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Saint-Venant's problem for elastic bodies with microstructure

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RIASSUNTO: *Nel contesto della teoria lineare dei corpi elastici con microstrutture, viene studiato il problema di Saint-Venant per un materiale anisotropo. Come nella teoria classica, il problema viene ridotto alla soluzione di certi problemi bidimensionali nella sezione trasversale del cilindro.*

ABSTRACT: *This paper is concerned with the linear theory of elastic bodies with microstructure. Saint Venant's problem for anisotropic materials is studied. As in the classical theory, the problem is reduced to the solution of certain two-dimensional problems in the cross section of the cylinder.*

KEY WORDS: *Saint-Venant's problem - Microstructure.*

A.M.S. CLASSIFICATION: 73S10 – 73C10

1 – Introduction

Saint-Venant's problem in classical elasticity consists in determining the equilibrium of an elastic cylinder, loaded only by surface forces distributed over its plane ends. Saint-Venant's approach to the problem is based on a relaxed statement in which the pointwise assignement of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. Justification of the procedure is twofold.

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First, it is difficult in practice to determine the actual distribution of applied stresses on the ends, although the resultant forces and moment can be measured accurately. Second, one invokes Saint-Venant's principle. For Saint-Venant's principle and Saint-Venant's problem we refer to the works [1 - 8]. Saint-Venant's formulation leads to the four basic problems of extension, bending, torsion and flexure. Saint-Venant has established the solutions of these problems for homogeneous and isotropic materials. It is obvious that the relaxed statement of the problem fails to characterize the solution uniquely. This fact led various authors to establish characterizations of Saint-Venant's solution. Thus, CLEBSCH [9] proved that Saint-Venant's solution can be derived from the assumption that the stress vector on any plane normal to the cross-sections of the cylinder is parallel to its generators. In [10], VOIGT rediscovered Saint-Venant's solution by using another assumption regarding the structure of the stress field. Thus, Saint-Venant's extension, bending and torsion solutions are derived from the hypothesis that the stress field is independent of the axial coordinate, and Saint-Venant's flexure solution is obtained if the stress field depends on the axial coordinate at most linearly. E. STERNBERG and J.K. KNOWLES [11] characterized Saint-Venant's solutions in terms of certain associated minimum strain-energy properties. Other intrinsic criteria that distinguish Saint-Venant's solutions among all the solutions of the relaxed problem were established in [13]. The results of CLEBSCH and VOIGT are restricted to the case of homogeneous and isotropic bodies and cannot be used for the study of nonhomogeneous or anisotropic elastic cylinders (see [12]). In [13], a rational scheme of deriving Saint-Venant's solutions is presented. The advantage of this method is that it does not involve artificial a priori assumptions. The method permits to construct a solution of the relaxed Saint-Venant's problem for other kinds of constitutive equations (anisotropic media, Cosserat continua, etc) where the physical intuition or semi-inverse method cannot be used.

In this paper an adaptation of the method of [13, 14] is used to study Saint-Venant's problem within Mindlin's linear theory of elastic bodies with microstructure [15]. In the linear case, the theory of an oriented elastic body with three directors [16], the theory of a dipolar elastic continuum [17] and the theory of simple microelastic solids [18] coincide with Mindlin's theory of elastic solids with microstructure (cf. [19]). Saint-Venant's principle in the linear theory of elastic bodies with microstructure has been studied in [20]. Throughout this paper we consider nonhomogeneous and anisotropic elastic bodies with microstructure where the constitutive coefficients are independent of the axial coordinate.

We denote by (P_1) the problem that corresponds to the loading cases of extension, bending and torsion and by (P_2) the flexure problem. The deformation field corresponding to any solution of the problem (P_1) will be called a primary solution. We prove that the partial derivative with respect to the axial coordinate of any primary solution gives rise to stresses having zero resultant force and moment. Moreover, we show that the partial derivative with respect to the axial coordinate of the deformation fields corresponding to any solution of the problem (P_2) is a primary solution. These results allow us to establish a solution of Saint-Venant's problem. As in classical theory the problem is reduced to the solution of certain two-dimensional problems in the cross section of the cylinder.

2 – Preliminaries

Throughout this paper, B denotes the interior of a right cylinder of lenght h with the open cross-section Σ and the lateral boundary Π. We assume that the generic cross-section Σ is a simply connected region and denote by Γ the boundary of Σ. We let $\overline{B}$ denote the closure of B, call ∂B the boundary of B, and designate by $\mathbf{n}$ the outward unit normal of ∂B. Letters in boldface stand for tensors of an order $p \geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots s}$ (p subscripts) for the components of $\mathbf{v}$ in the underlying rectangular Cartesian coordinate frame. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1,2), whereas Latin subscripts-unless otherwise specified-are confined to the range (1,2,3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding coordinate. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the x_3-axis is parallel to the generators of B, and the x_1Ox_2-plane contains one of the terminal cross-sections, while the other is in the plane $x_3 = h$. We denote by Σ_1 and Σ_2, respectively, the cross-sections located at $x_3 = 0$ and $x_3 = h$.

Assume now that the body accupying B is an anisotropic elastic material with microstructure. Let u_i denote the components of the displacement field, and let φ_{ij} denote the components of the microdeformation tensor. We introduce the twelve-dimensional vector $U = (u_1, u_2, u_3, \varphi_{11}, \varphi_{22}, \varphi_{33}, \varphi_{12}, \varphi_{23}, \varphi_{31}, \varphi_{21}, \varphi_{32}, \varphi_{13}) = (u_i, \varphi_{jk})$. The strain measures associated with U are defined by

$$(2.1)\quad e_{ij}(U) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \gamma_{ij}(U) = u_{j,i} - \varphi_{ij}, \quad \varkappa_{ijk}(U) = \varphi_{jk,i},$$

where $e_{ij}(U)$ is the macrostrain tensor, $\gamma_{ij}(U)$ is the relative deformation tensor and $\varkappa_{ijk}(U)$ is the microdeformation gradient tensor. We have

$$(2.2)\qquad e_{ij}(U) = 0, \quad \gamma_{ij}(U) = 0, \quad \varkappa_{ijk}(U) = 0,$$

if and only if

$$(2.3)\qquad u_i = a_i + \varepsilon_{ijk} b_j x_k, \qquad \varphi_{ij} = \varepsilon_{ijs} b_s.$$

Here ε_{ijk} is the alternating symbol and a_i and b_i are arbitrary constants.

