Rendiconti di Matematica, Serie VII
Volume 13 , Roma (1993), 701-731

On the physical indetermination of the response functions for general bodies of the differential type

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RIASSUNTO: *Per caratterizzare un corpo continuo $\mathcal{B}$ si assegna un sistema, $\hat{\sigma}$, di funzioni o funzionali costitutivi per stress, entropia, flusso di calore, ecc. In vari casi si è già studiata la indeterminazione fisica di $\hat{\sigma}$, ossia si sono stabilite, per un tale sistema $\hat{\sigma}$, delle condizioni necessarie o sufficienti a caratterizzare lo stesso corpo.*

In questo lavoro consideriamo un qualunque corpo $\mathcal{B}_{PQR}$ di tipo differenziale e di complessità $(P, Q, R) \in \{0, 1, 2, \dots\}^3$, ossia un corpo tale che gli argomenti delle sue funzioni costitutive possano comprendere le derivate materiali del gradiente di deformazione, della temperatura e del gradiente di temperatura fino agli ordini P, Q ed R rispettivamente.

Nell'ipotesi di invarianza Galileiana si dimostra un teorema di unicità per la funzione costitutiva dello stress in $\mathcal{B}_{PQR}$ per ogni (P, Q, R). Nella più forte ipotesi di invarianza Euclidea, per ogni P si dimostra un teorema di unicità per la funzione costitutiva dell'energia interna in $\mathcal{B}_{P00}$. Inoltre, per ogni $\mathcal{B}_{PQR}$, si stabilisce una condizione necessaria e sufficiente affinché la differenza tra due qualunque funzioni costitutive ammissibili per l'energia interna sia una funzione del punto materiale.

Nel caso tale condizione non valga, si mostra l'esistenza di una tripla indeterminazione nelle funzioni costitutive di energia interna, entropia e flusso di calore, che non può essere osservata sperimentalmente.

Per dimostrare i risultati accennati si usano le soluzioni, che godono la proprietà di indifferenza materiale, di certi sistemi simmetrici di equazioni differenziali alle derivate parziali a coefficienti costanti, involgenti una funzione incognita a valori ed argomenti tensoriali. Queste soluzioni vengono trovate nella prima parte del lavoro.

This work has been performed within the activity of the Consiglio Nazionale delle Ricerche, Group n. 3.

ABSTRACT: *To characterize a continuous body $\mathcal{B}$ one generally gives a system, $\hat{\sigma}$, of constitutive functions for e.g. stress, entropy, heat flux, etc. In various cases one has already studied the physical indetermination of $\hat{\sigma}$, that is, one has stated some conditions sufficient or necessary, for a similar system $\tilde{\sigma}$, to characterize the same body. In the present paper we consider any body $\mathcal{B}_{PQR}$ of the differential type and complexity $(P,Q,R) \in \{0,1,2,\dots,\}^3$, that is, such that the arguments of its constitutive functions include the material time derivatives of the deformation gradient, temperature, and temperature gradient; furthermore the possibly vanishing orders of these derivatives do not exceed P, Q and R respectively. Under the assumption of Galilean invariance, we prove a uniqueness theorem for the response function of the stress in $\mathcal{B}_{PQR}$ for any (P,Q,R). Under the stronger assumption of Euclidean invariance, for any P we also prove a uniqueness theorem for the response function of the internal energy in $\mathcal{B}_{P00}$. Furthermore, for any $\mathcal{B}_{PQR}$, we state a necessary and sufficient condition that the difference between any two admissible response functions for the internal energy be a function of the material point. We show that in case the aforementioned condition does not hold there is a threefold indetermination in the response functions for internal energy, entropy, and heat flux, which cannot be detected by experiments. To prove the above results we need the explicit forms of the frame-indifferent solutions to certain symmetric systems of linear P.D.E.s with constant coefficients in an unknown tensor-valued function with tensor arguments.*

KEY WORDS: *Symmetric systems of linear PDEs - Internal energy - Bodies of the differential type.*

A.M.S. CLASSIFICATION: 35Q20 - 15A69 - 73B05

1 – Introduction

A continuous body $\mathcal{B}$ is generally characterized by means of a system,

$$\hat{\sigma} = (\widehat{\mathbf{P}}, \hat{\mathbf{q}}, \hat{e}, \hat{\eta}),$$

of constitutive functions of functionals for stress, heat flux, internal energy, and entropy, respectively. In various cases one has already studied some physical indeterminations of $\hat{\sigma}$; that is, one has stated some conditions, for a similar system $\tilde{\sigma}$, to characterize the same body. Indeterminations in constitutive equations can really take place, and some authors have based on them some criticisms to certain approaches to thermodynamics[(1)]. For informations on the specific research field to which this paper belongs see the introduction of [14].

