Rendiconti di Matematica, Serie VII
Volume 13 , Roma (1993), 635-640

A 2nd-order differential equation for orthogonal polynomials

A.L.W. VON BACHHAUS

RIASSUNTO: *Usando la relazione differenziale:* $g_n(x)Y_n'(x)=f_n(n)Y_n(x)+Y_{n-1}(x)$, *verificata da tutti i polinomi ortogonali classici, e la relazione di ricorrenza a tre termini, soddisfatta da tutti i polinomi ortogonali, viene ricostruita l'equazione differenziale lineare omogenea del secondo ordine verificata dai polinomi classici considerati.*

L'equazione differenziale viene trovata mediante la seguente tecnica: usando la relazione differenziale suddetta, l'equazione differenziale $Y_n''(x)+p(x)Y_n'(x)+q(x)Y_n(x)=0$ *viene ridotta ad una formula di ricorrenza a due termini. Poiché i polinomi ortogonali non verificano una tale relazione di ricorrenza, bensì una relazione a tre termini, si deduce che entrambi i coefficienti di tale ricorrenza devono essere identicamente nulli. Ciò consente di determinare i coefficienti* $p(x)$, $q(x)$ *dell'equazione differenziale.*

ABSTRACT: *By using the differentiation formula:* $g_n(x)Y_n'(x) = f_n(x)Y_n(x) + Y_{n-1}(x)$, *which all orthogonal polynomials satisfying a 2nd-order linear homogeneous ordinary differential equation necessarily possess, and the three-term recurrence formula possessed by all orthogonal polynomials, one obtains, a 2nd-order differential equation.*

The differential equation is obtained by using the following technique: The general 2nd-order linear differential equation: $Y_n''(x) + p_n(x)Y_n'(x) + q_n(x)Y_n(x) = 0$ *is reduced to a two-term pure recurrence formula. Since orthogonal polynomials do not satisfy a recurrence formula of this type, but a three-term formula, both coefficients in the two-term formula must be identically zero. These coefficient relationships enable one to obtain the differential equation.*

KEY WORDS: *Orthogonal polynomials - Differential equation - Recurrence formula - Differentiation formula.*

A.M.S. CLASSIFICATION: 33C45

1 – The Differential Equation

A general 2nd-order linear homogeneous ordinary differential equation may, without loss of generality, be written in the form:

$$Y_n''(x) + p_n(x)Y_n'(x) + q_n(x)Y_n(x) = 0. \tag{1}$$

It was shown in [1] that all orthogonal polynomials which satisfy a differential equation (1) necessarily have a recurrence formula of the form:

$$g_n(x)Y_n'(x) = f_n(x)Y_n(x) + Y_{n-1}(x). \tag{2}$$

It is well known that a necessary and sufficient condition for orthogonal polynomials is a three-term recurrence formula (See chapter 9 of [2]) of the type:

$$(x - B_n)Y_n(x) = A_nY_{n+1}(x) + C_nY_{n-1}(x). \tag{3}$$

THEOREM. *The differential equation of all orthogonal polynomials satisfying a 2nd-order linear homogeneous ordinary differential equation is:*

$$Y_n''(x) + \left[\frac{g_n'(x)}{g_n(x)} - \frac{f_n(x)}{g_n(x)} - \frac{f_{n+1}(x)}{g_{n+1}(x)} - \frac{x - B_n}{C_n g_n(x)}\right] Y_n'(x) +$$

$$+\left\{\frac{1}{C_n g_n(x)}\left[\frac{A_n}{g_{n+1}(x)} - 1\right] + \frac{f_{n+1}(x)}{g_{n+1}(x)}\left[\frac{f_n(x)}{g_n(x)} + \frac{x - B_n}{C_n g_n(x)}\right] - \frac{f_n'(x)}{g_n(x)}\right\} Y_n(x) = 0.$$

The $g_n(x)$ and $f_n(x)$ are the coefficients of $Y_n'(x)$ and $Y_n(x)$ respectively in the differentiation formula (2). *The A_n, B_n and C_n are the coefficients in the three-term recurrence formula possessed by every orthogonal polynomial, i.e. equation* (3).