Let

$$(2.4)\quad \mathcal{P} = \{V^0 : V^0 = (v_i^0, \psi_{jk}^0), \quad v_i^0 = a_i + \varepsilon_{ijk} b_j x_k, \quad \psi_{jk}^0 = \varepsilon_{jks} b_s\},$$

where a_i and b_j are constants. A rigid motion is a twelve-dimensional vector field V^0 such that $V^0 \in \mathcal{P}$.

The strain energy density per unit volume corresponding to U is given by (cf. [15])

$$(2.5)\qquad \begin{aligned} \varepsilon(U) = {} & \frac{1}{2} C_{ijrs} e_{ij}(U) e_{rs}(U) + \frac{1}{2} B_{ijrs} \gamma_{ij}(U) \gamma_{rs}(U) + \\ & + \frac{1}{2} A_{ijkrmn} \varkappa_{ijk}(U) \varkappa_{rmn}(U) + D_{ijkrm} \gamma_{ij}(U) \varkappa_{krm}(U) + \\ & + F_{ijkrm} \varkappa_{ijk}(U) e_{rm}(U) + G_{ijkr} \gamma_{ij}(U) e_{kr}(U), \end{aligned}$$

where A_{ijkrmn}, B_{ijrs}, C_{ijrs}, D_{ijkrm}, F_{ijkrm} and G_{ijkr} are smooth functions on $\overline{B}$ such that

$$(2.6)\qquad \begin{aligned} & A_{ijkrmn} = A_{rmnijk}, \quad B_{ijrs} = B_{rsij}, \quad C_{ijrs} = C_{jirs} = C_{jirs}, \\ & F_{ijkrs} = F_{ijksr}, \quad G_{ijrs} = G_{ijsr}. \end{aligned}$$

We assume that the strain energy density is a positive-definite quadratic form in the components of the strain measures. The constitutive equations are

$$
(2.7)\qquad
\begin{aligned}
\tau_{ij}(U) &= C_{ijrs}e_{rs}(U) + G_{rsij}\gamma_{rs}(U) + F_{pqrij}\varkappa_{pqr}(U)\,,\\
\sigma_{ij}(U) &= G_{ijrs}e_{rs}(U) + B_{rsij}\gamma_{rs}(U) + D_{ijpqr}\varkappa_{pqr}(U)\,,\\
\mu_{ijk}(U) &= F_{ijkrs}e_{rs}(U) + D_{rsijk}\gamma_{rs}(U) + A_{ijkpqr}\varkappa_{pqr}(U)\,,
\end{aligned}
$$

where $\tau_{ij}(U)$ denotes the stress tensor, $\sigma_{ij}(U)$ means the relative stress tensor, and $\mu_{ijk}(U)$ is the double stress tensor associated with U.

We call a vector field $U = (u_i, \varphi_{jk})$ an equilibrium vector field for B if u_i, $\varphi_{jk} \in C^1(\overline{B}) \cap C^2(B)$ and

$$
(2.8)\qquad (\tau_{ij}(U) + \sigma_{ij}(U))_{,i} = 0\,, \quad (\mu_{ijk}(U))_{,i} + \sigma_{jk}(U) = 0\,,
$$

hold on B. The traction and the double-traction corresponding to U at regular points of ∂B are given by

$$
(2.9)\qquad T_i(U) = (\tau_{ji}(U) + \sigma_{ji}(U))n_j\,, \quad M_{ij}(U) = \mu_{rij}(U)n_r\,.
$$

The strain energy $E(U)$ corresponding to a smooth vector field U on B is

$$
(2.10)\qquad E(U) = \int_B \varepsilon(U)dv\,.
$$

The functional $E(\cdot)$ generates the bilinear functional

$$
(2.11)\qquad
\begin{aligned}
E(U,V) = \frac{1}{2}\int_B \{&C_{ijrs}e_{ij}(U)e_{rs}(V) + B_{ijrs}\gamma_{ij}(U)\gamma_{rs}(V)+\\
&+ A_{ijkrmn}\varkappa_{ijk}(U)\varkappa_{rmn}(V) + D_{ijkrm}\left[\gamma_{ij}(U)\varkappa_{krm}(V)+\right.\\
&\left.+ \gamma_{ij}(V)\varkappa_{krm}(U)\right] + F_{ijkrm}\left[\varkappa_{ijk}(U)e_{rm}(V) + \varkappa_{ijk}(V)e_{rm}(U)\right]+\\
&+ G_{ijkr}\left[\gamma_{ij}(U)e_{kr}(V) + \gamma_{ij}(V)e_{kr}(U)\right]\Big\}dV\,.
\end{aligned}
$$

We introduce the notation

$$
(2.12)\qquad \langle U,V\rangle = 2E(U,V)\,.
$$

For any two equilibrium vector fields $U = (u_i, \varphi_{jk})$ and $V = (v_i, \psi_{jk})$ one has

$$\langle U, V\rangle = \int_{\partial B} [v_i T_i(U) + \psi_{jk} M_{jk}(U)]\, da\,, \tag{2.13}$$

and

$$\int_{\partial B} [u_i T_i(V) + \varphi_{jk} M_{jk}(V)]\, da = \int_{\partial B} [v_i T_i(U) + \psi_{jk} M_{jk}(U)]\, da\,. \tag{2.14}$$

We assume for the remainder of this chapter that the functions A_{ijkrmn}, B_{ijrs}, C_{ijrs}, D_{ijkrs}, F_{ijkrm}, G_{ijrs} are independent of the axial coordinate and belong to $C^\infty(\overline{\Sigma})$. Moreover, we assume that Σ is C^∞-smooth.