[(1)]For instance DAY [4] proves that a particular rigid heat conductor with memory has infinitely many entropy functionals pairwise differing by a nonlinear function; on this basis he criticizes the use of entropy as a primitive variable in continuum thermodynamics and he justifies his unusual approach proposed in [3] - read ftn. 2, p. 253, in [4]. In [5] GREEN and NAGHDY set up a thermodynamic theory for simple bodies in which

In this paper, divided in two parts, we consider a body $\mathcal{B}_{PQR}$ of the differential type, having the complexity $(P,Q,R) \in \{0,1,2,\dots\}^3$; briefly this means that the arguments of $\mathcal{B}_{PQR}$'s constitutive functions include the material time derivatives of the deformation gradient $\mathbf{F}$, temperature Θ, and temperature gradient $\mathbf{G}$, where their possibly vanishing orders do not exceed P, Q and R respectively.

In paper [14] we have stated some theorems on the indeterminations of $\hat{\mathbf{q}}$ for various complexities of the body. In particular, with regard to any choice of $\mathcal{B}_{PQR}$ there is an indetermination for $\hat{\mathbf{q}}$ given by the additive function

$$\widehat{\mathbf{Q}}(\Theta, \mathbf{G}, \mathbf{X}) := \mathbf{G} \times \operatorname{grad} \hat{\varphi}(\Theta, \mathbf{X})$$

for any scalar function $\hat{\varphi}$ of class C^3 with

$$\operatorname{grad} \hat{\varphi}(\Theta, \mathbf{X}) \times \mathbf{N}(\mathbf{X}) = \mathbf{0} \quad \text{for each} \quad \Theta \in \mathbf{R}^+ \quad \text{and} \quad \mathbf{X} \in \partial\mathcal{B}_{PQR},$$

where $\widehat{\mathbf{N}}(\mathbf{X})$ in the unit outward normal to the boundary $\partial\mathcal{B}_{PQR}$ at the material point $\mathbf{X}$. For $Q = 0 = R$ this indetermination is maximal and just coincides with the one found in [8] for the thermoelastic case.

The first part of this paper has a purely mathematical and preliminary character: we solve certain symmetric systems of linear P.D.E.s with constant coefficients, which involve tensor-valued functions with tensor arguments. More details on the contents of this part can be found at its outset in § 2.

In the second part we consider choices of $\mathcal{B}_{PQR}$ and, following [1], [8], [9] and [14], we use the Mach-Painlevé point of view.

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there is a response functional for the specific internal rate of entropy production and the Clausius-Duhem inequality is replaced by a certain balance law for the entropy. They point out an indetermination in their constitutive equations, suggest a way to eliminate it, and assert that Day's criticisms on the use of entropy does not apply to their theory. This way is criticized in [7] and is replaced in some cases by an other way. Lastly, MONTANARO [8] proves that in any simple body there is an additive indetermination in the response function $\hat{\mathbf{q}}$ for the heat flux which depends on the choice of an arbitrary scalar function and is unobservable by means of experiments; furthermore the maximal indetermination of $\hat{\mathbf{q}}$ is fully characterized for any thermoelastic body; in order to select a physically privileged response function for the heat flux in any thermoelastic body, a physical criteria is described which involves (unusual) experiments of cut and contact with a nonconducting body. This physical criteria is the extension to thermoelasticity of the second part of paper [1], which refers to the purely mechanical theories of simple bodies.

For $(P, Q, R) \in \{0, 1, 2, \dots\}^3$ we say that $\hat{\sigma} = (\widehat{\mathbf{P}}, \hat{\mathbf{q}}, \hat{e}, \hat{\eta})$ is a *(generalized) system of response functions* for $\mathcal{B}_{PQR}$ if the balance equations of linear momentum, angular momentum, and energy, as well as the Clausius-Duhem inequality, identically hold along any physically possible thermokinetic process of the body(2), provided the (Piola) stress $\mathbf{P}$, heat flux $\mathbf{q}$, internal energy e, and entropy η are determined, along the process, by means of the response functions in $\hat{\sigma}$.

Note that in this definition the balance laws and the entropy inequality are regarded as conditions of the four fields $\mathbf{P}$, $\mathbf{q}$, e and η. From this point of view contact forces, heat flux, specific internal energy, and specific entropy are not primitive notions of the theory. Moreover, at least a priori, by the definition above it is not excluded that $\mathcal{B}_{PQR}$ may have distinct systems of response functionals connected with the same reference configuration, that we can call physically equivalent - for more details on this point read the introductions of [1], [8], [9] and [14].

