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Foliated differentiable spaces
Stability and quotient structure

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RIASSUNTO: *Si introducono nozioni molto generali di foliazioni $\mathcal{F}$ su spazi differenziabili X (ridotti o no) e si inizia, in questo ambito generale, lo sviluppo di una teoria delle foliazioni stabili, estendendo ampiamente e unificando risultati classici. Ad una foliazione $\mathcal{F}$ si associano lo spazio delle foglie $X/\mathcal{F}$ e due gruppi di olonomia (che coincidono nel caso classico). Per casi del tutto generali ma non troppo "selvaggi" si collega la stabilità di $\mathcal{F}$ con la finitezza dei gruppi di olonomia e con il fatto che lo spazio delle foglie $X/\mathcal{F}$ sia uno spazio differenziabile (avente in generale singolarità anche nel caso in cui X sia una varietà).*

ABSTRACT: *We introduce very general notions of foliations $\mathcal{F}$ on differentiable spaces X (reduced or not) and start to develop a theory of stable foliations in this general frame, extending largely and unifying classical results. To $\mathcal{F}$ there are associated the leaf space $X/\mathcal{F}$ and two holonomy groups (which coincide in classical cases). For quite general, but not too "wild" cases the stability of $\mathcal{F}$ is connected with the finiteness of the holonomy groups and with the leaf space $X/\mathcal{F}$ being a differentiable space (having in general singularities even for manifolds X).*

KEY WORDS: *Foliations – Singularities – Integral manifolds – Stability – Differentiable spaces and manifolds.*

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– Introduction

CH. EHRESMANN proposed in [3] some quite general theory of foliations. A rigorous development of such a theory however is not so easily carried out. One runs for example into problems of singularities, even if one - as up to now - starts with manifolds: The leafspace has in general singularities. So one should start right from the beginning with objects, which may include singularities, i.e. with differentiable spaces: reduced, but even non reduced ones [18]. This general level of foliated spaces has important impact also with other fields ([11], [1]).

We propose to develop a quite general theory of foliated spaces, which might have been already in the mind of Ch. Ehresmann. For the convenience of the reader, we fix our intention to the C^∞-case. But all works also for the C^ω-case (for example for subanalytic spaces) and the $C^{\omega*}$-case (especially for complex spaces). Our technics are those from differentiable spaces, which are well developed by now (after the times of Ehresmann). To avoid too lengthy and involved papers for the readers convenience, we proceed stepwise and discuss in this paper foliations only of *reduced* spaces into *manifolds*, however of (possibly) different dimensions, so called *coherent foliations* from [22] (extending [8], where all manifolds have the same dimension; extending [10], [2], where only manifolds are foliated, see also [12] with some other extension; extending [14], [15], where the foliations are obtained by Lie groups operating on spaces). Some other general type of foliations of (sometimes even only topological) spaces, where the leaves even may be spaces, is discussed in [17], [4], [5]. In another paper we extend our results to some most general situation (still with reasonable results), which will cover all different relevant cases as those mentioned above.

In 1-6 we establish the stability-theorem (main theorem) for coherent foliations (6.2). Besides other technics from the theory of spaces, especially notions and results about locally integrable vectorfields ([20], [22]) and coherent foliations ([9], [22]) are relevant. Also the technics in [8], developed for foliations of differentiable *spaces*, are important in our generalizing procedures. In our case of foliated *spaces*, two *different* holonomy groups (describing the neighbourhoods of the leaves) are involved right from the beginning: The analogue of the classical group, which we call "geometric" (due to its nature; see 5.6) and some bigger group, which we call formal (due to its more formal nature, see 2.4). Only for manifolds, foliated into manifolds, but all of them with the same dimension (classical case!), both groups always coincide ([8]; see [21] for the complicated possibilities in the case of space).

In 7 we first complement 1-6 by describing without proof some additional results to coherent foliations, especially their quotient structure (see 7.1). After this, we describe in 8 a far reaching extension of the notion of foliated spaces from 1-6 and indicate further results in this direction and some relations to different fields (for ex. [11]: mixed manifolds and supermanifolds, [1]: control theory).

We should mention, that our coherent foliations are (since quite recently) also called *Stefan foliations* in the special case of foliated manifolds, sometimes also *singular foliations*. But this is misleading, because we introduce in this paper what are really singular foliations, namely where the leaves themselves may have singularities. Our name "coherence" indicates, that along any leaf the geometry of the foliation does not really change. Especially, the leaves are then manifolds (of possibly different dimensions however). And some analogue of our formal holonomy-groups are called *transverse holonomy* groups in the case more complicated foliated *manifolds* (note however: *both* of our holonomy groups are in fact "transverse". So we prefer our more "intentional" names), some version also appeared in [5].

1 – Preliminaries

Let X be a reduced differentiable space ([18]). Assume that X is locally compact. We give some fundamental definitions and facts. For the convenience of the reader, we may suppose: $X \subset \mathbb{R}^n$ (embedded situation).

DEFINITION 1.1. ([22]) *A smooth foliation $\mathcal{F}$ of X is a family of connected manifolds $\{L_j; j \in J\}$ with 1-1-immersions $i : L_j \to X$, such that the following holds: $\bigcup_{j\in J} L_j = X$, $L_j \cap L_k = \emptyset$ for $j \neq k$. The manifolds L_j are called leaves. If $x \in X$, then the leaf containing x is denoted by L_x.*

The set-sheaf of all germs of locally integrable vector fields on X is denoted by $\mathcal{V}^i(X)$. $\mathcal{V}^i(X)$ is a C^N-sheaf ([19]).

DEFINITION 1.2. ([22]) *A distribution $\mathcal{V}$ on X is a C^N-subsheaf of $\mathcal{V}^i(X)$. $\mathcal{V}$ is called a Lie distribution if $\mathcal{V}$ is a C^N-sheaf of Lie algebras* *(i.e. $[\mathcal{V},\mathcal{V}] \subset \mathcal{V}$).*

DEFINITION 1.3. ([22]) *A distribution $\mathcal{V}$ is called integrable if, for each point $x \in X$, there is a manifold $M \subset X$ such that $x \in M$ and, for each $y \in M$, we have*

$$T_yM = \mathcal{V}(y) := \{v \in T_yX; (\exists V \in \mathcal{V})V(y) = v\}.$$

PROPOSITION 1.4. ([20]) *If $\mathcal{V}$ is an integrable distribution, then the local integral manifolds of $\mathcal{V}$ "stick together" to a smooth foliation $\mathcal{F}$ of X. We have*

$$(\forall L \in \mathcal{F})(\forall y \in L)T_yL = \mathcal{V}(y).$$

DEFINITION 1.5. ([22]) *A foliation induced by an integrable distribution $\mathcal{V}$ as in Proposition 1.4 is called coherent and denoted by $\mathcal{F}(\mathcal{V})$.*

Let σ_0 and T_0 be the natural differentiable structure and the natural topology on X, respectively. Let $\mathcal{F}$ be a coherent foliation of X. $\mathcal{F}$ induces on X some new differentiable structure $\sigma_{\mathcal{F}}$ and some new topology $T_{\mathcal{F}}$.

We have, by [22] and [9]:

PROPOSITION 1.6. *For a smooth foliation $\mathcal{F} = \{L_j; j \in J\}$, the following conditions are equivalent:*

i) *$\mathcal{F}$ is coherent,*
ii) *for any $L \in \mathcal{F}$, $x \in L$ and $v \in T_xL$, there exists in a neighbourhood U of x a vector field V such that $V(x) = v$ and V is tangent to leaves of $\mathcal{F}$ in each point of U,*
iii) *for each $x \in X$, if $x \in L_{j_0}$, $r = \dim L_{j_0}$, there exist $U \in T_0$ with $x \in U$, an open connected neighbourhood $W \subset \mathbb{R}^r$ of w, a set $A \subset \mathbb{R}^{n-r}$, $0 \in A$, and a diffeomorphism $\varphi : U \to A \times W$, such that*
   a) *$\varphi(x) = (0, w)$,*
   b) *for each $L' \in \mathcal{F}$ we have $\varphi(L' \cap U) = S' \times W$, where $S' := \{a \in A; \varphi^{-1}(a, w) \in L'\}$,*
   c) *if $\dim L' = r$, then S' is countable,*
   d) *if $\dim L' = r$, then $\varphi|U \cap L' : (U \cap L', \sigma_{\mathcal{F}}) \to (S' \times W, \sigma_d \times \sigma_0)$ is a diffeomorphism, where σ_d is the differentiable structure of the*

0-dimensional manifold of A and σ_0 is the natural differentiable structure of W,

e) *for each $a \in S'$ and for $\dim L' = r$, the set $\varphi^{-1}(\{a\} \times W)$ is a connected component of $U \cap L'$ in $T_{\mathcal{F}}$ and in T_0, as well.*

DEFINITION 1.7. *$(U, \varphi, A \times W)$ described in Proposition 1.6* iii) *is called an adapted chart of $\mathcal{F}$ around x. The set A is called a transversal of the adapted chart. Each connected component of the set $U \cap L'$ is called a plaque of U in L'. From* e) *it follows that $\varphi^{-1}(\{0\} \times W)$ is a plaque which is called the central plaque of U.*

Denote by $\mathcal{F}|U$ the foliation of U whose leaves are plaques of U. Proposition 1.6 easily implies the following.