PROOF. From (2):

$$Y_n'(x) = \frac{f_n(x)}{g_n(x)} Y_n(x) + \frac{Y_{n-1}(x)}{g_n(x)}. \tag{4}$$

Let $n \longrightarrow n+1$ in (4):

$$Y_{n+1}'(x) = \frac{f_{n+1}(x)}{g_{n+1}(x)} Y_{n+1}(x) + \frac{Y_n(x)}{g_{n+1}(x)}. \tag{5}$$

Eliminating $Y_{n-1}(x)$ from (3) and (4) gives:

$$Y_n'(x) = \frac{f_n(x)}{g_n(x)} Y_n(x) + \frac{x - B_n}{C_n g_n(x)} Y_n(x) - \frac{A_n Y_{n+1}(x)}{C_n g_n(x)}. \tag{6}$$

Differentiating (6) gives:

$$\begin{aligned} Y_n''(x) &= \frac{g_n(x) f_n'(x) - f_n(x) g_n'(x)}{g_n^2(x)} Y_n(x) + \frac{f_n(x)}{g_n(x)} Y_n'(x) + \\ &+ \frac{g_n(x) - (x - B_n) g_n'(x)}{C_n g_n^2(x)} Y_n(x) + \frac{x - B_n}{C_n g_n(x)} Y_n'(x) + \\ &- \frac{A_n}{C_n} \frac{g_n(x) Y_{n+1}'(x) - Y_{n+1}(x) g_n'(x)}{g_n^2(x)}. \end{aligned} \tag{7}$$

Now solve (3) for $Y_{n-1}(x)$:

$$Y_{n-1}(x) = \frac{1}{C_n} \left[(x - B_n) Y_n(x) - A_n Y_{n+1}(x) \right]. \tag{8}$$

Substitute (8) into (4):

$$Y_n'(x) = \left[\frac{f_n(x)}{g_n(x)} + \frac{x - B_n}{C_n g_n(x)} \right] Y_n(x) - \frac{A_n}{C_n g_n(x)} Y_{n+1}(x). \tag{9}$$

Now substitute (5) and (9) into (7):

$$
\begin{aligned}
Y_n''(x) = &\left[\frac{f_n'(x)}{g_n(x)} - \frac{f_n(x)g_n'(x)}{g_n^2(x)}\right] Y_n(x)+\\
&+\frac{f_n(x)}{g_n(x)}\left[\frac{f_n(x)}{g_n(x)}Y_n(x)+\frac{x-B_n}{C_ng_n(x)}Y_n(x)-\frac{A_nY_{n+1}(x)}{C_ng_n(x)}\right]+\\
&+\left[\frac{1}{C_ng_n(x)}-\frac{(x-B_n)g_n'(x)}{C_ng_n^2(x)}\right]Y_n(x)+\\
&+\frac{x-B_n}{C_ng_n(x)}\left[\frac{f_n(x)}{g_n(x)}Y_n(x)+\frac{x-B_n}{C_ng_n(x)}Y_n(x)-\frac{A_nY_{n+1}(x)}{C_ng_n(x)}\right]+\\
&-\frac{A_n}{C_ng_n(x)}\left[\frac{f_{n+1}(x)}{g_{n+1}(x)}Y_{n+1}(x)+\frac{1}{g_{n+1}(x)}Y_n(x)\right]+\\
&+\frac{A_ng_n'(x)}{C_ng_n^2(x)}Y_{n+1}(x).
\end{aligned}
\tag{10}
$$

Now substitute (6) and (10) into (1) and collect terms:

$$
\begin{aligned}
&\left[\frac{f_n'(x)}{g_n'(x)}-\frac{f_n(x)g_n'(x)}{g_n^2(x)}+\frac{f_n^2(x)}{g_n^2(x)}+\frac{f_n(x)(x-B_n)}{C_ng_n^2(x)}+\frac{1}{C_ng_n(x)}+\right.\\
&-\frac{(x-B_n)g_n'(x)}{C_ng_n^2(x)}+\frac{f_n(x)(x-B_n)}{C_ng_n^2(x)}+\frac{(x-B_n)^2}{C_n^2g_n^2(x)}+\\
&\left.-\frac{A_n}{C_ng_n(x)g_{n+1}(x)}+\frac{p_n(x)f_n(x)}{g_n(x)}+\frac{p_n(x)(x-B_n)}{C_ng_n(x)}+q_n(x)\right]Y_n(x)+\\
&+\left[-\frac{A_nf_n(x)}{C_ng_n^2(x)}-\frac{A_n(x-B_n)}{C_n^2g_n^2(x)}-\frac{A_nf_{n+1}(x)}{C_ng_n(x)g_{n+1}(x)}+\right.\\
&\left.+\frac{A_ng_n'(x)}{C_ng_n^2(x)}-\frac{A_np_n(x)}{C_ng_n(x)}\right]Y_{n+1}(x)=0.
\end{aligned}
\tag{11}
$$

Formula (11) is a pure recurrence formula, but not of the three-term type possessed by orthogonal polynomials. The only way for (11) to be true is for the coefficients of $Y_n(x)$ and $Y_{n+1}(x)$ to be identically zero.