3 – The generalized plane strain

The state of generalized plane strain of B is characterized by

$$u_i = u_i(x_1, x_2), \quad \varphi_{jk} = \varphi_{jk}(x_1, x_2), \qquad (x_1, x_2) \in \Sigma\,. \tag{3.1}$$

It follows from (2.1) and (3.1) that $e_{33}(U) = 0$, $\varkappa_{3jk}(U) = 0$ and

$$\begin{gathered} e_{\alpha\beta}(U) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad e_{\alpha 3}(U) = \frac{1}{2} u_{3,\alpha}\,; \\ \gamma_{\alpha i}(U) = u_{i,\alpha} - \varphi_{\alpha i}, \quad \gamma_{3i}(U) = -\varphi_{3i}, \quad \varkappa_{\alpha jk}(U) = \varphi_{jk,\alpha}\,. \end{gathered} \tag{3.2}$$

By (2.7) and (3.2),

$$\begin{aligned} \tau_{\alpha i}(U) &= C_{\alpha ij\beta} e_{j\beta}(U) + G_{kj\alpha i}\gamma_{kj}(U) + F_{\beta rs\alpha i}\varkappa_{\beta rs}(U)\,, \\ \sigma_{ij}(U) &= G_{ijr\beta} e_{r\beta}(U) + B_{krij}\gamma_{kr}(U) + D_{ij\beta rs}\varkappa_{\beta rs}(U)\,, \\ \mu_{\alpha ij}(U) &= F_{\alpha ijr\beta} e_{r\beta}(U) + D_{rs\alpha ij}\gamma_{rs}(U) + A_{\alpha ij\beta rs}\varkappa_{\beta rs}(U)\,. \end{aligned} \tag{3.3}$$

The equations of equilibrium (2.8) in the presence of the body force f_i and body double-force L_{ij} take the form

$$\left(\tau_{\alpha i}(U) + \sigma_{\alpha i}(U)\right)_{,\alpha} + f_i = 0\,, \quad \left(\mu_{\alpha ij}(U)\right)_{,\alpha} + \sigma_{ij}(U) + L_{ij} = 0\,. \tag{3.4}$$

We assume that on the lateral boundary we have the conditions

$$(3.5) \qquad (\tau_{\alpha i}(U) + \sigma_{\alpha i}(U))n_\alpha = P_i\,, \quad (\mu_{\alpha ij}(U))n_\alpha = Q_{ij}\,,$$

where P_i and Q_{ij} are prescribed functions.

Clearly, the state of generalized plane strain demands that f_i, L_{ij}, P_i and Q_{ij} be independent of the axial coordinate.

The generalized plane strain problem consists in finding a vector field $U \in C^1(\overline{\Sigma}) \cap C^2(\Sigma)$ which satisfies the equations (3.4) on Σ and the boundary conditions (3.5) on Γ. The functions $\tau_{3i}(U)$ and $\mu_{3ij}(U)$ can be calculated after the determination of U.

The conditions of equilibrium for the cylinder B are

$$(3.6) \qquad \begin{aligned} &\int_\Sigma f_i da + \int_\Gamma P_i ds = 0\,, \\ &\int_\Sigma \varepsilon_{\alpha\beta}(x_\alpha f_\beta + L_{\alpha\beta}) da + \int_\Gamma \varepsilon_{\alpha\beta}(x_\alpha P_\beta + Q_{\alpha\beta}) ds = 0\,, \end{aligned}$$

and

$$(3.7) \qquad \begin{aligned} &\int_\Sigma (x_2 f_3 + L_{23} - L_{32}) da + \int_\Gamma (x_2 P_3 + Q_{23} - Q_{32}) ds + \\ &- \int_\Sigma (\tau_{32}(U) + \sigma_{32}(U)) da = 0\,, \\ &\int_\Sigma (x_1 f_3 + L_{13} - L_{31}) da + \int_\Gamma (x_1 P_3 + Q_{13} - Q_{31}) ds + \\ &- \int_\Sigma (\tau_{31}(U) + \sigma_{31}(U)) da = 0\,. \end{aligned}$$

Here we have used the notation $\varepsilon_{\alpha\beta} = \varepsilon_{3\alpha\beta}$.

The conditions (3.7) are identically satisfied on the basis of (3.4) and

(3.5). Indeed, we have

$$\begin{aligned}\int_\Sigma (\tau_{32}(U)+\sigma_{32}(U))da &= \int_\Sigma [\tau_{23}(U)+\sigma_{23}(U)+\sigma_{32}(U)-\sigma_{23}(U)]da = \\ &= \int_\Sigma [\tau_{23}(U)+\sigma_{23}(U)+x_2(\tau_{\alpha 3}(U)_{,\alpha}+\sigma_{\alpha 3}(U)_{,\alpha}+f_3)+ \\ &+ L_{23}-L_{32}+(\mu_{\alpha 23}(U)-\mu_{\alpha 32}(U))_{,\alpha}]da = \\ &= \int_\Sigma \Big\{[x_2(\tau_{\alpha 3}(U)+\sigma_{\alpha 3}(U)]_{,\alpha}+x_2 f_3+L_{23}-L_{32}+ \\ &+ (\mu_{\alpha 23}(U)-\mu_{\alpha 32}(U))_{,\alpha}\Big\}da = \int_\Gamma (x_2 P_3+Q_{23}-Q_{32})ds+ \\ &+ \int_\Sigma (x_2 f_3+L_{23}-L_{32})da\,.\end{aligned}$$

In a similar way we can prove that the second condition in (3.7) is satisfied.

Following [21, 22] we are led to the following

THEOREM 1. *The boundary-value problem* (3.4), (3.5) *has a solution belonging to* $C^\infty(\overline{\Sigma})$ *if and only if the* C^∞ *functions* f_i, L_{ij}, P_i *and* Q_{ij} *satisfy the conditions* (3.6).

4 – The relaxed Saint-Venant problem

By a solution of the relaxed Saint-Venant's problem we mean an equilibrium vector field for B that satisfies the conditions

$$(4.1)\qquad \begin{aligned} (\tau_{\alpha i}(U)+\sigma_{\alpha i}(U))n_\alpha = 0\,, &\quad \mu_{\alpha ij}(U)n_\alpha = 0 \quad \text{on} \quad \Pi\,, \\ \mathbf{R}(U)=\mathbf{F}\,, &\quad \mathbf{H}(U)=\mathbf{M}\,, \end{aligned}$$

where $\mathbf{F}$ and $\mathbf{M}$ are prescribed vectors representing the resultant force and the resultant moment about O of the tractions acting on Σ_1. Accordingly,

$\mathbf{R}(\cdot)$ and $\mathbf{H}(\cdot)$ are the vector-valued linear functionals defined by

$$R_i(U) = -\int_{\Sigma_1} (\tau_{3i}(U) + \sigma_{3i}(U))da\,,$$

$$(4.2) \quad H_\alpha(U) = -\int_{\Sigma_1} \varepsilon_{\alpha\beta}\Big[x_\beta(\tau_{33}(U) + \sigma_{33}(U)) + \mu_{3\beta 3}(U) - \mu_{33\beta}(U)\Big]da\,,$$

$$H_3(U) = -\int_{\Sigma_1} \varepsilon_{\alpha\beta}\Big[x_\alpha(\tau_{3\beta}(U) + \sigma_{3\beta}(U)) + \mu_{3\alpha\beta}(U)\Big]da\,.$$

We denote by (P) the relaxed Saint-Venant's problem corresponding to the resultants $\mathbf{F}$ and $\mathbf{M}$. Let $K(\mathbf{F},\mathbf{M})$ denote the class of solutions to the problem (P). It is convenient to use the decomposition of the problem (P) into problems (P_1) and (P_2) where (P_1) corresponds to $F_\alpha = 0$ and (P_2) corresponds to $F_3 = M_i = 0$. Clearly, (P_1) is the extension-bending-torsion problem and (P_2) is the flexure problem. We denote by $K_I(F_3, M_1, M_2, M_3)$ the class of solutions to the problem (P_1) and by $K_{II}(F_1, F_2)$ the class of solutions to the problem (P_2).