PROPOSITION 1.8. *If $(U, \varphi, A \times W)$ is an adapted chart around x of $\mathcal{F}$, then the decomposition of A into connected components of the sets S' described in Proposition 1.6 iii) is a coherent foliation $\mathcal{F}_A$ of A. Moreover, $\varphi : (U, \sigma_{\mathcal{F}|U}) \to (A \times W, \sigma_{\mathcal{F}_A} \times \sigma_0)$ is a diffeomorphism.*

Denote by ρ, ρ_U and ρ_A the equivalence relations induced, respectively, by $\mathcal{F}, \mathcal{F}|U$ and $\mathcal{F}_A$, and let π, π_U and and π_A be the respective canonical projections. By Corollary II., 2.12 from [9], we have

PROPOSITION 1.9. *The equivalence relations ρ, ρ_U and ρ_A are open.*

It is clear that the spaces $A/_{\rho_A}$ and $U/_{\rho_U}$ are homeomorphic. Denote this homeomorphism by Λ_U. From [9] we have

PROPOSITION 1.10. *Let $\mathcal{F}$ be a coherent foliation of X and let $L \in \mathcal{F}$. For any $x, x' \in L$, there are open neighbourhoods $U(x)$, $U(x')$ and a diffeomorphism $\psi : U(x) \to U(x')$, such that*

i) $\psi(x) = x'$,

ii) $z \in L' \iff \psi(z) \in L' \quad \forall z \in U(x) \quad \forall L' \in \mathcal{F}$,

iii) *ψ is a diffeomorphism relative to $\sigma_{\mathcal{F}}$.*

Let $L \in \mathcal{F}$.

DEFINITION 1.11. *L is called stable if there is a basis of ρ-saturated neighbourhoods of L. L is called transversely stable if, for some adapted chart $(U, \varphi, A \times W)$ around some $x \in L$, the leaf $\{0\}$ of the foliation $\mathcal{F}_A$ is stable.*

By Proposition 1.10, the above definition does not depend on the choice of an adapted chart. In a simple way, using Proposition 1.8, we obtain the following

PROPOSITION 1.12. *Let L be a transversely stable leaf of $\mathcal{F}$ and $(U, \varphi, A \times W)$ - an adapted chart around $x \in L$, $\varphi(x) = (0, w)$. Set $O := \varphi^{-1}(C \times V)$ with C being a ρ_A-saturated open subset of A, $\overline{C} \subset A$, V - an open connected neighbourhood of w in W. such that $\overline{V} \subset W$, $x \in O$ and $\overline{O} \subset U$. Then $(O, \varphi|O, C \times V)$ is an adapted chart around x and, for each plaque $P = \varphi^{-1}(S \times V)$ of O, the equality*

$$\overline{P} = \varphi^{-1}(S \times \overline{V})$$

holds. Here the closure $\overline{P}$ is taken with respect to $T_{\mathcal{F}}$.

By using the openness of the equivalence relation induced by a coherent foliation, it is easy to show the following properties satisfied by the coherent foliation near a transversely stable leaf:

PROPOSITION 1.13. *Let L be a transversely stable leaf of $\mathcal{F}$, $x \in L$, and let $(U, \varphi, A \times W)$ be an adapted chart around x. Let Ω be a ρ_A-saturated neighbourhood of 0 in A and V - a connected neighbourhood of w in W. Then the mapping*

$$i_{\widehat{U},U} : \widehat{U}/_{\rho_{\widehat{U}}} \ni [x']_{\rho_{\widehat{U}}} \longmapsto [x']_{\rho_U} \in U/_{\rho_U}$$

with $\widehat{U} := \varphi^{-1}(\Omega \times V)$ is a homeomorphism onto an open neighbourhood of $[x]$ in $U/_{\rho_U}$.

By the above proposition, the set $\Omega/_{\rho_\Omega}$ can be considered as an open neighbourhood of $[0]_{\rho_A}$ in $A/_{\rho_A}$ or equivalently, the set $\widehat{U}/_{\rho_{\widehat{U}}}$ - as an open neighbourhood of $[x]_{\rho_U}$ in $U/_{\rho_U}$.

Definition 1.14. *A leaf $L \in \mathcal{F}$ is called proper if $T_0|L = T_{\mathcal{F}}|L$, i.e., for each $x \in L$, there is an adapted chart $(U, \varphi, A \times W)$ around x with $U \cap L = \varphi^{-1}(\{0\} \times W)$.*

By the methods used in [10], we obtain

Proposition 1.15. *Each closed leaf $L \in \mathcal{F}$ is proper.*

2 – Formal holonomy group

Let L be a transversely stable leaf of $\mathcal{F}$. Now, we construct a formal holonomy group of L.

Definition 2.1. *Let $(U, \varphi, A \times W)$ be an adapted chart around $x \in L$. A homeomorphism $h : \Gamma \to \tilde{\Gamma}$ of open neighbourhoods of $[x]$ in $U/_{\rho_U}$ is called $\mathcal{F}$-faithful at $[x]$ if and only if*
i) $h([x]) = [x]$,
ii) $(\forall c \in \Gamma)(\exists L' \in \mathcal{F})\pi_U^{-1}(c), \pi_U^{-1}(h(c))$ *are plaques of U in L'.*

Let $H(L)$ denote the set of all germs $[h]_{[x]}$ of $\mathcal{F}$-faithful homeomorphisms at $[x]$.

Remark 2.2. Condition i) gives a possibility to compose representative elements of two germs and the composition has a germ in $H(L)$ which does not depend on the choice of the respective elements. In such a way we get a structure of a group in $H(L)$.

We want to prove that the isomorphism class of $H(L)$ depends only on the leaf L.

Choose $x' \in L$ and an adapted chart $(U', \varphi', A' \times W')$ around x'. With these data, we construct the group $H'(L)$ of all germs $[h']_{[x']}$ of $\mathcal{F}$-faithful homeomorphisms h' of neighbourhoods $\Gamma', \tilde{\Gamma}'$ of $[x']$ in $U'/_{\rho_{U'}}$.

Proposition 2.3. *$H(L)$ and $H'(L)$ are isomorphic.*

PROOF. By Proposition 1.10, we have a diffeomorphism $\psi : U(x) \to U(x')$ which maps x into x' and is compatible with the foliation. We can restrict the charts $(U, \varphi, A\times W)$, $(U', \varphi', A'\times W')$ to charts $(\widehat{U}, \widehat{\varphi}, \widehat{A}\times\widehat{W})$, $(\widehat{U}', \widehat{\varphi}', \widehat{A}'\times\widehat{W}')$ so that $\widehat{U} \subset U(x), \widehat{U}' \subset U(x')$, and $\widehat{A}$, $\widehat{A}'$ are ρ_A - and $\rho_{A'}$ - saturated, respectively.

The diffeomorphism ψ induces the homeomorphism

$$\widehat{\psi} : \widehat{U}/_{\rho_{\widehat{U}}} \to \widehat{U}'/_{\rho_{\widehat{U}'}}$$

with the properties that

i) $\widehat{\psi}([x]_{\rho_{\widehat{U}}}) = [x']_{\rho_{\widehat{U}'}}$,

ii) $(\forall c \in \widehat{U}/_{\rho_{\widehat{U}}})(\exists L' \in \mathcal{F})\pi_{\widehat{U}}^{-1}(c)$, $\pi_{\widehat{U}'}^{-1}(\hat{\psi}(c))$ are plaques in L'.

□

Define

$$H(L) \ni [h]_{[x]} \longmapsto [\widehat{\psi} \circ h \circ \widehat{\psi}^{-1}]_{[x']} \in H'(L)\,.$$

It is easy to see that this mapping is an isomorphism of the groups.

DEFINITION 2.4. *The group $H(L)$ is called formal holonomy group of L.*

3 – Technical coverings

We extend some notions and results from [8] and first we have the following topological results:

LEMMA 3.1. *Let M be a paracompact topological space and $\{U_i\}_{i\in j}$ - an open covering of M. Then there exists a locally finite open refinement $\{Y_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ with a refinement mapping $\tau : J \to I$, such that if $\overline{Y}_r \cap \overline{Y}_s \neq \emptyset$, then $\overline{Y}_r \cup \overline{Y}_s \subset U_{\tau(r)} \cap U_{\tau(s)}$.*

For the proof see for example [8].

DEFINITION 3.2. *$\{Y_j\}_{j\in J}$ is called a star-refinement of $\{U_i\}_{i\in I}$.*

Let $\mathcal{F}$ be a coherent foliation of X and $L \in \mathcal{F}$ be a closed transversely stable leaf.