From [9] and [14, § 3] we recall the notion of ordinary system of response functions for the body $\mathcal{B}_{PQR}$. Let $\mathcal{B}_{te}$ be a thermoelastic body. The generalized system of response functions $\bar{\sigma}_{te} = (\overline{\mathbf{P}}, \bar{\mathbf{q}}, \bar{e}_{te}, \bar{\eta}_{te})$ for $\mathcal{B}_{te}$ is *boundary-ordinary* if $\overline{\mathbf{P}}_{te}$ and $\bar{\mathbf{q}}_{te}$ give rise to zero normal stresses and heat fluxes, respectively, on any portion of $\mathcal{B}_{te}$'s boundary, whenever this portion is put in contact with vacuum. We say that the generalized system of response functions $\hat{\sigma} = (\widehat{\mathbf{P}}, \hat{\mathbf{q}}, \hat{e}, \hat{\eta})$ for the differential body $\mathcal{B}$ is *ordinary* if for any point $\mathbf{X} \in \partial\mathcal{B}$ the conditions $\widehat{\mathbf{P}}\mathbf{N} = \overline{\mathbf{P}}_{te}\mathbf{N}$ and $\hat{\mathbf{q}} \cdot \mathbf{N} = \bar{\mathbf{q}}_{te} \cdot \mathbf{N}$ hold on some surface neighbourhood σ $(\subset \partial\mathcal{B})$ in case $\mathcal{B}$ is put in contact through σ with a thermoelastic body $\mathcal{B}_{te}$ in a certain way (specified in [14], Axiom 3.1]), and $\overline{\mathbf{P}}_{te}$ and $\bar{\mathbf{q}}_{te}$ belong to a boundary-ordinary system of response functions for $\mathcal{B}_{te}$. Obviously, a response function, e.g. for the stress, is said to be generalized or ordinary if it belongs to a generalized or ordinary system of response function for the body.

In this paper for any complexity (P, Q, R) we also extend to $\mathcal{B}_{PQR}$ the uniqueness theorems for both the generalized and the ordinary response functions of the stress, proved in [1] and [9] for elastic and thermoelastic bodies, respectively: we prove that the difference between any two gen-

(2) Under suitable regularity conditions for the thermokinetic process, the external and body forces (possibly giving rise to integro-differential equations), and the external heat supply.

eralized Cauchy function for the stress is an Eulerian pressure which is constant on the whole body; furthermore the ordinary function for the stress in unique.

For any $\mathcal{B}_{P00}$, with $P \geq 0$, we also prove a uniqueness theorem for the response function of the internal energy: the difference between any two such functions only depends on the material point - see Theorem 9.1 (a). Furthermore if both

$$\hat{\sigma} = (\hat{\mathbf{P}}, \hat{\mathbf{q}}, \hat{e}, \hat{\eta}) \quad \text{and} \quad \tilde{\sigma} = (\tilde{\mathbf{P}}, \tilde{\mathbf{q}}, \tilde{e}, \tilde{\eta})$$

are generalized systems of response functions for $\mathcal{B}_{PQR}$, referred to the same reference configuration, and $Q \geq 1$, $R = 0$, then the equalities

$$\text{(1.1)} \qquad \partial(\hat{e} - \tilde{e})/\partial \mathbf{G} \equiv \mathbf{0} \quad \text{and} \quad \partial(\hat{\mathbf{q}} - \tilde{\mathbf{q}})/\partial\dot{\Theta} \equiv \mathbf{0},$$

are mutually equivalent and imply that the difference $\hat{e} - \tilde{e}$ depends at most on the material point - see Theorem 9.1 (b).

To prove the uniqueness of the stress it suffices to assume that the response functions are Galilean invariant, whereas to prove the uniqueness of the internal energy the stronger hypothesis of Euclidean invariance must be used.

Lastly, we exhibit an indetermination in the response functions for any body $\mathcal{B}_{PQ0}$ with $Q \geq 1$. Note that in this case $\dot{\Theta}$ is an argument of the response functions. Bodies of this kind are in effect used by some theorists; for instance MÜLLER's thermoelastic materials in [15] have complexity $(0, 1, 0)$. The indetermination shown here simultaneously involves the three response functions for the internal energy, entropy, and heat flux. Following [1, Part 2] and [9], one might easily show that this indetermination vanishes by using experiments of cut and contact with some bodies. There experiments are generally not considered (authors generally follow TRUESDELL's program [18, p. 121]. For more details read the introductions of [1] and [9].