Let Z denote the set of all equilibrium vector fields U that satisfy the conditions

$$(4.3) \qquad (\tau_{\alpha i}(U) + \sigma_{\alpha i}(U))n_\alpha = 0\,, \quad \mu_{\alpha ij}(U)n_\alpha = 0 \quad \text{on} \quad \Pi\,.$$

THEOREM 2. *If $U \in Z$ and $U_{,3} \in C^1(\overline{B}) \cap C^2(B)$ then $U_{,3} \in Z$ and*

$$(4.4) \qquad \mathbf{R}(U_{,3}) = \mathbf{0}\,, \quad H_\alpha(U_{,3}) = \varepsilon_{\alpha\beta}R_\beta(U)\,, \quad H_3(U_{,3}) = 0\,.$$

PROOF. Since the constitutive coefficients are independent of x_3, it follows from (2.1) and (2.7) that $\tau_{ij}(U_{,3}) = (\tau_{ij}(U))_{,3}$, $\sigma_{ij}(U_{,3}) = (\sigma_{ij}(U))_{,3}$, $\mu_{rij}(U_{,3}) = (\mu_{rij}(U))_{,3}$. These relations and the hypotheses imply the first assertion. Using the equations of equilibrium, we have

$$
\begin{aligned}
&\tau_{3i}(U_{,3}) + \sigma_{3i}(U_{,3}) = -\left[\tau_{\alpha i}(U) + \sigma_{\alpha i}(U)\right]_{,\alpha}, \\
&\varepsilon_{\alpha\beta}\left[x_\beta(\tau_{33}(U_{,3}) + \sigma_{33}(U_{,3})) + \mu_{3\beta 3}(U_{,3}) - \mu_{33\beta}(U_{,3})\right] = \\
&= \varepsilon_{\alpha\beta}\left[-x_\beta(\tau_{\rho 3}(U) + \sigma_{\rho 3}(U))_{,\rho} + (\mu_{3\beta 3}(U))_{,3} - (\mu_{33\beta}(U))_{,3}\right] = \\
&= \varepsilon_{\alpha\beta}\Big\{-\left[x_\beta(\tau_{\rho 3}(U) + \sigma_{\rho 3}(U))\right]_{,\rho} + \tau_{\beta 3}(U) + \sigma_{\beta 3}(U) - (\mu_{\rho\beta 3}(U))_{,\rho} + \\
(4.5)\quad & - \sigma_{\beta 3}(U) + (\mu_{\rho 3\beta}(U))_{,\rho} + \sigma_{3\beta}(U)\Big\} = \\
&= -\varepsilon_{\alpha\beta}\left[x_\beta(\tau_{\rho 3}(U) + \sigma_{\rho 3}(U)) + \mu_{\rho\beta 3}(U) - \mu_{\rho 3\beta}(U)\right]_{,\rho} + \\
&+ \varepsilon_{\alpha\beta}(\tau_{3\beta}(U) + \sigma_{3\beta}(U)), \\
&\varepsilon_{\alpha\beta}\left[x_\alpha(\tau_{3\beta}(U_{,3}) + \sigma_{3\beta}(U_{,3})) + \mu_{3\alpha\beta}(U_{,3})\right] = \\
&= -\varepsilon_{\alpha\beta}\left[x_\alpha(\tau_{\rho\beta}(U) + \sigma_{\rho\beta}(U))_{,\rho} + (\mu_{\rho\alpha\beta}(U))_{,\rho} + \sigma_{\alpha\beta}(U)\right] = \\
&= -\varepsilon_{\alpha\beta}\left[x_\alpha(\tau_{\rho\beta}(U) + \sigma_{\rho\beta}(U)) + \mu_{\rho\alpha\beta}(U)\right]_{,\rho}.
\end{aligned}
$$

By (4.2), (4.5) and the divergence theorem we arrive to desired result. □

In view of theorem 2, if $U \in K_I(F_3, M_1, M_2, M_3)$ and $U_{,3} \in C^1(\overline{B}) \cap C^2(B)$ then $\mathbf{R}(U_{,3}) = \mathbf{0}$ and $\mathbf{H}(U_{,3}) = \mathbf{0}$. Thus, we are led to look for a solution V of the problem (P_1) such that $V_{,3} \in \mathcal{P}$.

THEOREM 3. *Let X be the set of all vector fields $U \in C^1(\overline{B}) \cap C^2(B)$ such that $U_{,3} \in \mathcal{P}$. Then, there exists a vector field $V \in X$ such that $V \in K_I(F_3, M_1, M_2, M_3)$.*

PROOF. Let $V \in X$, $V = (v_i, \psi_{jk})$. Then

$$v_{i,3} = a_i + \varepsilon_{ijk} b_j x_k, \quad \psi_{jk,3} = \varepsilon_{jks} b_s.$$

It follows that

$$
\begin{aligned}
&v_\alpha = \varepsilon_{\alpha\beta}\left(\frac{1}{2} b_\beta x_3^2 - b_3 x_\beta x_3\right) + w_\alpha, \\
(4.6)\quad &v_3 = (\varepsilon_{\alpha\beta} b_\alpha x_\beta + b_4) x_3 + w_3, \\
&\psi_{jk} = \varepsilon_{jks} b_s x_3 + \omega_{jk},
\end{aligned}
$$

except for an additive rigid motion. Here $W = (w_i, \omega_{jk})$ is an arbitrary vector field independent of x_3 and we used the notation $b_4 = a_3$. We shall prove that the functions w_i and ω_{jk} and the constants b_s $(s = 1,2,3,4)$ can be determined such that $V \in K_I(F_3, M_1, M_2, M_3)$.

