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On the asymptotic behavior of second order nonlinear differential equations

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RIASSUNTO: *Si assegnano condizioni sufficienti affinché le soluzioni di una equazione differenziale non lineare del secondo ordine ammettano un asintoto.*

ABSTRACT: *We give sufficient conditions under which the solutions of a second order non linear differential equation are asymptotic to* $at + b$ *where,* a, b *are real constants.*

KEY WORDS: *Asymptotic behavior.*

A.M.S. CLASSIFICATION: 34D05

Consider the differential equation

$$u'' + f(t, u, u') = 0. \tag{1}$$

As a particular case of (1) we have

$$u'' + f(t, u) = 0. \tag{2}$$

The asymptotic behavior of the solutions (2) has been discussed by COHEN [1], TONG [3] and TRENCH [4] using the method of Bellman-Bihari inequalities. In this note we will use the same method to prove similar results for the general case of the nonlinear differential equation (1).

THEOREM 1. *Assume the following hypotheses:*

(i) *the function* $f(t,u,v)$ *is continuous on* $D = \{(t,u,v) : t \geq 1, u, v \in \mathbb{R}\}$;

(ii) *there are functions* $h_1, h_2, h_3 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *continuous and such that*

$$\int_1^\infty h_1(s)ds < \infty, \quad \int_1^\infty h_2(s)ds < \infty, \quad \int_1^\infty h_3(s)ds < \infty,$$

with the property that

$$|f(t,u,v)| \leq h_1(t)g\Big(\frac{|u|}{t}\Big) + h_2(t)|v| + h_3(t), \ (t,u,v) \in D,$$

where $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a continuous, nondecreasing function such that* $g(x) > 0$ *for* $x > 0$ *and if we denote*

$$G(x) = \int_1^x \frac{ds}{g(s)}, \quad x > 0,$$

then $\lim_{x \to \infty} G(x) = \infty$.

Then for every solution $u(t)$ *of* (1) *we have that* $u(t) = at + b + o(t)$ *as* $t \to \infty$ *where* a, b *are real constants.*

PROOF. By (i) we have that equation (1) has solutions $u(t)$ corresponding to arbitrary given initial values $u(1) = c_1$, $u'(1) = c_2$.

Integrating (1) twice from 1 to t we get

$$u'(t) = c_2 - \int_1^t f(s,u(s),u'(s))ds, \quad 1 \leq t, \tag{3}$$

$$u(t) = c_1 + c_2(t-1) - \int_1^t (t-s)f(s,u(s),u'(s))ds, \quad 1 \leq t. \tag{4}$$

By (ii) we obtain that

$$\begin{aligned} |u'(t)| \leq |c_2| + \int_1^t h_1(s)g\Big(\frac{|u(s)|}{s}\Big)ds + \int_1^t h_2(s)|u'(s)|ds + \\ + \int_1^t h_3(s)ds, \qquad 1 \leq t, \end{aligned} \tag{5}$$

$$\frac{|u(t)|}{t} \leq |c_1| + |c_2| + \int_1^t h_1(s) g\left(\frac{|u(s)|}{s}\right) ds + \int_1^t h_2(s)|u'(s)| ds + \tag{6}$$
$$+ \int_1^t h_3(s) ds, \qquad 1 \leq t.$$

Let us denote

$$x(t) = |c_1| + |c_2| + \int_1^t h_1(s) g\left(\frac{|u(s)|}{s}\right) ds + \int_1^t h_2(s)|u'(s)| ds +$$
$$+ \int_1^t h_3(s) ds, \quad 1 \leq t.$$

By (5) and (6) we obtain that

$$x(t) \leq a(t) + \int_1^t h_1(s) g(x(s)) ds + \int_1^t h_2(s) x(s) ds, \quad 1 \leq t,$$

where $a(t) = 1 + |c_1| + |c_2| + \int_1^t h_3(s) ds$, $1 \leq t$.