Precisely, we show that if $\hat{\sigma} = (\hat{\mathbf{P}}, \hat{\mathbf{q}}, \hat{e}, \hat{\eta})$ is a generalized system of response functions for $\mathcal{B}_{PQ0}$ $(Q \geq 1)$, then there are certain Euclidean invariant functions $\widehat{E}$, $\widehat{\mathbf{Q}}$, and $\widehat{\mathbf{N}}$, depending on nine arbitrary scalar functions, such that $\tilde{\sigma} = (\hat{\mathbf{P}}, \hat{\mathbf{q}} + \widehat{\mathbf{Q}}, \hat{e} + \widehat{E}, \hat{\eta} + \widehat{N})$ is a similar system for $\mathcal{B}_{PQ0}$ and moreover the two response functions $\hat{\mathbf{q}}$ and $\hat{\mathbf{q}} + \widehat{\mathbf{Q}}$ give rise to the same normal heat flux on $\partial\mathcal{B}_{PQ0}$. Hence $\hat{\sigma}$ and $\tilde{\sigma}$ can be said to be physically equivalent.

The indetermination functions $\widehat{\mathbf{Q}}$ and $\widehat{E}$ violate equalities $(1.1)_1$ and $(1.1)_2$. By Theorem 9.1 (b) and by the existence of this threefold indetermination, with regard to any body $\mathcal{B}_{PQ0}$ with $Q \geq 1$ the condition $(1.1)_1$ or $(1.1)_2$ is necessary and sufficient for the difference between any two response functions of the internal energy to be a function of the material point.

Incidentally, note that the theorems proved here do not involve the Clausius-Duhem inequality or any consequence of it, in that their proofs only use the balance laws and a natural axiom of physical possibility, Axiom 6.1, which usually is assumed and used in an implicit manner. Hence the uniqueness theorems proved here hold for materials of the differential type also in any theory which has a dissipation inequality that differs from the Clausius-Duhem one.

2 – Solutions of certain multiple symmetric systems

Assume that (i) $\mathcal{V}$ is an inner product space of dimension three on the real field R, referred to an orthonormal basis; (ii) $\mathcal{T}^\nu(\mathcal{V})$ is the linear space of tensors of order $\nu \geq 1$, equipped with the topology induced by the associated inner product; (ii) u^p is an open connected subset of $\mathcal{T}^\nu(\mathcal{V})$ for $p = 0, 1, \dots, P$. Let

$$(2.1)\quad \widehat{\mathbf{Q}} : u^0 \times u^1 \times \dots \times u^P \to \tau^\nu(\mathcal{V}), \quad \mathbf{Q} = \widehat{\mathbf{Q}}(\overset{0}{\mathbf{Y}}, \overset{1}{\mathbf{Y}}, \dots, \overset{P}{\mathbf{Y}}), \quad (\nu \geq 1),$$

be a smooth function. The next theorem will state the equivalence of the assertions (2.A) through (2.C) below.

(2.A) [(2.B)]. *The function* (2.1) *is a* $C^2[C^\infty]$*-solution on* $u \times u^1 \times \dots \times u^P$ *to the system of symmetric equations*

$$(2.2)\quad \begin{array}{l} \partial Q^{a(A} / \partial \overset{p}{Y}_{bB)} = 0^{(3)} \ \textit{for}\ p \in \{0, 1, \dots, P\}, b, A, B \in \{1, 2, 3\}\ \textit{and} \\ a \in \{1, 2, 3\}^{\nu - 1} \qquad (\{1, 2, 3\}^0 = \emptyset)\,. \end{array}$$

(3) For any tensor X its *symmetric* and *skew-symmetric parts* with respect to a and b are denoted by $X^{\cdots(a\dots b)\cdots} = (X^{\cdots a \dots b} + X^{\cdots b \dots a \cdots})/2$ and $X^{\cdots[a\dots b]\cdots} = (X^{\cdots a \dots b \cdots} - X^{\cdots b \dots a \cdots})/2$, respectively.

(2.C). *For* $i, j = 0, 1, \ldots, P$ *there are tensors* $\overset{[0]}{\sigma}, \overset{[ii]}{\tau} \in \mathcal{T}^{\nu}(\mathcal{V})$ *and* $\overset{[i]}{\tau}, \overset{[ij]}{\tau} \in \mathcal{T}^{\nu+1}(\mathcal{V})$ *such that*

$$
\begin{aligned}
Q^{aA} = {\overset{[0]}{\sigma}}{}^{aA} + \sum_{i=0}^{P} {\overset{[i]}{\tau}}{}^{abC} \varepsilon^{ABC} \overset{i}{Y}_{bB} + \sum_{i=0}^{P} {\overset{[ii]}{\tau}}{}^{ah} \varepsilon^{hbc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{i}{Y}_{cC} + \\
+ \sum_{\substack{ij=0 \\ i \neq j}}^{P} {\overset{[ij]}{\tau}}{}^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{i}{Y}_{cC} \qquad (\overset{0}{Y}_{aA} = Y_{aA}) .
\end{aligned}
\tag{2.3}
$$

THEOREM 2.1. *The three assertions* (2.A), (2.B) *and* (2.C) *are equivalent.*

For the proof see [13, § 5].