It follows from (4.6) and (2.7) that

$$
\begin{aligned}
&\tau_{ij}(V) = (C_{ij33} + G_{33ij})(\varepsilon_{\alpha\beta} b_\alpha x_\beta + b_4) + \\
&+ (C_{ij\alpha3} + G_{3\alpha ij})\varepsilon_{\beta\alpha} b_3 x_\beta + F_{3mnij}\varepsilon_{mnr} b_r + \tau_{ij}(W)\,, \\
&\sigma_{ij}(V) = (G_{ij33} + B_{33ij})(\varepsilon_{\alpha\beta} b_\alpha x_\beta + b_4) + \\
&+ (B_{3\alpha ij} + G_{ij\alpha3})\varepsilon_{\beta\alpha} b_3 x_\beta + D_{ij3mn}\varepsilon_{mnr} b_r + \sigma_{ij}(W)\,, \\
&\mu_{ijk}(V) = (F_{ijk33} + D_{33ijk})(\varepsilon_{\alpha\beta} b_\alpha x_\beta + b_4) + \\
&+ (F_{ijk\alpha3} + D_{3\alpha ijk})\varepsilon_{\beta\alpha} b_3 x_\beta + A_{ijk3mn}\varepsilon_{mnr} b_r + \mu_{ijk}(W)\,.
\end{aligned}
\tag{4.7}
$$

The equations of equilibrium and the conditions on the lateral boundary reduce to

$$
\begin{aligned}
&\left(\tau_{\alpha i}(W) + \sigma_{\alpha i}(W)\right)_{,\alpha} + f_i = 0\,, \\
&\left(\mu_{\alpha ij}(W)\right)_{,\alpha} + \sigma_{ij}(W) + L_{ij} = 0 \quad \text{on} \quad \Sigma\,, \\
&\left(\tau_{\alpha i}(W) + \sigma_{\alpha i}(W)\right) n_\alpha = P_i\,,\ \mu_{\alpha ij}(W) n_\alpha = Q_{ij} \text{ on } \Gamma\,,
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
&f_i = \sum_{s=1}^{4} b_s f_i^{(s)}\,, \quad L_{jk} = \sum_{s=1}^{4} b_s L_{jk}^{(s)}\,, \\
&P_i = \sum_{s=1}^{4} b_s P_i^{(s)}\,, \quad Q_{jk} = \sum_{s=1}^{4} b_s Q_{jk}^{(s)}\,, \\
&f_i^{(\beta)} = \big[(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})\varepsilon_{\beta\nu} x_\nu + \\
&\quad + (D_{\alpha i3mn} + F_{3mn\alpha i})\varepsilon_{mn\beta}\big]_{,\alpha}\,, \\
&f_i^{(3)} = \big[(C_{\alpha i\rho3} + G_{3\rho\alpha i} + G_{\alpha i\rho3} + B_{3\rho\alpha i})\varepsilon_{\beta\rho} x_\beta + \\
&\quad + (D_{\alpha i3\rho\nu} + F_{3\rho\nu\alpha i})\varepsilon_{\rho\nu}\big]_{,\alpha}\,, \\
&f_i^{(4)} = (C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})_{,\alpha}\,, \\
&L_{ij}^{(\beta)} = \big[(F_{\alpha ij33} + D_{33\alpha ij})\varepsilon_{\beta\nu} x_\nu + A_{\alpha ij3mn}\varepsilon_{mn\beta}\big]_{,\alpha} + \\
&\quad + (G_{ij33} + B_{33ij})\varepsilon_{\beta\nu} x_\nu + D_{ij3mn}\varepsilon_{mn\beta}\,, \\
&L_{ij}^{(3)} = \big[(F_{\alpha ij\rho3} + D_{3\rho\alpha ij})\varepsilon_{\beta\rho} x_\beta + A_{\alpha ij3\eta\rho}\varepsilon_{\eta\rho}\big]_{,\alpha} + \\
&\quad + (B_{3\alpha ij} + G_{ij\alpha3})\varepsilon_{\beta\alpha} x_\beta + D_{ij3\alpha\beta}\varepsilon_{\beta\alpha}\,,
\end{aligned}
\tag{4.9}
$$

$$L_{ij}^{(4)} = (F_{\alpha ij33} + D_{33\alpha ij})_{,\alpha} + G_{ij33} + B_{33ij}\,,$$
$$P_i^{(\beta)} = \big[(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})\varepsilon_{\nu\beta}x_\nu + \\ + (D_{\alpha i3mn} + F_{3mn\alpha i})\varepsilon_{nm\beta}\big]n_\alpha\,,$$
$$P_i^{(3)} = \big[(C_{\alpha i\rho 3} + G_{3\rho\alpha i} + G_{\alpha i\rho 3} + B_{3\rho\alpha i})\varepsilon_{\rho\beta}x_\beta + \\ + (D_{\alpha i3\rho\nu} + F_{3\rho\nu\alpha i})\varepsilon_{\nu\rho}\big]n_\alpha\,,$$
$$P_i^{(4)} = -(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})n_\alpha\,,$$
$$Q_{ij}^{(\beta)} = \big[(F_{\alpha ij33} + D_{33\alpha ij})\varepsilon_{\nu\beta}x_\nu + A_{\alpha ij3mn}\varepsilon_{nm\beta}\big]n_\alpha\,,$$
$$Q_{ij}^{(3)} = \big[(F_{\alpha ij\rho 3} + D_{3\rho\alpha ij})\varepsilon_{\rho\beta}x_\beta + A_{\alpha ij3\rho\nu}\varepsilon_{\nu\rho}\big]n_\alpha\,,$$
$$Q_{ij}^{(4)} = -(F_{\alpha ij33} + D_{33\alpha ij})n_\alpha\,.$$

Clearly, (4.8) is a generalized plane strain problem for the unknown functions w_i and ω_{jr}. It is a simple matter to verify that the necessary and sufficient conditions for the existence of a solution to this problem are satisfied for any constants b_1, b_2, b_3 and b_4.