LEMMA 3.3. *Let $\{(U_i, \varphi_i, A_i \times W_i); i \in I\}$ be a family of adapted charts around points of L, where $\overline{U}_i$ are compact and $\bigcup_{i \in I} P_i = L$ with P_i being the central plaque of U_i. Then there exist a family $\{O_j, \psi_j, C_j \times V_j); j \in J\}$ of adapted charts around points of L and a refinement mapping $\tau : J \longrightarrow I$, such that*

i) *J is countable,*

ii) *$\{Q_j\}_{j \in J}$ is a locally finite $T_{\mathcal{F}}$-open covering of L, where Q_j is the central plaque of O_j,*

iii) $(\forall r, s \in J)\overline{O}_r \cap \overline{O}_s \neq \emptyset \implies \overline{O}_r \cup \overline{O}_s \subset U_{\tau(r)} \cap U_{\tau(s)}$,

iv) $(\forall j \in J)\overline{O}_j$ *is compact,* $\psi_j = \varphi_{\tau(j)}|O_j$,

v) $(\forall j \in J)C_j$ *is a $\rho_{A_{\tau(j)}}$-saturated open subset of $A_{\tau(j)}$ and $\overline{C}_j$ is compact.*

Moreover, it follows from ii)-v) *that:*

vi) *if $Q_r \cap Q_s \neq \emptyset$, then,*
for open $\Gamma_{rs} := \pi_{O_r}(O_r \cap O_s) \subset O_r/\rho_{O_r} \subset U_{\tau(r)}/\rho_{U_{\tau(r)}}$, there exists a (uniquely defined) homeomorphism

$$h_{rs} : \Gamma_{rs} \rightarrow \Gamma_{sr}$$

such that

$$h_{rs} \circ \pi_{O_r}|O_r \cap O_s = \pi_{O_s}|O_r \cap O_s,$$

and

$$(\forall c \in \Gamma_{rs})\pi_{O_r}^{-1}(c), \pi_{O_s}^{-1}(h_{rs}(c))$$

are plaques of the same leaf. Additionally, $(h_{rs})^{-1} = h_{sr}$.

PROOF. By Proposition 1.15, we can assume that $U_i \cap L = P_i$. Let $\{G_t\}_{t \in T}$ be a family of open sets such, that $\{U_i\} \cup \{G_t\}$ is a covering of X and $G_t \cap L = \emptyset$ for $t \in T$. To the covering of X obtained, we can apply Lemma 3.1 (X is paracompact, since it is locally compact and has a countable basis ([13])). From the star-refinement we choose the family $\{Y_k\}_{k \in K}$ with $Y_k \cap L \neq \emptyset$. If σ is a refinement mapping, then, for each $k \in K$, $\sigma(k) \in I$. For each $k \in K$, the set $\overline{Y}_k$ is compact. Since X is paracompact, thus normal ([13]), we can choose a covering $\{Z_k\}$

of L with Z_k being relatively compact in Y_k and $Z_k \cap L \neq \emptyset$. For each $k \in K$, the compact subset $\overline{Z}_k \cap L$ of Y_k can be covered by a finite family $\{(O_{(k,t)}, \varphi_{(k,t)}, A_{(k,t)} \times W_{(k,t)}\}_{t=1}^{n(k)}$ of adapted charts around points of L. We can assume that $\varphi_{(k,t)}$ is a restriction of $\varphi_{\sigma(k)}$, $A_{(k,t)}$ is relatively compact and $\rho_{A_{\sigma(k)}}$-saturated in $A_{\sigma(k)}$, and $\overline{W}_{(k,t)}$ is compact. Let $J := \bigcup_{k \in K} M_k$ where $M_k := \{(k,t); 1 \leq t \leq n(k)\}$ and let $\tau : J \ni (k,t) \longmapsto \sigma(k) \in I$ and $\psi_{(k,t)} = \varphi_{\sigma(k)} | O_{(k,t)}$.

Property i) follows from the existence of a countable basis. For the proof of ii), we use the local finiteness of the star-refinement of the covering $\{U_i\} \cup \{G_t\}$ and the finiteness of M_k. Properties iii)-v) are obvious by construction.

To prove vi), assume that $O_r \cap O_s \neq \emptyset$. Then $\Gamma_{rs} := \pi_{O_r}(O_r \cap O_s) \neq \emptyset$. By Propositions 1.9 and 1.13, Γ_{rs} is open in $U_{\tau(r)}/\rho_{U_{\tau(r)}}$. In view of iii), we have $O_r \cup O_s \subset U_{\tau(r)} \cap U_{\tau(s)}$. For $c \in \Gamma_{rs}, \pi_{O_r}^{-1}(c) \cap O_s \neq \emptyset$. By the $T_{\mathcal{F}}$-connectednes of $\pi_{O_r}^{-1}(c)$, it is contained in the unique plaque of $U_{\tau(s)}$. Thus $h_{rs}(c) := \pi_{U_{\tau(s)}} \pi_r^{-1}(c)$ is a correctly defined mapping with values in Γ_{sr}. By iv), we have

$$h_{rs} \circ \pi_{O_r} | O_r \cap O_s = \pi_{O_s} | O_r \cap O_s .$$

It is easy to see that the above equality determines the mapping h_{rs} uniquely. It is obvious that $\pi_{O_r}^{-1}(c)$ and $\pi_{O_s}^{-1}(h_{rs}(c))$ lie in the same leaf since they lie in the same plaque of $U_{\tau(s)}$.

Finally, h_{rs} is a homeomorphism by Proposition 1.9. □

PROPOSITION 3.4. *Let L be a closed transversely stable leaf and, for $x_0 \in L$, let $(U, \varphi, A \times D)$ be a fixed adapted chart around x_0, where D is the open disc in $\mathbb{R}^r$ with centre 0 and radius 1, and $\varphi(x_0) = (0,0)$.*
i) Then, for each $x \in L$, there is an adapted chart $(U_x, \varphi_x, A_x \times D_x)$ around $x \in U_x, U_{x_0} = U$, such that

- a) $\varphi_x(L \cap U_k) = \{0\} \times D_x$,
- b) $\varphi_x(x) = (0,0) \in A_x \times D_x, A_x \times D_x$ *is open in* $A \times D$ *and* A_x *is* ρ_A*-saturated,*
- c) *for each* $\zeta \in (A_x/\rho_{A_x}) \cap (A_y/\rho_{A_y}) \subset A/\rho_A$, $\pi_{U_x}^{-1} \Lambda_{U_x}(\zeta)$, $\pi_{U_y}^{-1} \Lambda_{U_y}(\zeta)$ *lie in the same leaf of* $\mathcal{F}$.

Moreover, we can assume that, for each $x \in L$, the set D_x is the open disc with centre 0 and radius 1.

ii) *If L is compact, then there are $x_1, \dots, x_n \in L$ with the properties*

α) $\bigcup_{i=1}^{n} U_{x_i} \supset L$,

β) $A_{x_i} = A_{x_j} =: C(\forall i, j)$,

γ) a), b) *and* c) *from* i) *hold.*

iii) *There exists a family $\{(O_j, \psi_j, C_j \times V_j); j \in J\}$ of adapted charts for the family $\{(U_x, \varphi_x, A_x \times D_x); x \in L\}$ as in Lemma 3.3, such that*

a) $\bigcup_{j \in J} O_j \supset L$,

b) $\psi_j(L \cap O_j) = \{0\} \times V_j (\forall j \in J)$,

c) *there is a mapping $\tau : J \to L$ such that O_j is relatively compact in $U_{\tau(j)}$, C_j is an open relatively compact and $\rho_{A_{\tau(i)}}$-saturated neighbourhoods of* 0 *in $A_{\tau(j)}$, V_j-relatively compact in $D_{\tau(j)}$ and $\psi_j = \varphi_{\tau(j)}|O_j$,*

d) vi) *from Lemma 3.3 holds.*

Moreover, we can assume that V_j is an open disc with centre 0 *and radius* $0 < \varepsilon_j < 1$. *If L is compact, then we can assume that* card $J < \infty$.

PROOF. Choose an adapted chart $(U, \varphi, A \times D)$ around x_0 with $L \cap U = \varphi^{-1}(\{0\} \times D)$ and $\varphi(x_0) = (0, 0)$, where D is the open disc with centre 0 and radius 1.

To prove i), take the neighbourhoods $U(x)$, $U(x_0)$ and the diffeomorphism $\psi : U(x) \to U(x_0)$ as in Proposition 1.10. Choose a ρ_A-saturated open neighbourhood A_x of 0, such that $\varphi^{-1}(A_x \times D_x) \subset U(x_0)$. Define

$$U_x := \psi^{-1}\varphi^{-1}(A_x \times D_x), \varphi_x := \varphi \circ \psi .$$

We obtain the adapted chart $(U_x, \varphi_x, A_x \times D_x)$ which fulfils a)-c). We can apply the homothety in the last variables so that D_x be the disc with radius 1.

To prove ii), choose $x_1, .., x_n \in L$ with α) and set $C := \bigcap_{i=1}^{n} A_{x_i}$. The charts obtained by the restriction fulfil ii).

Finally, iii) is obvious. □

DEFINITION 3.5. *A covering $\{O_j\}$ of L as in Proposition 3.4 is called a technical covering of L.*

PROPOSITION 3.6. *Let L be a closed transversely stable leaf of $\mathcal{F}$. Then there is a covering $\{U_i; i \in I\}$ of L by charts such that, for $i \in I$, the set $U_i \cap L$ is one plaque of U_i. For this covering, there exists a technical covering $\{(O_j, \psi_j, C_j \times V_j); j \in J\}$ of L as in Proposition 3.4* iii) *such that*

α) $(0,0) \in C_j \times V_j (\forall j \in J)$,

β) for each $j \in J$, V_j is the open disc in $\mathbb{R}^r$ with centre 0 *and radius* 1.

Fully analogous considerations as in [8] prove the assertion of the above proposition and the assertion of the following.