3 – Physically remarkable solutions of the above symmetric systems

Next we characterize the class of the solutions to the multiple symmetric system (2.2), which satisfy conditions (3.A) and (3.B) below. The first condition expresses the property of Galilean invariance, the second the balance law of angular momentum, provided the function (2.1) be interpreted as the difference between any two admissible response functions for the stress in a body of the differential type with complexity $(P, 0, 0)$ - see § 6 - connected with the same reference configuration. Let $\mathbf{Y}^{\mathsf{T}}$ be the transpose of tensor $\mathbf{Y}$.

(3.A). $\hat{\mathbf{Q}}(\mathbf{RY}, \mathbf{R}\overset{1}{\mathbf{Y}}, \ldots, \mathbf{R}\overset{P}{\mathbf{Y}}) = \mathbf{R}\hat{\mathbf{Q}}(\mathbf{Y}, \overset{1}{\mathbf{Y}}, \ldots, \overset{P}{\mathbf{Y}})$ *for each* $\mathbf{R} \in \mathrm{Orth}^{+}$ *and* $(\mathbf{Y}, \overset{1}{\mathbf{Y}}, \ldots, \overset{P}{\mathbf{Y}}) \in u \times u^{1} \times \ldots \times u^{P}$ - see (2.1).

(3.B). $\hat{\mathbf{Q}}(\mathbf{Y}, \overset{1}{\mathbf{Y}}, \ldots, \overset{P}{\mathbf{Y}})\mathbf{Y}^{\mathsf{T}} = [\hat{\mathbf{Q}}(\mathbf{Y}, \overset{1}{\mathbf{Y}}, \ldots, \overset{P}{\mathbf{Y}})\mathbf{Y}^{\mathsf{T}}]^{\mathsf{T}}$ *for each* $(\mathbf{Y}, \overset{1}{\mathbf{Y}}, \ldots, \overset{P}{\mathbf{Y}}) \in u \times u^{1} \times \ldots, \times u^{P}$.

The next Theorems 3.1-3.2 will relate the following assertions (3.C) through (3.F).

(3.C) [(3.D)]. *For* $\nu = 2$ *the function* (2.1) *is a* C^2*-solution on* $u \times u^1 \times \ldots \times u^P$ *to system* (2.2) *and satisfies condition* (3.A) [*both conditions* (3.A) *and* (3.B)].

(3.E). *For* $i, j = 0, 1, \ldots, P$ *there are vectors* $\overset{[i]}{\mathbf{d}}$ *and scalars* $\overset{[ij]}{d}$ *such that*

$$Q^{aA} = \sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{ABC} \overset{i}{Y}_{aB} + \sum_{i,j=0}^{P} \overset{[ij]}{d} \varepsilon^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{cC} \tag{3.2}$$
$$(A = 1, 2, 3)\,, \qquad \overset{[ij]}{d} = \overset{[ji]}{d}.$$

(3.F). *There is a scalar* d *such that*

$$Q^{aA} = d \varepsilon^{abc} \varepsilon^{ABC} Y_{bB} Y_{cC} \qquad (a, A = 1, 2, 3)\,. \tag{3.3}$$

THEOREM 3.1. *The two assertions* (3.C) *and* (3.E) *are equivalent.*

THEOREM 3.2. *The two assertions* (3.D) *and* (3.F) *are equivalent.*

PROOF OF THEOREM 3.1. [(3.C) $\Longrightarrow$ (3.E)] Assume (3.C); by Theorem 2.1 assertion (2.C) holds. Hence by (2.3) and (3.A) we have

$$\begin{aligned}
&\overset{[0]}{\sigma}{}^{aA} + \sum_{i=0}^{P} \overset{[i]}{T}{}^{abC} \varepsilon^{ABC} R_{bl} \overset{i}{Y}_{lB} + \sum_{i=0}^{P} \overset{[ij]}{T}{}^{ah} \varepsilon^{hbc} \varepsilon^{ABC} R_{bl} R_{cs} \overset{i}{Y}_{lB} \overset{i}{Y}_{sC} + \\
&+ \sum_{\substack{ij=0 \\ i \neq j}}^{P} \overset{[ij]}{T}{}^{abc} \varepsilon^{ABC} R_{bl} R_{cs} \overset{i}{Y}_{lB} \overset{j}{Y}_{sC} = R_{at} \overset{[0]}{T}{}^{tA} + \sum_{i=0}^{P} R_{at} \overset{[i]}{T}{}^{tlC} \varepsilon^{ABC} \overset{i}{Y}_{lB} + \\
&+ \sum_{i=0}^{P} R_{at} \overset{[ii]}{T}{}^{th} \varepsilon^{hls} \varepsilon^{ABC} \overset{i}{Y}_{lB} \overset{i}{Y}_{sC} + \sum_{\substack{ij=0 \\ i \neq j}}^{P} R_{at} \overset{[ij]}{T}{}^{tls} \varepsilon^{ABC} \overset{i}{Y}_{lB} \overset{j}{Y}_{sC}\,.
\end{aligned} \tag{3.4}$$