We denote by $W^{(j)}$ a solution of the problem (4.8) when $b_i = \delta_{ij}$, $b_4 = 0$, and by $W^{(4)}$ a solution of the problem (4.8) corresponding to $b_i = 0$, $b_4 = 1$. Thus the vector fields $W^{(s)}$ $(s = 1,2,3,4)$, $W^{(s)} = (w_i^{(s)}, \omega_{ij}^{(s)})$, are characterized by

$$(4.10)\quad \begin{aligned} &\big(\tau_{\alpha i}(W^{(s)}) + \sigma_{\alpha i}(W^{(s)})\big)_{,\alpha} + f_i^{(s)} = 0\,,\\ &\big(\mu_{\alpha ij}(W^{(s)})\big)_{,\alpha} + \sigma_{ij}(W^{(s)}) + L_{ij}^{(s)} = 0 \quad \text{on} \quad \Sigma\,,\\ &\big(\tau_{\alpha i}(W^{(s)}) + \sigma_{\alpha i}(W^{(s)})\big)n_\alpha = P_i^{(s)}\,,\ \mu_{\alpha ij}(W^{(s)})n_\alpha = Q_{ij}^{(s)} \ \text{on}\ \Gamma\,. \end{aligned}$$

We assume that $W^{(s)}$ are known. We have

$$(4.11)\qquad W = \sum_{s=1}^{4} b_s W^{(s)}\,,$$

so that the vector field V can be written in the form

$$(4.12)\qquad V = \sum_{s=1}^{4} b_s V^{(s)}\,,$$

where $V^{(s)} = (v_i^{(s)}, \psi_{jk}^{(s)})$ are defined by

$$
\begin{aligned}
(4.13)\qquad & v_\alpha^{(\beta)} = \frac{1}{2}\varepsilon_{\alpha\beta}x_3^2 + w_\alpha^{(\beta)}, \quad v_3^{(\beta)} = \varepsilon_{\beta\alpha}x_\alpha + w_3^{(\beta)}, \\
& v_\alpha^{(3)} = -\varepsilon_{\alpha\beta}x_\beta x_3 + w_\alpha^{(3)}, \quad v_3^{(3)} = w_3^{(3)}, \\
& v_\alpha^{(4)} = w_\alpha^{(4)}, \quad v_3^{(4)} = x_3 + w_3^{(4)}, \\
& \psi_{jk}^{(s)} = \varepsilon_{jks}x_3 + \omega_{jk}^{(s)}, \quad \psi_{jk}^{(4)} = \omega_{jk}^{(4)}.
\end{aligned}
$$

It follows from (4.7) and (4.11) that

$$
\begin{aligned}
(4.14)\qquad & \tau_{ij}(V) = \sum_{s=1}^{4} b_s \tau_{ij}(V^{(s)}), \quad \sigma_{ij}(V) = \sum_{s=1}^{4} b_s \sigma_{ij}(V^{(s)}), \\
& \mu_{ijk}(V) = \sum_{s=1}^{4} b_s \mu_{ijk}(V^{(s)}),
\end{aligned}
$$

where

$$
\begin{aligned}
& \tau_{ij}(V^{(\beta)}) = (C_{ij33} + G_{33ij})\varepsilon_{\beta\nu}x_\nu + F_{3mnij}\varepsilon_{mn\beta} + \tau_{ij}(W^{(\beta)}), \\
& \tau_{ij}(V^{(3)}) = (C_{ij\alpha 3} + G_{3\alpha ij})\varepsilon_{\beta\alpha}x_\beta + F_{3\rho\nu ij}\varepsilon_{\rho\nu} + \tau_{ij}(W^{(3)}), \\
& \tau_{ij}(V^{(4)}) = C_{ij33} + G_{33ij} + \tau_{ij}(W^{(4)}), \\
& \sigma_{ij}(V^{(\beta)}) = (G_{ij33} + B_{33ij})\varepsilon_{\beta\nu}x_\nu + D_{ij3mn}\varepsilon_{mn\beta} + \sigma_{ij}(W^{(\beta)}), \\
(4.15)\qquad & \sigma_{ij}(V^{(3)}) = (B_{3\alpha ij} + G_{ij\alpha 3})\varepsilon_{\beta\alpha}x_\beta + D_{ij3\rho\nu}\varepsilon_{\rho\nu} + \sigma_{ij}(W^{(3)}), \\
& \sigma_{ij}(V^{(4)}) = G_{ij33} + B_{33ij} + \sigma_{ij}(W^{(4)}), \\
& \mu_{ijk}(V^{(\beta)}) = (F_{ijk33} + D_{33ijk})\varepsilon_{\beta\nu}x_\nu + A_{ijk3mn}\varepsilon_{mn\beta} + \mu_{ijk}(W^{(\beta)}), \\
& \mu_{ijk}(V^{(3)}) = (F_{ijk\alpha 3} + D_{3\alpha ijk})\varepsilon_{\beta\alpha}x_\beta + A_{ijk3\rho\nu}\varepsilon_{\rho\nu} + \mu_{ijk}(W^{(3)}), \\
& \mu_{ijk}(V^{(4)}) = F_{ijk33} + D_{33ijk} + \mu_{ijk}(W^{(4)}).
\end{aligned}
$$

Since $V_{,3} \in \mathcal{P}$, by Theorem 2 we obtain $R_\alpha(V) = \varepsilon_{\beta\alpha}H_\beta(V_{,3}) = 0$. The conditions $R_3(V) = F_3$, and $\mathbf{H}(V) = \mathbf{M}$ reduce to

$$
(4.16)\qquad \sum_{s=1}^{4} K_{\alpha s}b_s = \varepsilon_{\alpha\beta}M_\beta, \quad \sum_{s=1}^{4} K_{3s}b_s = -F_3, \quad \sum_{s=1}^{4} K_{4s}b_s = -M_3,
$$

where

$$K_{\alpha s}=\int_{\Sigma}\Big\{x_\alpha\big[\tau_{33}(V^{(s)})+\sigma_{33}(V^{(s)})\big]+\mu_{3\alpha 3}(V^{(s)})-\mu_{33\alpha}(V^{(s)})\Big\}da\,,$$

$$(4.17)\quad K_{3s}=\int_{\Sigma}\big[\tau_{33}(V^{(s)})+\sigma_{33}(V^{(s)})\big]da\,,$$

$$K_{4s}=\int_{\Sigma}\varepsilon_{\alpha\beta}\Big\{x_\alpha\big[\tau_{3\beta}(V^{(s)})+\sigma_{3\beta}(V^{(s)})\big]+\mu_{3\alpha\beta}(V^{(s)})\Big\}da\,.$$

In the view of (2.10) and (4.12) we have

$$(4.18)\qquad E(V)=\frac{1}{2}\sum_{i,j=1}^{4}\langle V^{(i)},V^{(j)}\rangle b_i b_j\,.$$

By (2.13), (2.14) and (4.13) we arrive at

$$(4.19)\qquad \begin{aligned}&\langle V^{(\alpha)},V^{(s)}\rangle=h\varepsilon_{\alpha\beta}K_{\beta s}\,,\quad \langle V^{(3)},V^{(s)}\rangle=hK_{4s}\,,\\ &\langle V^{(4)},V^{(s)}\rangle=hK_{3s}\,.\end{aligned}$$

It follows from (4.18) and (4.19) that the system (4.16) can always be solved for b_1, b_2, b_3 and b_4. □

We denote by $V\{\hat{b}\}$ the solution of the problem (P_1) established in the proof of Theorem 3. Let Y denote the set of all vector fields U such that

$$(4.20)\qquad U=\int_0^{x_3}V\{\hat{c}\}dx_3+V\{\hat{d}\}+W_0\,,$$

where $\hat{c}=(c_1,c_2,c_3,c_4)$ and $\hat{d}=(d_1,d_2,d_3,d_4)$ are two constant four-dimensional vectors, and W_0 is a vector field independent of x_3.