PROPOSITION 3.7. *Let L be a compact transversely stable leaf and $s \in \mathbb{N}$ - a fixed number. Then there exists a technical covering $\{(O_j, \psi_j, C_j \times V_j); j \in J\}$, $J = \{1, .., m\}$, of L with*

$$(\forall j_1, .., j_s \in \{1, .., m\}) O_{j1} \cap \ldots \cap O_{j_s} \neq \emptyset \implies O_{j1} \cap \ldots \cap O_{j_s} \cap L \neq \emptyset$$

DEFINITION 3.8. *The technical covering of L which fulfils the condition of Proposition 3.7 is called a technical covering of power s with respect to L.*

4 – Chains

DEFINITION 4.1. *Let $O_1, .., O_r$ be domains of adapted charts and let $L \in \mathcal{F}$.*

i) *$O_1, .., O_r$ is called a chain on L of length r with a base x if*

a) *for each $1 \le t \le r$, there is a plaque P_t of O_t in L,*

b) $x \in P_1$,

c) *for each $1 \le t \le r - 1$, $P_t \cap P_{t+1} \neq \emptyset$.*

ii) *Let $P_1, ., P_r$ be plaques of $O_1, ., O_r$ respectively. The sequence $P_r, ., P_r$ is called a chain of plaques of length r with a base x if*

a) $x \in P_1$,

b) *for each $1 \le t \le r - 1$, $P_t \cap P_{t+1} \neq \emptyset$.*

iii) *The chain $O_1,..,O_r$ (resp. $P_1,.,P_r$) is called simple if*

$$O_t \cap O_s \neq \emptyset \quad (\textit{resp. } P_t \cap P_s \neq \emptyset) \iff |t-s| \leq 1.$$

From the above definition we easily get

REMARK 4.2. Let $P_1,..,P_r$ be a chain of plaques with a base x, P_i being a plaque of a domain O_i of an adapted chart around a point of a fixed leaf L. Then $O_1,..,O_r$ is a chain on L_x with a base x.
Exactly as in [8] we can show the following

REMARK 4.3. Let $O_1,..,O_r$ be domains of adapted charts around points of L. Let $O_1,..,O_r$ be a chain on L with a base x. Then there is an open ρ_{O_1}-saturated set $S \subset O_1$, $x \in S$, such that $O_1,..,O_r$ is a chain on L_y with a base y for each $y \in S$.
We now have

PROPOSITION 4.4. *Let L be a compact transversely stable leaf of $\mathcal{F}$, let $\{(O_k, \psi_k, C_k \times V_k); k \in K\}$ with $K = \{1,..,r\}$ be a technical covering of L of power 2 with respect to L, $x \in L$ (we can assume that $x \in O_1$) and $M \in \mathbb{N}$ - a fixed number. Then there exists an open ρ_{O_1}-saturated neighbourhood $S \subset O_1$ of x such that, for each chain $Q_{i_1},..,Q_{i_n}$ of plaques of $O_{i_1},..,O_{i_n} \in \{O_1,..,O_r\}$ of length $\leq M$ and a base in S, the inclusion*

$$\overline{Q}_{i_t} \subset O_1 \cup \ldots \cup O_r =: O, \; 1 \leq t \leq s,$$

holds.

PROOF. Let P_i be a plaque of O_i in L, $1 \leq i \leq r$. We first prove the following assertion:

1) For each $y \in P_i$, there exists an open neighbourhood $V(y)$ of y in O_i such that, for each $z \in V(y)$, if P is the plaque of O_i through z, then $\overline{P} \subset O$.

Assume that this assertion is false. Thus there are $y \in P_i$ and a sequence (z_n) of points in O_i with $\lim_{n\to\infty} z_n = y$, such that if P_n is the plaque of O_i through z_n, then $\overline{P}_n$ is not contained in O. Therefore, for each $n \in \mathbb{N}$, there exists $r_n \in \overline{P}_n$ such that $r_n \notin O$. Remark that r_n are contained in the compact set $\overline{O}_i \setminus O \subset U_{\tau(i)}$ since $\overline{P}_n \subset \overline{O}_i$. thus we can assume $\lim_{n\to\infty} r_n = r \in \overline{O}_i \setminus O \subset U_{\tau(i)}$. We now show that

$$*) \qquad \mathrm{pr}_1\, \varphi_{\tau(i)}(r) = \lim_{n\to\infty} \mathrm{pr}_1\, \varphi_{\tau(i)}(r_n) = 0\,.$$

Assume that $\lim_{n\to\infty} \mathrm{pr}_1\, \varphi_{\tau(i)}(r_n) \neq 0$. Then there exists a $\rho_{A_{\tau(i)}}$-saturated neighbourhood G of 0 in $A_{\tau(i)}$ with $\mathrm{pr}_1\, \varphi_{\tau(i)}(r_n) \notin G$ for an infinite number of $n \in \mathbb{N}$. Thus $\mathrm{pr}_1\, \varphi_{\tau(i)}(z_n) \notin G$ for an infinite number of $n \in \mathbb{N}$ since r_n and z_n lie in the same plaque of $U_{\tau(i)}$, and G is $\rho_{A_{\tau(i)}}$-saturated. Therefore $\lim_{n\to\infty} z_n \neq y$. The contradiction obtained proves *).

It follows from *) that r lies in the plaque of $U_{\tau(i)}$ through y, so $r \in L \subset O$, which contradicts $r \in \overline{O}_i \setminus O$. Thus 1) holds.

Choose now $y \in P_i$ and $V(y)$ as in 1). Then $S_j := \pi_{O_i}^{-1}\pi_{O_i}(V(y))$ is open in O_i, and $P_i \subset S_i$. Denote by P_z the plaque of O_i through z. Then, for $z \in S_i$, we have $\overline{P}_z \subset O$.

We now prove the following assertion:

2) Each chain $Q_{i_1}, .., Q_{i_n}$ of plaques of $O_1, .., O_r$ of length $\leq M$ induces a chain $O_{i_1}, .., O_{i_n}$ on L of length $\leq M$.

Since $\{O_1, .., O_r\}$ is of power 2 with respect to L, we have the following sequence of implications:

$$\begin{aligned} Q_{i_t} \cap Q_{i_{t+1}} \neq \emptyset \implies O_{i_t} \cap O_{i_{t+1}} \neq \emptyset \implies O_{i_t} \cap O_{i_{t+1}} \cap L \neq \emptyset \\ \implies P_{i_t} \cap P_{i_{t+1}} \neq \emptyset, 1 \leq t \leq n-1\,, \end{aligned}$$

which implies 2).

Moreover, we have:

3) For each chain $O_{i_1}, .., O_{i_n}$ on L, there exists an open $\rho_{O_{i_1}}$-saturated set $S(i_1, .., i_n) \subset O_{i_1}$ containing P_{i_1} such that, for each chain of plaques $Q_{i_1}, .., Q_{i_n}$ with $Q_{i_1} \subset S(i_1, .., i_n)$, the inclusion $\overline{Q}_{i_t} \subset O$ for $1 \leq t \leq n$ holds.

Indeed, let $Y_{n-1} := \pi_{O_{i_{n-1}}}^{-1}\pi_{O_{i_{n-1}}}(S_{i_{n-1}} \cap S_{i_n})$. Define inductively $Y_{n-t} := \pi_{O_{i_{n-t}}}^{-1}\pi_{O_{i_{n-t}}}(Y_{n-t+1} \cap S_{i_{n-t}})$ for $2 \leq t \leq n-1$. Then Y_{n-t} is open in $S_{i_{n-t}}$, and $P_{i_t} \subset Y_t$. The set $S(i_1, .., i_n) := Y_1$ has properties asserted in 3).

If $Q_{i_1} \cap Q_{i_2} \neq \emptyset$, then there is a plaque P of $U_{\tau(i_2)}$ with $P \cap O_{i_2} = Q_{i_2}$ since O_{i_2} has a $\rho_{A_{\tau(i_2)}}$- saturated transversal.

Moreover, we have $Q_{i_1} \cap Q_{i_2} \subset P$. Since $Q_{i_1} \subset Y_1$, the set $Q_{i_1} \cap Y_2$ is nonempty, and $\emptyset \neq P \cap Y_2 = P \cap Q_{i_2} \cap Y_2 = Q_{i_2} \cap Y_2$. Thus $Q_{i_2} \subset Y_2$ since Y_2 is $\rho_{O_{i_2}}$-saturated in O_{i_2}.

Inductively, we get in this manner $Q_{i_t} \subset S_{i_t}$, $1 \leq t \leq n$. Therefore $\overline{Q}_{i_t} \subset O$.

By 2), each chain $Q_{i_1}, .., Q_{i_n}$ of plaques induces a chain on L. Since $\{i_1, .., i_n\} \subset \{1, .., r\}$, there are only a finite number of chains on L of length $\leq M$ which begin with i_1, say, k chains. For each of these k chains on L, there is a set as in 3). The intersection of these sets is the required set S. □

5 – Geometric holonomy group

We define as in [8] the notions of a *pseudogroup of homeomorphisms* of a topological space Y, a *subpseudogroup* of a pseudogroup, a *system of generators* of a pseudogroup, a *symmetric system of generators* and a *finitely* and *countably generated pseudogroup*.

Let $\{(O_j, \psi_j, C_j \times V_j); j \in J\}$ be a technical covering of L for the family $\{(U_i, \varphi_i, A_i \times W_i); i \in I\}$ with a refinement mapping $\tau : J \to I$.