By equating the terms of equal degree in the two sides of (3.4), by the arbitrariness of the independent variables, we find

$$(3.5)\qquad \begin{gathered} \overset{[0]}{\sigma}{}^{aA} = R_{at}\overset{[0]}{\sigma}{}^{tA}\,, \qquad \overset{[i]}{T}{}^{abC}R_{bl} = R_{at}\overset{[i]}{T}{}^{tlC}\,, \\ \overset{[ii]}{T}{}^{ah}\varepsilon^{hbc}R_{bl}R_{cs} = R_{at}\overset{[ii]}{T}{}^{th}\varepsilon^{hls}\,, \qquad \overset{[ij]}{T}{}^{abc}R_{bl}R_{cs} = R_{at}\overset{[ij]}{T}{}^{tls}\,. \end{gathered}$$

Now the arbitrariness of $\mathbf{R}$ in Orth^+, Lemma 2.1 and equality $(3.5)_1$ yield

$$(3.6)\qquad \overset{[0]}{\sigma}{}^{aA} = 0\,.$$

Multiply both the sides of equality $(3.5)_2$ by R_{as}; then $\overset{[i]}{T}{}^{abC}R_{bl}R_{as} = \overset{[i]}{T}{}^{tlC}$ and Lemma 2.1 yields

$$(3.7)\qquad \overset{[i]}{T}{}^{abC} = \overset{[i]}{d}{}^{C}\delta^{ab} \quad \text{for some} \quad \overset{[i]}{\mathbf{d}} \in \mathcal{T}_1(\mathcal{V})\,.$$

Multiply both the sides of equality $(3.5)_3$ by $R_{ar}\varepsilon^{kls}$; we find

$$\begin{gathered} \overset{[ii]}{T}{}^{ah}\varepsilon^{hbc}\varepsilon^{kls}R_{bl}R_{cs}R_{ar} = R_{ar}R_{at}\overset{[ii]}{T}{}^{th}\varepsilon^{hls}\varepsilon^{kls}\,, \quad \text{i.e.} \\ \det\mathbf{R}\,\overset{[ii]}{T}{}^{ah}R_{hk}R_{ar} = \overset{[ii]}{T}{}^{rk}\,; \end{gathered}$$

hence[4] we have

$$(3.8)\qquad \overset{[ii]}{T}{}^{ah} = \overset{[ii]}{b}\delta^{ah} \quad \text{for some} \quad \overset{[ii]}{b} \in \mathbf{R} \quad (i = 1,2,3)\,.$$

Lastly multiply both the sides of equality $(3.5)_3$ by R_{ar}; we find

$$(3.9)\qquad \overset{[ij]}{T}{}^{abc} = \overset{[ij]}{b}\varepsilon^{abc} \quad \text{for some} \quad \overset{[ij]}{b} \in \mathbf{R} \quad (i,j = 1,2,3)\,.$$

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[4] Recall that (i) [(ii)] the tensor $\mathbf{V} \in \mathcal{T}^2(\mathcal{V})$ $[\mathbf{U} \in \mathcal{T}_1(\mathcal{V})]$ satisfies $V_{ab}R^a_i R^b_j = V_{ij}$ $[U_a R^a_i = U_i]$ $\forall\mathbf{R} \in \mathrm{Orth}^+$ if and only if $V_{ij} = d\delta_{ij}$ for some $d \in R$ $[\mathbf{U} = \mathbf{0}]$; (iii) the tensor $\mathbf{V} \in \mathcal{T}^3(\mathcal{V})$ satisfies $V_{abc}R^a_i R^b_j R^c_k = V_{ijk}$ $\forall\mathbf{R} \in \mathrm{Orth}^+$ if and only if $V_{ijk} = d\varepsilon_{ijk}$ for some $d \in R$ ($i,j,k = 1,2,3$ δ_{ij} = Kronecker delta, ε = Ricci tensor) – for instance see [12].