Fix $T > 1$. By the preceding inequality we obtain that

$$x(t) \leq a(T) + \int_1^t h_1(s) g(x(s)) ds + \int_1^t h_2(s) x(s) ds, \quad 1 \leq t \leq T. \tag{7}$$

Put $y(t) = a(T) + \int_1^t h_1(s) g(x(s)) ds + \int_1^t h_2(s) x(s) ds$, $1 \leq t \leq T$. In view of (7) we get

$$y'(t) \leq h_1(t) g(y(t)) + h_2(t) y(t), \quad 1 \leq t \leq T.$$

Now, putting

$$w(t) = y(t) \exp\left(-\int_1^t h_2(s) ds\right), \quad 1 \leq t \leq T,$$

we can write the last inequality as

$$w'(t) \le h_1(t) g(y(t)) \exp\left(-\int_1^t h_2(s)ds\right), \quad 1 \le t \le T.$$

Integrating this from 1 to t we obtain

$$y(t) \le a(T) \exp\left(\int_1^t h_2(s)ds\right) +$$

$$+ \exp\left(\int_1^t h_2(s)ds\right) \int_1^t h_1(s) \exp\left(-\int_1^s h_2(r)dr\right) g(y(s))ds, \quad 1 \le t \le T,$$

thus

$$y(t) \le a(T) \exp\left(\int_1^T h_2(s)ds\right) +$$

$$+ \exp\left(\int_1^T h_2(s)ds\right) \int_1^t h_1(s) \exp\left(-\int_1^s h_2(r)dr\right) g(y(s))ds, \quad 1 \le t \le T.$$

Applying Bihari's inequality (see [2]) we deduce that

$$y(t) \le G^{-1}\left\{G\left[a(T) \exp\left(\int_1^T h_2(s)ds\right)\right] +\right.$$

$$\left.+ \exp\left(\int_1^T h_2(s)ds\right) \int_1^t h_1(s) \exp\left(-\int_1^s h_2(r)dr\right) ds\right\}, \quad 1 \le t \le T.$$

By (7) we have that

$$x(t) \le G^{-1}\left\{G\left[a(T) \exp\left(\int_1^T h_2(s)ds\right)\right] +\right.$$

$$\left.+ \exp\left(\int_1^T h_2(s)ds\right) \int_1^t h_1(s) \exp\left(-\int_1^s h_2(r)dr\right) ds\right\}, \quad 1 \le t \le T,$$

and since $T > 1$ was arbitrarily choosen, we deduce that

$$x(t) \leq G^{-1}\left\{G\left[a(t)\exp\left(\int_1^t h_2(s)ds\right)\right] + \right.$$
$$\left. + \exp\left(\int_1^t h_2(s)ds\right)\int_1^t h_1(s)\exp\left(-\int_1^s h_2(r)dr\right)ds\right\}, \quad 1 \leq t.$$

Our hypotheses imply that the right-hand term in the preceding inequality is bounded thus there exists an $M > 0$ such that

$$x(t) \leq M, \qquad 1 \leq t.$$

From (5) and (6) one obtains

$$|u'(t)| \leq M, \qquad 1 \leq t,$$
$$\frac{|u(t)|}{t} \leq M, \qquad 1 \leq t.$$

By (ii) we have

$$\int_1^t \left|f(s,u(s),u'(s))\right|ds \leq \int_1^t h_1(s)g\left(\frac{|u(s)|}{s}\right)ds + \int_1^t h_2(s)|u'(s)|ds +$$
$$+ \int_1^t h_3(s)ds \leq x(t) \leq M, \qquad 1 \leq t,$$

thus the integral $\int_1^t f(s,u(s),u'(s))ds$ is absolutely convergent and consequently

$$\lim_{t\to\infty}\int_1^t f(s,u(s),u'(s))ds < \infty.$$

By (3) we obtain that there exists $a \in \mathbb{R}$ such that $\lim_{t\to\infty} u'(t) = a$. Applying de l'Hospital's rule we deduce that

$$\lim_{t\to\infty}\frac{u(t)}{t} = \lim_{t\to\infty} u'(t) = a.$$

This completes the proof of Theorem 1.