Let $Y := \sum_{j \in J} O_j/\rho_{O_j}$ denote the disjoint union of the spaces O_j/ρ_{O_j}.

Let $\tilde{\Pi}$ denote the set of all homeomorphisms $h_{rs} : \Gamma_{rs} \to \Gamma_{sr}$ from Lemma 3.3 vi). Let Π be the pseudogroup of local homeomorphisms of Y generated by $\tilde{\Pi}$, i.e. an element of Π is a finite composition of elements from $\tilde{\Pi}$. Then $\tilde{\Pi}$ is a symmetric and countable system of generators of Π.

For $a \in Y$ with $\pi_{O_j}^{-1}(a) \subset L$, j being a suitable index, let $\Pi_a = \{h \in \Pi; h(a) = a\}$ be the isotropy pseudogroup of Π at a. The set of all germs of elements from Π_a at the point a will be denoted by $G(a)$.

Lemma 3.3 vi) easily implies:

REMARK 5.1. Let $h \in \Pi_a$ with the domain $D(h)$ and the range $W(h)$, contained in O_j/ρ_{O_j}. Then, for each $y \in D(h)$, there exists $L' \in \mathcal{F}$ such that $\pi_{O_j}^{-1}(y), \pi_{O_j}^{-1}(h(y)) \subset L'$.

As a corollary we obtain.

REMARK 5.2. $G(a)$ is a subgroup of $H(L)$.

We need the following.

LEMMA 5.3. *Let $(U_1, \varphi_1, A_1 \times W_1), (U_2, \varphi_2, A_2 \times W_2)$, be adapted charts around points of L and let $x \in U_1 \cap U_2 \cap L$. Then there are open neighbourhoods Γ_1, Γ_2 of $[x]$ in U_1/ρ_{U_1}, U_2/ρ_{U_2} respectively, and a homeomorphism $f : \Gamma_1 \to \Gamma_2$, satisfying the following condition:*

$$(\forall \zeta \in \Gamma_1)(\exists y \in U_1 \cap U_2)\pi_{U_1}^{-1}(\zeta) \subset L_y \iff \pi_{U_2}^{-1}(f(\zeta)) \subset L_y .$$

PROOF. We can restrict both charts to relatively compact neighbourhoods of x which are contained in $U_1 \cap U_2$ and have ρ_{A_1}-saturated and ρ_{A_2}-saturated transversals, respectively. Then we get a similar situation as in vi) of Lemma 3.3 and f is defined in the same way as h_{rs}.

We now show that the isomorphism class of $G(a)$ depends only on the plaque P which represents a.

Let $\{(O'_k, \psi'_k, C'_k \times V'_k); k \in K\}$ be another technical covering of L (for some covering $\{(U'_r, \varphi'_r, A'_r \times W'_r); r \in R\}$), $Y' := \sum_{k \in K} O'_k/\rho'_{O_k}$, $\widetilde{\Pi}'$ - the countable symmetric set of homeomorphisms $h'_{rs} : \Gamma'_{rs} \to \Gamma'_{sr}$ as in Lemma 3.3 and Π' - the pseudogroup generated by $\widetilde{\Pi}'$.

We first consider the case when $\{O'_k\}$ is a topological refinement of $\{O_j\}$. Let $\lambda : K \to J$ be a refinement mapping. Let $h_{rs} \in \widetilde{\Pi}$, $h_{rs} : O_r/\rho_{O_r} \supset \Gamma_{rs} \to \Gamma_{sr} \subset O_s/\rho_{O_s}$ with $h_{rs}(b) = c$ where b is a fixed base point defining a plaque of O_r in L. Choosing $x \in O_r \cap O_s$ with $\tau_{O_r}(x) = b$, let P be the plaque of O_s containing x and let $y \in P$. We can find a simple chain P_{k_i} of plaques of O'_{k_i}, $0 \le i \le v$ joining x to y and such that $P_{k_i} \cap P \neq \emptyset$, $i = 0, .., v$. Set $\hat{b} := \pi_{O'_{k_0}}(x) \in O'_{k_0}/\rho_{O'_{k_0}}$, $\hat{c} := \pi_{O'_{k_v}}$, $(y) \in O'_{k_v}/\rho_{O'_{k_v}}$. By Lemma 5.3, we have homeomorphisms

$$f_{k_0 r} : O'_{k_0}/\rho_{O'_{k_0}} \supset \Gamma_{k_0} \to \Gamma_r \subset O_r/\rho_{O_r}, f_{s k_v} : O_s/\rho_{O_s} \supset \Gamma_s \to \Gamma_{k_v} \subset O'_{k_v}/\rho_{O'_{k_v}}$$

such that $\hat{b} \in \Gamma_{k_0}$, $f_{k_0 r}(\hat{b}) = b$, $c \in \Gamma_s$, $f_{s k_v}(c) = \hat{c}$ and they "map plaques of a leaf into plaques" of the same leaf.

Let $Y_{v-1} := \pi_{O'_{k_{v-1}}}^{-1} \pi_{O'_{k_{v-1}}}(O'_{k_v} \cap O'_{k_{v-1}})$. For $2 \le t \le v$, we define inductively $Y_{v-t} := \pi_{O'_{k_{v-t}}}^{-1} \pi_{O'_{k_{v-t}}}(Y_{v-t+1} \cap O'_{k_{v-t}})$.

Then $x \in Y_0$, and Y_0 is open in O'_{k_0}.

For an arbitrary point $z \in Y_0$, we have a chain of plaques $P'_0, .., P'_v$ of $O'_{k_0}, .., O'_{k_v}$ with $z \in P'_0$, $P'_t \subset Y_t, 0 \le t \le v$, where $Y_v := \pi_{O'_{k_v}}^{-1} \pi_{O'_{k_v}}(O'_{k_{v-1}} \cap O'_{k_v})$.

We have $O'_{k_t} \cap O_s \neq \emptyset$ since $P_{k_t} \cap P \neq \emptyset$. Therefore $O_{\lambda(k_t)} \cap O_s \neq \emptyset$ and $O'_{k_t} \subset O_{\lambda(k_t)} \subset U_{\tau(s)}$. There is a plaque $\widehat{P}$ of $U_{\tau(s)}$ with $z \in \widehat{P}$. In particular, $P'_0 \cup \ldots \cup P'_v \subset \widehat{P}$. Thus we have:

For each $z \in Y_0$, there exists exactly one plaque $\widehat{P}$ of $U_{\tau(s)}$ such that, for any $0 \leq t \leq v$, the inclusion $P'_t \subset \widehat{P} \cap O'_{k_t}$ holds.

We now choose on open neighbourhood $\Gamma \subset O'_{k_0}/\rho_{O'_{k_0}}$ of $\hat{b}$ such that:

i) $\Gamma \subset \Gamma_{k_0}$, $f_{k_0 r}(\Gamma) \subset D(h_{rs})$,
ii) $h_{rs}(f_{k_0 r}(\Gamma)) \subset \Gamma_s$,
iii) $\pi^{-1}_{O'_{k_v}}(f_{sk_v} h_{rs} f_{k_0 r}(\Gamma)) \subset Y_v$,
iv) $\Gamma \subset D(h'_{k_{v-1}k_v} \circ \ldots \circ h'_{k_0 k_1})$,
v) for each $\zeta \in \Gamma$, there is a plaque $N(\zeta)$ of O_r such that $\emptyset \neq \pi^{-1}_{O'_{k_0}}(\zeta) \cap O_r \subset N(\zeta)$.

Let $\zeta \in \Gamma$, P'_0 be a plaque of O'_{k_0} with $\pi_{O'_{k_0}}(P'_0) = \zeta$ and let $P'_0, .., P'_v$ be the respective chain. By iii), we have $\pi_{O'_{k_v}}(P'_v) \in D(f^{-1}_{sk_v})$. Then

$$f^{-1}_{sk_v}(\pi_{O'_{k_v}}(P'_v)) = \pi_{O_s}(P'_v \cap O_s) = \pi_{O_s}(\widehat{P}(\zeta) \cap O_s) = \pi_{U_{\tau(s)}}(\widehat{P}(\zeta)),$$

so

$$\pi_{O'_{k_v}}(P'_v) = f_{sk_v}(\pi_{U_{\tau(s)}}(\widehat{P}(\zeta))).$$

We get

$$\begin{aligned} h'_{k_{v-1}k_v} \circ \ldots \circ h'_{k_0 k_1}(\zeta) &= h'_{k_{v-1}k_v} \circ \ldots \circ h'_{k_1 k_2}(\pi_{O'_{k_1}}(P'_0 \cap O'_{k_1})) = \\ &= h'_{k_{v-1}k_v} \circ \ldots \circ h'_{k_1 k_2}(\pi_{O'_{k_1}}(P'_1)) = \\ &= h'_{k_{v-1}k_v} \circ \ldots \circ h'_{k_2 k_3}(\pi_{O'_{k_2}}(P'_2)) = \ldots = \\ &= \pi_{O'_{k_v}}(P'_v) = f_{sk_v}(\pi_{U_{\tau(s)}}(\widehat{P}(\zeta)) \\ &= f_{sk_v}(\pi_{U_{\tau(s)}}(N(\zeta))) = \\ &= f_{sk_v}(\pi_{U_{\tau(s)}} \pi^{-1}_{O_r} \pi_{O_r}(\pi^{-1}_{O'_{k_0}}(\zeta) \cap O_r)) = \\ &= f_{sk_v} \circ h_{rs} \circ f_{k_0 r}(\zeta) \end{aligned}$$

and therefore,

$$h'_{k_{v-1}k_v} \circ \ldots \circ h'_{k_0 k_1}|\Gamma = f_{sk_v} \circ h_{rs} \circ f_{k_0 r}|\Gamma \in \Pi'.$$

Let now $h_{i_{n-1}i_n} \circ \ldots \circ h_{i_0 i_1} \in \Pi_a$. Choose $y_0 \in O_{i_0} \cap O_{i_1} \cap L$ with $\pi_{O_{i_0}}(y_0) = a$ and $y_1 \in P_1 := \pi^{-1}_{O_{i_1}}(h_{i_0 i_1}(a))$ with $y_1 \in O_{i_2}$.