Thus equalities (2.3) and (3.6-9) yield

$$Q^{aA} = \sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{ABC} \overset{i}{Y}_{aB} + \sum_{i=0}^{P} \overset{[ii]}{b} \varepsilon^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{i}{Y}_{cC} + \sum_{\substack{i,j=0 \\ i \neq j}}^{P} \overset{[ij]}{b} \varepsilon^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{i}{Y}_{cC} . \tag{3.10}$$

The symmetry of $\varepsilon^{abc}\varepsilon^{ABC}\overset{i}{Y}_{bB}\overset{i}{Y}_{cC}$ in i and j implies that, in the last addend of the right-hand side of equality (3.10), $\overset{[ij]}{b}$ can be replaced with its symmetric part $\overset{[ij]}{d} = (\overset{[ij]}{b} + \overset{[ji]}{b})/2$; thus equality (3.10) yields (3.2). □

PROOF OF THEOREM 3.2. [(3.D) $\Longrightarrow$ (3.F)] Assume (3.D); (2.2) and (3.B) yield

$$Q^{aA}\overset{0}{Y}_{tA} = \sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{ABC} \overset{i}{Y}_{aB} \overset{0}{Y}_{tA} + \sum_{i,j=0}^{P} \overset{[ij]}{d} \varepsilon^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{cC} \overset{0}{Y}_{tA} =$$
$$= \sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{ABC} \overset{i}{Y}_{tB} \overset{0}{Y}_{aA} + \sum_{i,j=0}^{P} \overset{[ij]}{d} \varepsilon^{tbc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{cC} \overset{0}{Y}_{aA} = Q^{tA} \overset{0}{Y}_{aA} .$$

By equating the terms of equal degree in the two sides of the second equality above we find

$$\sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{ABC} \overset{i}{Y}_{aB} \overset{0}{Y}_{tA} = \sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{ABC} \overset{i}{Y}_{tB} \overset{0}{Y}_{aA} ,$$
$$\sum_{i,j=0}^{P} \overset{[ij]}{d} \varepsilon^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{cC} \overset{0}{Y}_{tA} = \sum_{i,j=0}^{P} \overset{[ij]}{d} \varepsilon^{tbc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{cC} \overset{0}{Y}_{aA} . \tag{3.11}$$

By taking the derivatives of both the sides of equality $(3.11)_1$ first with respect to $\overset{0}{Y}_{mM}$ and then with respect to $\overset{i}{Y}_{nN}$ for $i \neq 0$, we find

$$\sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{MBC} \overset{i}{Y}_{aB} \delta_{tm} + \overset{[0]}{d}{}^{C} \varepsilon^{AMC} \overset{0}{Y}_{tA} \delta_{am} = \sum_{i=0}^{P} \overset{[i]}{d}{}^{C} \varepsilon^{MBC} \overset{i}{Y}_{tB} \delta_{am} + \overset{[0]}{d}{}^{C} \varepsilon^{AMC} \overset{0}{Y}_{aA} \delta_{tm} \tag{3.12}$$

and

$$\overset{[i]}{d}{}^C \varepsilon^{MNC} \delta_{an} \delta_{tm} = \overset{[i]}{d}{}^C \varepsilon^{MNC} \delta_{tn} \delta_{am} .$$

The last equality multiplied by ε^{MNH} yields

$$\overset{[i]}{d}{}^H \delta_{an} \delta_{tm} = \overset{[i]}{d}{}^H \delta_{tn} \delta_{am} \quad (i = 1, \dots, P) . \tag{3.13}$$

By taking the derivatives of both the sides of equality (3.12) with respect to $\overset{0}{Y}_{nN}$ we find

$$\overset{[0]}{d}{}^C \delta_{an} \delta_{tm} (\varepsilon^{MNC} - \varepsilon^{NMC}) = \overset{[0]}{d}{}^C \delta_{tn} \delta_{am} (\varepsilon^{MNC} - \varepsilon^{NMC}) ;$$

hence (3.13) holds for $i = 0$ too. Now by the choice $a = n$, $t = m$ $a \neq t$, (3.13) yields

$$\overset{[i]}{d}{}^H = 0 \quad \text{for} \quad i = 0, \dots, P . \tag{3.14}$$

By taking the derivatives of both the sides of equality $(3.11)_2$ first with respect to $\overset{h}{Y}_{mM}$ and then with respect to $\overset{k}{Y}_{nN}$, by the symmetry of $\overset{[hk]}{d}$ in h and k, we find

$$\overset{[hk]}{d} \varepsilon^{amn} \overset{0}{Y}_{tA} = \overset{[hk]}{d} \varepsilon^{tmn} \overset{0}{Y}_{aA} \qquad (h, k = 1, \dots, P) . \tag{3.15}$$

