 2$, and $\Omega_k^-=\phi$ for $k\geq 1$. Now we can apply Theorem A.1 to W:

$$\text{(A.1)}\qquad \int_{\Omega\times\mathbb{R}^1_+} Q^2|\partial_t\chi_W| = \int_{\partial W\cap(\Omega\times\mathbb{R}^1_+)} Q^2|\nu_t|d\mathcal{H}^n.$$

Noting that $\partial W\cap(\Omega\times\mathbb{R}^1_+)=\partial W\cap(\Omega\,(w>0)\times\mathbb{R}^1)$, we have

$$\text{(A.2)}\qquad \int_{\partial W\cap(\Omega\times\mathbb{R}^1_+)} Q^2|\nu_t|d\mathcal{H}^n = \int_{\Omega(w>0)\times\mathbb{R}^1} Q^2|\nu_t|d\mathcal{H}^n\lfloor\partial W = \int_\Omega Q^2\chi_{w>0},d\mathcal{L}^n.$$

Hence, from (A.1) and (A.2) we arrive at the conclusion. □

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