Inductively, choose $y_t \in O_{i_{t+1}} \cap O_{i_t}$ such that y_{t-1} and y_t lie in the same plaque of O_{i_t}, $1 \leq t \leq n-1$. The points y_0, y_n lie in the same plaque P of $O_{i_0} = O_{i_n}$ since $h_{i_{n-1}i_n} \circ \ldots \circ h_{i_0 i_1}(a) = a$. Choose now $x \in P$. Let $x \in O'_k$. As above, we can define a "mapping chain" $h'_{k^t_{v-1}k^t_v} \circ \ldots \circ h'_{k^t_0 k^t_1}$ $(1 \leq t \leq n-1)$ from y_{t-1} to y_t, $h'_{k^0_{v-1}k^0_v} \circ \ldots \circ h'_{k^0_0 k^0_1}$ from x to y_0 $h'_{k^n_{v-1}k^n_v} \circ \ldots \circ h'_{k^n_0 k^n_1}$ from y to x. There exists a neighbourhood Γ of a such that

$$f_{i_n k^n_v} \circ h_{i_{n-1}i_n} \circ \ldots \circ h_{i_0 i_1} \circ f_{k^0_0 i_0}|\Gamma =$$

$$= f_{i_n k^n_v} \circ h_{i_{n-1}i_n} \circ f_{k^{n-1}_v i_{n-1}} \circ f_{i_{n-1}k^{n-1}_v} \circ \ldots \circ f_{i_1 k^1_v} \circ h_{i_0 i_1} \circ f_{k_0 i_0}|\Gamma =$$

$$= (h'_{k^n_{v-1}k^n_v} \circ \ldots \circ h'_{k^n_0 k^n_1}) \circ \ldots \circ (h'_{k^0_{v-1}k^0_v} \circ \ldots \circ h'_{k^0_0 k^0_1})|\Gamma =$$

$$= h'_{k^n_{v-1}k^n_v} \circ \ldots \circ h'_{k^0_0 k^0_1}|\Gamma \in \Pi'_d,\, d := \pi_{O'_{k^0_0}}(x)\,.$$

Thus $[f_{i_n k^n_v}]_a \cdot [h_{i_{n-1}i_n} \circ \ldots \circ h_{i_0 i_1}]_a \cdot [f_{k^0_0 i_0}]_d = [h'_{k^n_{v-1}k^n_v} \circ \ldots \circ h'_{k^0_0 k^0_1}]_d$.

In this way, we get a homomorphism $j : G(a) \to G'(d)$ where $G'(d)$ is the group of germs of homeomorphisms from Π'_d. The homomorphism j is, of course , injective. Moreover for each $h' \in \Pi'_d$, there is an element $h \in \Pi_a$ such that $j([h]_a) = [h']_d$ since $\{O'_k\}$ is a refinement of $\{O_j\}$. Therefore j is an isomorphism.

Now, we can consider a general case. Let $\{O'_k\}$ be an arbitrary technical covering of L. The family $\{Z_{kj} := O_j \cap O'_k; j \in J, k \in K, O_j \cap O'_k \neq \emptyset\}$ is an open covering of L which is a refinement of $\{O'_k\}$ as well as of $\{O_j\}$. For this covering, there exists a refinement $\{U''_t; t \in M\}$ whose elements are domains of adapted charts. A technical covering $\{O''_i; i \in I\}$ for $\{U''_t\}$ is a topological refinement of $\{O'_k\}$ as well as of $\{O_j\}$. Now, the isomorphism class of $G(a)$ depends only on the plaque P by the particular case considered above. □

We now have to show that $G(a)$ depends only on the leaf L.

REMARK 5.4. If $x, y \in L$, then there are $j_0, j_n \in J$ such that, for $x \in O_{j_0}$, $y \in O_{j_n}$ and $a := \pi_{O_{j_0}}(x)$, $b := \pi_{O_{j_n}}(y)$, there exists $h \in \Pi$ with $h(a) = b$.

PROOF. There is a simple chain of plaques P_{j_t} of O_{j_t} $(0 \leq t \leq n)$ from x to y.

The mapping $h := h_{j_{n-1}j_n} \circ \ldots \circ h_{j_0 j_1}$ is the required homeomorphism. $\square$

If $f \in \Pi_a$, then $hfh^{-1} \in \Pi_b$, and $[f]_a \longmapsto [hfh^{-1}]_b$ is an isomorphism $G(a) \to G(b)$.

In this way we have proved the following

PROPOSITION 5.5. *The group $G(a)$ is determined by the leaf L uniquely up to an isomorphism.*

Let $\mathrm{Hol}(L)$ denote this group.

DEFINITION 5.6. $\mathrm{Hol}(L)$ *is called a geometric holonomy group of L.*

6 – Stability theorem

Let Y be a topological space, R - an equivalence relation on Y and let $y \in Y$. Denote by $H(R, y)$ the group of all germs of local homeomorphisms h of open neighbourhoods of y such that $h(y) = y$ and, for each $z \in D(h)$, we have $zRh(z)$.

To prove the stability theorem, we need the following

LEMMA 6.1. *Let $G \subset H(R, y)$ be a finite subgroup. Then there exist an open neighbourhood U of y and a finite group $G(U)$ of homeomorphisms of U onto itself, such that*

i) $(\forall g \in G(U)) g(y) = y$,
ii) $(\forall u \in U)(\forall g \in G(U)) uRg(u)$,
iii) $G = \{[g]_y; g \in G(U)\}$,
iv) $\mathrm{ord}\, G = \mathrm{ord}\, G(U)$.

Moreover, if O is an open neighbourhood of y, then U can be chosen in such a way that $U \subset O$.

For the proof of the above lemma, see, for example [8].

We now have

THEOREM 6.2. *Let L be a compact transversely stable leaf of $\mathcal{F}$ with the finite geometric holonomy group* $\mathrm{Hol}(L)$. *Then L is stable.*

PROOF. Let $\{O_j; j \in J\}$ be a technical covering of L of power 2 respective to L. Let $J = \{1, \dots, r\}$; then $L \subset O_1 \cup \dots \cup O_r =: O$. Choose $x \in O_1 \cap L$, $\pi_{O_1}(x) = a \in O_1/\rho_{O_1}$ and, for each $h \in \mathrm{Hol}(L)$, choose $g \in \Pi_a$ with $[g]_a = h$, $\mathrm{ord}\,\mathrm{Hol}(L) =: m$. By Lemma 6.1, there are open neighbourhood $U \subset O_1/\rho_{O_1}$ of a and $g_1, .., g_m \in \Pi_a$, such that

i) $\mathrm{Hol}(L) = \{[g_i]_a; 1 \leq i \leq m\}$,
ii) $\{g_i|U : U \to U; 1 \leq i \leq m\}$ is a group of homeomorphisms.

Set $G := \{h \in \Pi; h = h_{i_{t-1}i_t} \circ \dots \circ h_{i_0 i_1}, t \leq r, i_o = 1, (\forall j)\ i_j \in \{1, \dots, r\}\}$. Then the set G is finite. Choose an open neighbourhood $S \subset O_1$ of x as in Proposition 4.4 for $M := r \cdot m \cdot \mathrm{card}\, G$, i.e. for each chain $Q_{i_1}, .., Q_{i_n}$ of plaques of $O_{i_1}, .., O_{i_n}$ of length $\leq M$ and with a base in S, the inclusion $\overline{Q}_{i_t} \subset O$ holds for each $1 \leq t \leq n$. By Lemma 6.1, U can be chosen in such a way that $\pi_{O_1}^{-1}(U) \subset S$. Now, the group $\mathrm{Hol}(L)$ acts on the set U by $g_i|U$, $i = 1, .., m$.

Let $\mathrm{Or}(\zeta)$ be the orbit of ζ in this action. In particular, $\mathrm{card}\,\mathrm{Or}(\zeta) \leq m = \mathrm{ord}\,\mathrm{Hol}(L)$. Then

$$\mathcal{P}(\zeta) := \bigcup_{\substack{y \in h(\mathrm{Or}(\zeta)) \\ h \in G, 1 \leq j \leq r}} \pi_{O_j}^{-1}(y)$$

is the union of at most M plaques.

Then we have:

If $L' \in \mathcal{F}$ with $L' \supset \pi_{O_1}^{-1}(\zeta)$,then $L' = \mathcal{P}(\zeta)$.