In a similar way we can prove

THEOREM 2. *Assume the following hypotheses:*

(i) *the function* $f(t,u,v)$ *is continuous on* $D = \{(t,u,v) : t \geq 1, u, v \in \mathbb{R}\}$;

(ii) *there are functions* $h_1, h_2, h_3 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *continuous and such that*

$$\int_1^\infty h_1(s)ds < \infty, \quad \int_1^\infty h_2(s)ds < \infty, \quad \int_1^\infty h_3(s)ds < \infty,$$

with the property that

$$|f(t,u,v)| \leq h_1(t)\frac{|u|}{t} + h_2(t)g(|v|) + h_3(t), \quad (t,u,v) \in D,$$

where $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is continuous nondecreasing function such that* $g(x) > 0$ *for* $x > 0$ *and if we denote*

$$G(x) = \int_1^x \frac{ds}{g(s)}, \qquad x > 0,$$

then $\lim_{x \to \infty} G(x) = \infty$.

Then for every solution $u(t)$ *of* (1) *we have that* $u(t) = at + b + o(t)$ *as* $t \to \infty$ *where* a, b *are real constants.*

EXAMPLE 1. Let us consider the differential equation

$$u'' + \frac{u\sin(u)}{t^3 + t} + \frac{g(u')}{t^2} + \frac{1}{t^2 + 1} = 0 \tag{8}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(x) = 0$, $x \leq 0$; $g(x) = x\,ln(x+1)$, $x \in \mathbb{R}_+$. An application of Theorem 2 enables us to deduce that for every solution $u(t)$ of (8) there exist $a, b \in \mathbb{R}$ such that $u(t) = at + o(t)$ as $t \to \infty$.

Consider now the nonlinear differential equation

$$u'' + f(t,u) = 0. \tag{2}$$

COROLLARY 1 [1]. *Suppose $f(t,u)$ satisfies the following conditions:*

(i) *$f(t,u)$ is continuous on $D: t \geq 1,\ u \in \mathbb{R}$;*

(ii) *the derivative $f_u(t,u)$ exists on D and $f_u(t,u) > 0$ on D;*

(iii) *$|f(t,u)| < f_u(t,0)|u|$ on D.*

In addition, suppose that

$$\int_1^\infty s\, f_u(s,0)ds < \infty .$$

Then every solution $u(t)$ of (7) is asymptotic to a line $at+b$ as $t \to \infty$.

Corollary 1 generalizes a result of TRENCH [4].

COROLLARY 2. *Let $f(t,u)$ be continuous on $D: t \geq 1,\ u \in \mathbb{R}$. Suppose there exists $\varphi, u \in C(\mathbb{R}_+, \mathbb{R}_+)$, w nondecreasing on $\mathbb{R}_+$, $w(x) > 0$ for $x > 0$, such that*

$$|f(t,u)| \leq \varphi(t) w\left(\frac{|u|}{t}\right), \quad t \geq 1, \quad u \in \mathbb{R},$$

and

$$\int_1^\infty \varphi(s)ds < \infty , \qquad \int_1^\infty \frac{ds}{w(s)} = \infty .$$

Then if $u(t)$ is a solution of (7) we have that $u(t) = at + b + o(t)$ as $t \to \infty$ where a, b are constants.

REMARK. We observe (see Example 2) that the results of TONG [3] is false. Corollary 2 - we added the condition

$$\int_1^\infty \frac{ds}{w(s)} = \infty ,$$

might be considered as a specification of the conditions under which the assertions of TONG's theorem holds.

EXAMPLE 2. Let us consider the differential equation

$$u'' - \frac{3}{t^5} u^2 = 0 \tag{10}$$

with the initial conditions $u(1) = 2$, $u'(1) = 6$. If the result of TONG [3] would be valid without other assumptions we would have that every solution of equation (10) is asymptotic to some $at + b$ as $t \to \infty$ (with $a = 0$ allowed). It is easy to verify that $u(t) = 2t^3$ is a solution of (10) which is not asymptotic to a line as $t \to \infty$.

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