This gives:

$$p_n(x) = \frac{g_n'(x)}{g_n(x)} - \frac{f_n(x)}{g_n(x)} - \frac{f_{n+1}(x)}{g_{n+1}(x)} - \frac{x - B_n}{C_n g_n(x)} \tag{12}$$

and

$$\begin{aligned} &\frac{f_n'(x)}{g_n(x)} - \frac{f_n(x)g_n'(x)}{g_n^2(x)} + \frac{f_n^2(x)}{g_n^2(x)} + \frac{f_n(x)(x - B_n)}{C_n g_n^2(x)} + \\ &+ \frac{1}{C_n g_n(x)} - \frac{(x - B_n)g_n'(x)}{C_n g_n^2(x)} + \frac{f_n(x)(x - B_n)}{C_n g_n^2(x)} + \\ &+ \frac{(x - B_n)^2}{C_n^2 g_n^2(x)} - \frac{A_n}{C_n g_n(x) g_{n+1}(x)} + p_n(x)\frac{f_n(x)}{g_n(x)} + \\ &+ p_n(x)\frac{x - B_n}{C_n g_n(x)} + q_n(x) = 0. \end{aligned} \tag{13}$$

When (12) is substituted into (13) and the result is simplified one obtains:

$$\begin{aligned} q_n(x) = &-\frac{f_n'(x)}{g_n(x)} - \frac{1}{C_n g_n(x)} + \frac{A_n}{C_n g_n(x) g_{n+1}(x)} + \\ &+ \frac{f_n(x) f_{n+1}(x)}{g_n(x) g_{n+1}(x)} + \frac{f_{n+1}(x)(x - B_n)}{C_n g_n(x) g_{n+1}(x)}. \end{aligned} \tag{14}$$

Finally substitute (12) and (14) into (1) to obtain the differential equation:

$$\begin{aligned} &Y_n''(x) + \left[\frac{g_n'(x)}{g_n(x)} - \frac{f_n(x)}{g_n(x)} - \frac{f_{n+1}(x)}{g_{n+1}(x)} - \frac{x - B_n}{C_n g_n(x)}\right] Y_n'(x) + \\ &+ \left\{\frac{1}{C_n g_n(x)}\left[\frac{A_n}{g_{n+1}(x)} - 1\right] + \frac{f_{n+1}(x)}{g_{n+1}(x)}\left[\frac{f_n(x)}{g_n(x)} + \right.\right. \\ &\left.\left. + \frac{x - B_n}{C_n g_n(x)}\right] - \frac{f_n'(x)}{g_n(x)}\right\} Y_n(x) = 0. \end{aligned} \tag{15}$$

Equation (15) is exactly the same equation as was obtained in [1] by a completely different technique (an operator technique).

2 – Conclusion

This differential equation was derived from a differentiation formula, which all orthogonal polynomials satisfying a 2nd-order linear homogeneous differential equation necessarily possess, and from a pure recurrence formula, which is a necessary and sufficient condition for orthogonal polynomials. Therefore it should be noted that any orthogonal polynomial, which satisfies a 2nd-order linear homogeneous ordinary differential equation, will satisfy this differential equation. Furthermore, all polynomial solutions of this differential equation will be orthogonal polynomials.

REFERENCES

[1] A.L.W. VON BACHHAUS: *The differential equation of orthogonal polynomials having the recurrence formula:* $g_n(x)Y_n'(x) = f_n(x)Y_n(x) + Y_{n-1}(x)$, presented at: The 4th International Symposium on Orthogonal Polynomials and their Applications, Evian-France, Oct. 19-23, 1992.

[2] EARL D. RAINVILLE: *Special Functions*, New York: The MacMillan Co., (1960).

Lavoro pervenuto alla redazione il 24 febbraio 1993
ed accettato per la pubblicazione il 10 novembre 1993

INDIRIZZO DELL'AUTORE:
A.L.W. von Bachhaus - Alter Weg 31 - 61440 Oberursel - Germany