Indeed, by the definition, $\mathcal{P}(\zeta) \subset L'$, $\mathcal{P}(\zeta) \neq \emptyset$ and $\mathcal{P}(\zeta)$ is $T_{\mathcal{F}}$-open. We have to show that $\mathcal{P}(\zeta)$ is $T_{\mathcal{F}}$-closed. Let z be a point of the $T_{\mathcal{F}}$-closure of $\mathcal{P}(\zeta)$. Then, for some $j \in J$, we have, by Proposition 1.12, $z \in \overline{\pi_{O_j}^{-1}(y)} = \varphi_{\sigma(j)}^{-1}(S_j \times \overline{V}_j)$ with $y = h(f(\zeta))$, $h \in G$, $f \in \{g_i|U\}$ and $\psi_j(\pi_{O_j}^{-1}(y)) = S_j \times V_j$. Therefore, by Proposition 4.4, $z \in O$. Thus there exists $s \in \{1, .., r\}$ with $z \in O_s$, i.e. $z \in \pi_{O_s}^{-1}(\eta)$ for some $\eta \in O_s/\rho_{O_s}$. We have $O_s \cap O_j \neq \emptyset$ since $z \in \overline{O}_j \cap O_s$. Consequently, $h_{js}(y) = \eta$. Since the plaques P_i of O_i in L cover L $(1 \leq i \leq r)$, there exists a simple chain of plaques $P_1, P_{j_1}, .., P_{j_t}, P_j, P_s$ with $1 \leq j_1, .., j_t \leq r$ of length $\leq r + 1$. Define

$$g^{-1} := h_{js} \circ h_{j_t j} \circ \dots \circ h_{1 j_1} \in G, \xi := g(\eta)\,.$$

Then $\xi = g(\eta) = g(h_{js}(y)) = g \circ h_{js} \circ h \circ f(\zeta)$, $g \circ h_{js} \circ h \circ f \in \Pi_a$ and consequently, there exists $k \in \{1, .., m\}$ such that $g \circ h_{js} \circ h \circ f|U = g_k|U$.

Therefore $\xi \in \mathrm{Or}(\zeta)$ and

$$\pi_{O_*}(z) = \eta = g^{-1}(\xi) \in g^{-1}(\mathrm{Or}(\zeta))$$

with $g^{-1} \in G$; consequently,

$$z \in \pi_{O_*}^{-1}(g^{-1}(\xi)) \subset \mathcal{P}(\zeta).$$

The set $V := \bigcup_{\zeta \in U} \mathcal{P}(\zeta)$ is a saturation of $\pi_{O_1}^{-1}(U)$, so it is an open and ρ-saturated neighbourhood of L with $U \subset O$. □

7 – Additional results, special cases

A coherent foliation $\mathcal{F}$ on a space X induces to each $L \in \mathcal{F}$, $x \in L$ and adapted chart $(U, \varphi, A \times W)$ with $x \in U \subset X$ by the projection $A \times W \to A$ a projection $\pi_U : U \to A$. The coherent foliation $\mathcal{F}_U$ respectively $\mathcal{F}_A$ induced on U respectively on A by $\mathcal{F}$ are connected by the relation: $\mathcal{F}_U = \pi_U^{-1}(\mathcal{F}_A)$ (see 1.6-1.8). We have now the following two more restricted subcases.

CASE 1. Let $\mathcal{F}_A$ consist only of isolated points. For the quotientspace we then have $U/\mathcal{F}_U := U/\pi_U = A$. If the (geometric) holonomy group $\mathrm{Hol}(L)$ of L is finite, it operates (without restriction) on A, and the quotientspace $A/\mathrm{Hol}(L)$ is again a differentiable space (i.e with singularities, even if X, hence A is a manifold - as in the classical case): See [14], [15]. Note for the space $(X/\mathcal{F})|U$ of global leaves passing through $U : (X/\mathcal{F})|U = U/\rho = A/\mathrm{Hol}(L)$. If all leaves are compact and have finite holonomies then these local quotients $A/\mathrm{Hol}(L)$ easily glue together to give a global quotientspace $X/\mathcal{F}$ with additional differentiable structure ([8]):

$X/\mathcal{F}$ is in a natural way a differentiable space with local charts $A/\mathrm{Hol}(L)$. Moreover, if the "local transversals" A are not too "wild" (as in the sense of [8], especially: if the A's are subanalytic or even manifolds) also the inverse of the stability theorem holds ([8]). With this one obtains in case 1 for a foliation with compact leaves of the same dimension:

THEOREM 7.1. *For $(X, \mathcal{F})$ the following conditions are equivalent:*

1) *The leaf space $X/\mathcal{F}$ is Hausdorff.*
2) *$X/\mathcal{F}$ is (in a natural way) a differentiable space.*
3) *The foliation $\mathcal{F}$ on X is stable.*

Moreover, many geometric properties of X carry over to the quotient $X/\mathcal{F}$ ([8]). In general, the leaf space $X/\mathcal{F}$ of a compactly foliated space is Hausdorff only on a "large" subset, and hence a differentiable space there. So we run here into some generalised notion of differentiable (:*almost differentiable*) space, which we study elsewhere.

CASE 2. Let the foliation $\mathcal{F}$ be induced by a Lie group G operating properly on X. The quotient space $X/\mathcal{F} = X/G$ is again a differentiable space [14], [15]. In this case the foliation $\mathcal{F}_A$ is induced by the connected components fix_x^* of the identity of the (compact!) isotropy groups $\mathrm{fix}_x \subset G$ at all points $x \in X$. Now, $A/\rho_A = A/\mathrm{fix}_x^*$ is a differentiable space [14]; for $x \in L$ (as above) we have differentiable "projections"

$$\pi_U^* : U \to A \to A/\rho_A =: A^* .$$

and $\mathrm{fix}_x / \mathrm{fix}_x^*$ operates differentiably as (finite!) holonomy group $\mathrm{Hol}(L)$ on A^*. Then the differentiable quotient space X/G has each $A^* / \mathrm{Hol}(L) = A/\mathrm{fix}_x$ as local differentiable chart.

PROBLEMS. 1) it is not known, under which quite general, "weak" assumptions the "transverse" coherent foliation $\mathcal{F}_A$ on each A, induced by $\mathcal{F}$ according to 1.8, leads to a differentiable space as quotientspace $A/\rho_A = A/\mathcal{F}_A$. If this is always the case for some $\mathcal{F}$, then $\mathrm{Hol}(L)$ operates differentiably on $A/\mathcal{F}_A$ and the quotient space becomes again a differentiable space; the analogue of theorem 7.1 then also holds (using [4]).

2) It is not known, under which "weak" assumptions a finite holonomy group $\mathrm{Hol}(L)$, which operates on A/ρ_A, can be "lifted" to a finite group $\mathrm{Hol}(L)^*$, operating differentiably on A. If this lifting is possible, we would have a differentiable space $A^* := A/\mathrm{Hol}(L)^*$ with a coherent foliation $\mathcal{F}_{A^*}$ on A^* such that each quotient $A^*/\mathcal{F}_{A^*}$ would be a local chart of $X/\mathcal{F}$. Then, one could try to build up a general theory of foliations on abstract spaces of the local type $A^*/\mathcal{F}_{A^*}$. For partial results to the lifting problem see [16].

It is clear now, that one needs still more, but not too(!) general versions as those mentioned up to here. By all this, what appeared above, the definitions in the following paragraph are partly motivated. Other parts are motivated by Jurchescus "mixed spaces" ([11]), still other parts by the theory of "super manifolds" in physics ([11]).

8 – Some generalizations

Let $X = (\underline{X}, \underline{\underline{X}})$ be a ringed space, where $\underline{X}$ is a (say Hausdorff) topological space, $S := \underline{\underline{X}}$ a sheaf of local K-algebras ($K = \mathbb{R}$ or $= \mathbb{C}$) on $\underline{X}$. Then the reduction $\operatorname{red} S$ is some sheaf of germs of K-valued functions on $\underline{X}$ (not necessarily continuous). See for example [18]. In the following way we consider X as a foliated space:

A local leaf L of X is a connected and locally connected subset $L \subset X$ such that the following holds: For any $x \in L$ and each $U(x)$ of some basis $\mathcal{U}(x)$ of neighbourhoods $U(x) \subset X$ of x in X we have:

*) $L \cap U(x) = \{y \in U(x) | f(x) = f(y) \text{ for each } f \in \Gamma(U, S)\}$
$L \cap U(x)$ is connected.

L is called local leaf of X passing through (any) $x \in L$. L may be just one point $x : L = \{x\}$. There also may be no leaf passing through some x in the just mentioned sense. Then we call $\{x\}$ a local leaf of X passing through $x \in \underline{X}$. The family of all of these local leaves we denote by $\mathcal{F}^l_X$, more simply by $\mathcal{F}^l$.

If two such leaves $L_1, L_2 \in \mathcal{F}^l$ pass through some common point x, they coincide in some neighbourhood $W(x) \subset X$ of x:

**) $L_1 \cap W(x) = L_2 \cap W(x)$.

Hence: This local foliation $\mathcal{F}^l$ induces a global foliation $\mathcal{F}$ (better: $\mathcal{F}_X$) on X into maximal connected leaves $L \hookrightarrow X$. Here $\hookrightarrow$ denotes an injective mapping, which locally is topological onto its local images.