In view of Theorem 2 we look for a solution of the flexure problem in the form (4.20).

THEOREM 4. *There exists a vector field $U_0\in Y$ such that $U_0\in K_{II}(F_1,F_2)$.*

PROOF. Let $U_0 \in Y$. To prove the statement, we have to determine the vectors $\hat{c}$ and $\hat{d}$ and the vector field $W_0 = (w_i^0, \psi_{jk}^0)$ such that $U_0 \in K_{II}(F_1, F_2)$. If $U_0 \in K_{II}(F_1, F_2)$, then by Theorem 2 we find that $V\{\hat{c}\} \in K_I(0, F_2, -F_1, 0)$. It follows from (4.16) that

$$\sum_{s=1}^{4} K_{\alpha s} c_s = -F_\alpha, \quad \sum_{s=1}^{4} K_{(2+\alpha)s} c_s = 0. \tag{4.21}$$

This system determines the constants c_1, c_2, c_3 and c_4. It follows from (4.12) and (4.20) that

$$\begin{aligned} u_\alpha^0 &= \varepsilon_{\alpha\beta}\Big(\frac{1}{6}c_\beta x_3^3 - \frac{1}{2}c_3 x_\beta x_3^2 + \frac{1}{2}d_\beta x_3^2 - d_3 x_\beta x_3\Big) + \\ &\quad + \sum_{s=1}^{4}(c_s x_3 + d_s) w_\alpha^{(s)} + w_\alpha^0, \\ u_3^0 &= \frac{1}{2}(\varepsilon_{\alpha\beta} c_\alpha x_\beta + c_4) x_3^2 + (\varepsilon_{\alpha\beta} d_\alpha x_\beta + d_4) x_3 + \\ &\quad + \sum_{s=1}^{4}(c_s x_3 + d_s) w_3^{(s)} + w_3^0, \\ \varphi_{ij}^0 &= \varepsilon_{ijs}\Big(\frac{1}{2}c_s x_3^2 + d_s x_3\Big) + \sum_{s=1}^{4}(c_s x_3 + d_s)\omega_{ij}^{(s)} + \psi_{ij}^0 . \end{aligned} \tag{4.22}$$

By (2.7), (4.14) and (4.22),

$$\begin{aligned} \tau_{ij}(U_0) &= \sum_{s=1}^{4}(c_s x_3 + d_s)\tau_{ij}(V^{(s)}) + \tau_{ij}(W_0) + g_{ij}, \\ \sigma_{ij}(U_0) &= \sum_{s=1}^{4}(c_s x_3 + d_s)\sigma_{ij}(V^{(s)}) + \sigma_{ij}(W_0) + h_{ij}, \\ \mu_{ijk}(U_0) &= \sum_{s=1}^{4}(c_s x_3 + d_s)\mu_{ijk}(V^{(s)}) + \mu_{ijk}(W_0) + m_{ijk}, \end{aligned} \tag{4.23}$$

where

$$g_{ij} = \sum_{s=1}^{4} c_s\left[(C_{ijk3} + G_{3kij})w_k^{(s)} + F_{3qrij}\omega_{qr}^{(s)}\right],$$

$$(4.24)\qquad h_{ij} = \sum_{s=1}^{4} c_s\left[(G_{ijk3} + B_{3kij})w_k^{(s)} + D_{ij3qr}\omega_{qr}^{(s)}\right],$$

$$m_{ijk} = \sum_{s=1}^{4} c_s\left[(F_{ijkr3} + D_{3rijk})w_r^{(s)} + A_{ijk3qr}\omega_{qr}^{(s)}\right].$$

The equations of equilibrium and the conditions on the lateral boundary reduce to

$$\left(\tau_{\alpha i}(W_0) + \sigma_{\alpha i}(W_0)\right)_{,\alpha} + f_i^* = 0,$$

$$(4.25)\qquad \left(\mu_{\alpha ij}(W_0)\right)_{,\alpha} + \sigma_{ij}(W_0) + L_{ij}^* = 0 \quad \text{on} \quad \Sigma,$$

$$\left(\tau_{\alpha i}(W_0) + \sigma_{\alpha i}(W_0)\right)n_\alpha = P_i^*, \quad \mu_{\alpha ij}(W_0)n_\alpha = Q_{ij}^* \text{ on } \Gamma,$$

where

$$f_i^* = g_{\alpha i,\alpha} + h_{\alpha i,\alpha} + \sum_{s=1}^{4} c_s\left[\tau_{3i}(V^{(s)}) + \sigma_{3i}(V^{(s)})\right],$$

$$(4.26)\qquad L_{ij}^* = m_{\alpha ij,\alpha} + h_{ij} + \sum_{s=1}^{4} c_s\mu_{3ij}(V^{(s)}),$$

$$P_i^* = -(g_{\alpha i} + h_{\alpha i})n_\alpha, \quad Q_{ij}^* = -m_{\alpha ij}n_\alpha.$$

In view of the divergence theorem and Theorem 2,

$$\int_\Sigma f_\alpha^* da + \int_\Gamma P_\alpha^* ds = -R_\alpha(V\{\hat{c}\}) = 0,$$

$$(4.27)\qquad \int_\Sigma f_3^* da + \int_\Gamma P_3^* ds = \sum_{s=1}^{4} K_{3s}c_s,$$

$$\int_\Sigma \varepsilon_{\alpha\beta}(x_\alpha f_\beta^* + L_{\alpha\beta}^*)da + \int_\Gamma \varepsilon_{\alpha\beta}(x_\alpha P_\beta^* + Q_{\alpha\beta}^*)ds = \sum_{s=1}^{4} K_{4s}c_s.$$