By **) , the intersections $U(x) \cap L$ from *) define a new topology $\mathcal{I}_{\mathcal{F}}$ on X. The connected components in this topology are just the leaves of $\mathcal{F}$.

Definition 8.1. *X together with $\mathcal{F}^l$ resp. $\mathcal{F}$ just described is called a foliated space. The elements of $\mathcal{F}$ are called global leaves of X.*

NOTE 8.2. α) The function germs of red S need not be continuous; thus for some $x \in X$ but no open $U(x) \subset X$, the leaves $L \cap U(x)$ may be closed in $U(x)$; the leaves may also not be locally compact. For example take $\mathcal{F}^l = \{\{0\}, \mathbb{R}^2 \setminus \{0\}\}$ in $\mathbb{R}^2$ or $\mathcal{F}^l = \mathbb{Q} \times \{0\} \cup \mathbb{R}^2 \setminus (\mathbb{Q} \times \{0\})$.

β) Instead of deriving foliations from given sheaves one also may start from given (local) foliations and associate to them in a natural way sheaves of germs of functions.

γ) To avoid too "wild" foliations, additional assumptions are necessary. For example "simple" ones: For each $x \in X$ and some neighbourhood $U(x) \subset X$ of x we require the equality *) for *each* $L \in \mathcal{F}^l$. Or we require, that red S should have only "nice" functions, for example continuous or even "better" ones. If one wants to obtain some "nice" global leaf structure of X, that means of $\mathcal{F}$, hence of the leaf space $X/\mathcal{F}$ (identifying global leaves to points), one will have to know in advance, that already the local leaf structure of X, i.e. of $\mathcal{F}^l$, hence of each $U(x)/(\mathcal{F}^l \cap U(x))$ is "nice" in a similar way.

So for example we may define:

DEFINITION 8.3. α) *A foliated space X is called C^N- transversely foliated, if the following holds: To each $x \in \underline{X}$ there exists a basis $\mathcal{U}(x)$ of neighbourhoods $U(x) \subset \underline{X}$ of x and to each $U \in \mathcal{U}(x)$ a (local) C^N-quotient morphism*

$$\phi^U : X|U \to X^U = (\underline{X}^U, \underline{\underline{X}}^U)\,, i.e.$$

i) *X^U is a ringed space, "situated" in an N-differentiable standard space ${}^N X^U$ such that: $\underline{X}^U = {}^N\underline{X}^U, \underline{\underline{X}}^U \subset {}^N\underline{\underline{X}}^U$ (write $X^U \hookrightarrow {}^N X$).*

ii) *ϕ^U is a morphism of ringed spaces.*

iii) *$\underline{\phi}^U : U \to \underline{X}^U$ is continuous, open and the inverse image $(\underline{\phi}^U)^{-1}(L) \subset \underline{X}^U$ of any global leaf L of X^U is connected and locally connected.*

iv) *$\underline{\phi}^U$ gives an isomorphism $\underline{\underline{X}}^U_y \to \underline{\underline{X}}_x$ for each $x \in U$, $y = \underline{\phi}(x)$.*

Moreover we require that these local C^N-quotient morphism are C^N-equivalent, i.e: If ϕ^U, ϕ^V are local C^N-quotient morphisms from our system above, then to each $x \in U \cap V$ we have a commutative diagramm in some neighbourhood $W(x) \subset U \cap V$:

$$
\begin{array}{ccc}
 & X|W & \\
 \swarrow^{\Phi^U|W} & & \searrow^{\Phi^U|W} \\
X^U|\underline{\Phi}^U(W) & \xrightarrow{\ *)\ } & X^V|\underline{\Phi}^V(W) \\
\downarrow & & \downarrow \\
{}^N X^U|\underline{\Phi}(W) & \longrightarrow & {}^N X^V|\underline{\Phi}^V(W)
\end{array}
$$

where the last line is a C^N-diffeomorphism and the line above a biomorphism induced by the last line.

β) Each X^U, satisfying α), i-iv) is called a local C^N- quotient of X.

Note, that 8.3 does not fully cover our earlier definitions. For this one has for example to drop the assumption $\underline{\underline{X}}^U \subset {}^N\underline{\underline{X}}^U$ under α), i) and to change the "C^N-equivalence" appropriately. We describe some special cases to make clear the meaning of 8.3:

CASE 1. α) $X^U = {}^N X^U$. Then each leaf of X^U is just one point and the local C^N-quotient morphisms are automatically connected by bimorphisms *). These are automatically already C^N-diffeomorphisms, because the ${}^N X^U$ are by our assumption standard spaces (for example reduced spaces, see [23]). If each X^U in addition is reduced, one can prove an analogue of theorem 7.1. However one may extend this result also to non-reduced spaces and obtain under slight geometric assumptions on the foliated space X (we discuss this in some forthcoming paper):

THEOREM 8.4. *X is stably foliated $\Longleftrightarrow$ The leafspace of X is a differentiable space $\Longleftrightarrow$* red X *is stably foliated $\Longleftrightarrow$ The leafspace of* red X *is a (reduced) differentiable space.*

In addition X may come from on C^M-differentiable space: ${}^M X \hookrightarrow X$ with $M \leq N$, and all ϕ^U differentiable.

CASE 2. X and all X^U are reduced. The foliation $\mathcal{F}^1$ on X^U is given by a stable coherent foliation on X^U as in 1-6. There will be similar results now as in 1-6. One obtains the situation in 1-6, if in addition

$X \hookrightarrow {}^N X$ for an C^N-differentiable space, and the foliation on X is given by a coherent foliation on ${}^N X$.

Other special cases are now obvious. To extend the last example, mentioned under case 1 for $M \leq N$ also for $M \geq N$, for example, we still need some more structured foliated spaces (however not yet the most general ones):

DEFINITION 8.5. *$X = (\underline{X}, S, S^*)$ is called a foliated space of mixed type (M, N) if the following holds:*

i) *(X, S) is a C^M-transversely foliated space, with $\underline{\underline{X}}^U = {}^M\underline{\underline{X}}^U$ in 8.3, i).*
ii) *For each local C^M-quotient morphism ϕ^U in 8.3, each $x \in \underline{\underline{X}}^U$ and each $n \in \mathbb{N}$, $n \leq M$ the n-th-order fiber $(X|U)|(\phi^U)^{-1}(x)^n := ((\underline{\phi}^U)^{-1}(x)$, the space $(S^*/\underline{\phi}^{-1}(\underline{\underline{m}}_x^n)|((\underline{\phi}^U)^{-1}(x))$ is an N-differentiable space. Here $m_x \subset \underline{\underline{X}}_x^U$ denotes the maximal ideal in x.*

Again, special cases make clearer our intentions:

CASE 1. Let $\mathcal{D}^k$ be the sheaf of C^k-function germs in (some) K^m, $k = 0, 1, .., \infty, \omega, \omega^*$, $K = \mathbb{R}$ or $\mathbb{C}$ (appropriately chosen), $\mathcal{D}^{l,k}$ the sheaf of germs of functions in $K^n \times K^m$ being "mixed" differentiable of class $C^{l,k}$ (C^l in K^n-, C^k in K^m-direction, see [19]).

The projection $\pi : K^n \times K^m \to K^n$ induces for $U \subset K^n$, $V \subset K^m$ open "mixed" differentiable morphisms

$$\pi : (U \times V, \mathcal{D}^{l,k}) \to (V, \mathcal{D}^k)$$

and for appropriate ideals also morphisms

$$\pi : (D^{l,k}, \mathcal{D}^{l,k}/\mathcal{I}^{l,k}) \to (D, \mathcal{D}^k/\mathcal{I}^k).$$

Now in 8.4 and 8.3 we may have in local charts

$$(\underline{X}, S^*)|U \simeq (D^{l,k}, \mathcal{D}^{l,k}/\mathcal{I}^{l,k})$$

$$X^U \simeq (D, \mathcal{D}^k/\mathcal{I}^k)$$

$$\phi^U \quad \text{induced by} \quad \pi.$$

More specifically, $(D^{l,k}, \mathcal{D}^{l,k}/\mathcal{I}^{l,k})$ may be a "mixed" product space $(D^l, \mathcal{D}^l/\mathcal{I}^l) \times (D^k, \mathcal{D}^k/\tilde{\mathcal{I}}^k)$ ([19]).

Stability results will hold in this context ([11]).

Special mixed spaces of this type are studied in Jurchescues theory of mixed spaces. The problem is, whether his type of results also hold in our more general cases.

CASE 2. Let $X = (\underline{X}, S^*)$ be an N-differentiable space of constant local embedding dimensions $\dim T_x X$, such that $(\underline{X}, S) = \text{red}\, X$ is a differentiable manifold. By [7] there exists a differentiable manifold M^* with $\dim M^* = \dim T_x X$ and a (highly non canonical!) embedding $X \hookrightarrow M^*$ and hence a retraction $M^* \to \text{red}\, X$, hence a retraction $X \hookrightarrow \text{red}\, X$, which makes $(\underline{X}, S, S^*)$ a foliated space of mixed type (N, N). If $N = \infty$ and S^*/m_x is a formal power series ring for each maximal ideal $m_x \subset S_x$, we obtain the space part of a supermanifold as foliated space (expressed in the language of differentiable spaces now: see [11] for more details).

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