It follows from (4.21) and (4.27) that the necessary and sufficient conditions for the existence of a solution of the boundary-value problem

(4.25) are satisfied. Thus, the vector field W_0 is characterized by the generalized plane strain problem (4.25). In view of Theorem 2, (4.2), (4.23) and (4.17) we obtain

$$R_\alpha(U_0) = \varepsilon_{\beta\alpha} H_\beta(U_{0,3}) = -\sum_{s=1}^{4} K_{\alpha s} c_s . \tag{4.28}$$

It follows from (4.21) and (4.28) that $R_\alpha(U_0) = F_\alpha$. The conditions $R_3(U_0) = 0$, $\mathbf{H}(U_0) = \mathbf{0}$ reduce to

$$\sum_{s=1}^{4} K_{rs} d_s = C_r , \tag{4.29}$$

where

$$\begin{aligned} C_\alpha = &- \int_\Sigma [(\tau_{33}(W_0) + \sigma_{33}(W_0) + g_{33} + h_{33}) x_\alpha + \mu_{3\alpha 3}(W_0) + \\ &- \mu_{33\alpha}(W_0) + m_{3\alpha 3} - m_{33\alpha}] da , \\ C_3 = &- \int_\Sigma [\tau_{33}(W_0) + \sigma_{33}(W_0) + g_{33} + h_{33}] da , \\ C_4 = &- \int_\Sigma \varepsilon_{\alpha\beta} \{ x_\alpha [\tau_{3\beta}(W_0) + \sigma_{3\beta}(W_0) + g_{3\beta} + h_{3\beta}] + \\ &+ \mu_{3\alpha\beta}(W_0) + m_{3\alpha\beta} \} da . \end{aligned}$$

Thus, the constants d_1, d_2, d_3 and d_4 are determined by (4.29). □

We note that the above results can be used to study the problems of Almansi and Michell. TRUESDELL's problem [23-26] rephased for elastic bodies with microstructure can be also studied.

REFERENCES

[1] M.E. Gurtin: *The linear theory of elasticity*, In vol. a/2 of the Handbuch der Physik, edited by C. Truesdell, Springer-Verlag, Berlin, (1972).

[2] G. Fichera: *Remarks on Saint Venant's principle, Complex Analysis and its Applications*, I.N. Vekua Anniversary Volume, Moscow, (1978), 543-554.

[3] C.O. Horgan - J.K. Knowles: *Recent developments concerning Saint-Venant's principle*, Adv. Appl. Mech., **23** (1983), 179-269.

[4] G. Fichera: *Problemi analitici nuovi nella fisica matematica classica*, Quaderni del Consiglio Nazionale delle Ricerche, Gruppo Nazionale di Fisica Matematica, Scuola Tipo-Lito "Istituto Anselmi", Napoli, (1985).

[5] A. Mielke: *On Saint Venant's problem for an elastic strip*, Proc. Roy. Soc. Edinburgh, **110** A (1988), 161-181.

[6] C.O. Horgan: *Recent developments concerning Saint Venant's principle: An update*, Appl. Mech. Rev., **42** (1989), 295-303.

[7] G. Grioli: *On the equilibrium of a cylindrical elastic solid: Comparisons with Saint-Venant's theory*, Arch. Rational Mech. Anal., **105** (1989), 191-204.

[8] R.J. Knops - S. Rionero - L.E. Payne: *Saint Venant's principle on unbounded regions*, Proc. Roy. Soc. Edinburgh, **115** A (1990), 319-336.

[9] A. Clebsch: *Theorie der Elasticität fester Körper*, B.G. Teubner, Leipzig, (1862).

[10] W. Voigt: *Theoretische Studien über die Elasticitätsverhältnisse der Krystalle*, Abh. Ges. Wiss. Göttingen, **34** (1887), 53-153.

[11] E. Sternberg - J.K. Knowles: *Minimum energy characterizations of Saint Venant's problem*, Arch. Rational Mech. Anal., **21** (1966), 89-107.

[12] S.G. Lekhnitskii: *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day, Inc., San Francisco, (1963).

[13] D. Iesan: *On Saint Venant's problem*, Arch. Rational Mech., Anal., **91** (1986), 363-373.

[14] D. Iesan: *Saint-Venant's problem*, Lecture Notes in Mathematics, Vol. 1279, Springer-Verlag, Berlin, (1989).

[15] R.D. Mindlin: *Microstructure in linear elasticity*, Arch. Rational Mech. Anal., **16** (1964), 51-77.

[16] R.A. Toupin: *Theories of elasticity with couple-stress*, Arch. Rational Mech. Anal., **17** (1964), 85-112.

[17] A.E. Green - R.S. Rivlin: *Multipolar continuum mechanics*, Arch. Rational Mech. Anal., **17** (1964), 113-147.

[18] A.C. Eringen - E.S. Suhubi: *Nonlinear theory of simple microelastic solids*, Int. J. Engng. Sci., **2** (1964), I: 189-203; II: 389-404.

[19] A.E. GREEN: *Micro-materials and multipolar continuum mechanics*, Int. J. Engng. Sci., **3** (1965), 533-537.

[20] R. BATRA: *Saint-Venant's principle in linear elasticity with microstructure*, J. Elasticity, **13** (1983), 165-173.

[21] I. HLAVACEK – M. HLAVACEK: *On the existence and uniqueness of solution and some variational principles in linear theories of elasticity with couple-stresses*, Aplikace Matematiky, **14** (1969), 387-410.

[22] G. FICHERA: *Existence Theorems in Elasticity*, In vol. VI a/2 of the Handbuch der Physik, edited by C. Truesdell, Springer-Verlag, Berlin (1972).

[23] C. TRUESDELL: *The rational mechanics of materials-past, present, future*, Appl. Mech. Reviews, **12** (1959), 75–80.

[24] C. TRUESDELL: *The rational mechanics of materials-past, present, future*, (corrected and modified reprint of [14]), of Applied Mechanics Surveys, Spartan Books, (1966), 225-236.

[25] C. TRUESDELL: *Some challenges offered to analysis by rational thermomechanics*, in Contemporary Developments in Continuum Mechanics and Partial Differential Equations, G.M. de la Penha and L.A. Medeiros Eds., North-Holland, (1978), 495-603.

[26] C. TRUESDELL: *History of Classical Mechanics*, Die Naturwissenschaften 63, Part I: to 1800, 53–62; Part II: the 19th and 20th Centuries, Springer-Verlag, (1976), 119-130.

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