Choose a, m, n and t such that $c \neq a \neq b \neq c$ and $t \neq a$; the last equality yields $\overset{[hk]}{d} \overset{0}{Y}_{tA} = 0$, which by the arbitrariness of $\overset{0}{Y}_{tA}$ yields

$$\overset{[hk]}{d} = 0 \quad \text{for} \quad h, k = 1, \dots, P . \tag{3.16}$$

Now by (3.15) equality $(3.11)_2$ rewrites as

$$\begin{aligned} &\overset{[oo]}{d} \varepsilon^{abc} \varepsilon^{ABC} \overset{0}{Y}_{bB} \overset{0}{Y}_{cC} \overset{0}{Y}_{tA} + 2 \sum_{i=0}^{P} \overset{[io]}{d} \varepsilon^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{0}{Y}_{cC} \overset{0}{Y}_{tA} = \\ &= \overset{[oo]}{d} \varepsilon^{tbc} \varepsilon^{ABC} \overset{0}{Y}_{bB} \overset{0}{Y}_{cC} \overset{0}{Y}_{aA} + 2 \sum_{i=0}^{P} \overset{[io]}{d} \varepsilon^{tbc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{0}{Y}_{cC} \overset{0}{Y}_{aA} . \end{aligned} \tag{3.17}$$

By taking the derivatives of both the sides of equality (3.17) first with respect to $\overset{h}{Y}_{mM}$, then with respect to $\overset{0}{Y}_{nN}$, by the symmetry of $\overset{[hk]}{d}$ in h and k, we find

$$\overset{[ho]}{d}\varepsilon^{amn}\varepsilon^{AMN}\overset{0}{Y}_{tA} = \overset{[ho]}{d}\varepsilon^{tmn}\varepsilon^{AMN}\overset{0}{Y}_{aA}\,.$$

This equality is similar to (3.15) and thus by the same procedure used to deduce (3.16) one finds

$$\overset{[ho]}{d} = 0 = \overset{[oh]}{d} \qquad (h = 1,\dots,P)\,. \tag{3.18}$$

By (3.18) equality (3.17) becomes

$$\overset{[oo]}{d}\varepsilon^{abc}\varepsilon^{ABC}\overset{0}{Y}_{bB}\overset{0}{Y}_{cC}\overset{0}{Y}_{tA} = \overset{[oo]}{d}\varepsilon^{tbc}\varepsilon^{ABC}\overset{0}{Y}_{bB}\overset{0}{Y}_{cC}\overset{0}{Y}_{aA}\,,$$

which identity holds because both its sides equal $2\overset{[oo]}{d}\det(\overset{0}{\mathbf{Y}})\delta_{at}$; hence this equality does not restrict the choice of $\overset{[oo]}{d}$. Lastly (3.14, 16, 18) yield (3.3). □

4 – More complex multiple symmetric systems

Next we consider smooth functions of the kind

$$\hat{\mathbf{Q}} : u^0 \times u^1 \times .. \times u^P \times u^{P+1} \times .. \times u^{P+R+1} \to \mathcal{T}^\nu(\mathcal{V}) \quad (\nu \geq 1)\,, \tag{4.1}$$
$$(\overset{0}{\mathbf{Y}},\dots,\overset{P}{\mathbf{Y}},\overset{0}{\mathbf{Z}},\dots,\overset{R}{\mathbf{Z}}) \to \mathbf{Q} = \hat{\mathbf{Q}}(\overset{0}{\mathbf{Y}},\dots,\overset{P}{\mathbf{Y}},\overset{0}{\mathbf{Z}},\dots,\overset{R}{\mathbf{Z}})\,,$$

where (a) $P \geq 0$, $R \geq 0$, (b) u^i is an open connected subset of $\mathcal{T}_2(\mathcal{V})$ ($i = 0, 1, \dots, P$) and (c) u^{P+1+i} is an open connected subset of $\mathcal{T}_1(\mathcal{V})$ ($\approx \mathcal{V}$) for $i = 0, \dots, R$.

Now consider the $(P + R + 2)$-fold system (4.2) and the assertions (4.A-B) below.

(4.A) [(4.B)]. *The function* (4.1) *is a* $C^2[C^\infty]$*-solution on* $u \times u^1 \times \dots \times u^{P+R+2}$ *to the system of symmetric equations*

$$\partial Q^{a(A}/\partial \overset{p}{Y}_{bB)} = 0 = \partial Q^{a(A}/\partial \overset{r}{Z}_{B)}\,, \tag{4.2}$$

with $(p,r) \in \{0,\dots,P\} \times \{0,\dots,R\}$, $a \in \{1,2,3\}^{\nu-1}$, $b, A, B \in \{1,2,3\}$, *and furthermore satisfies conditions* (3.A) - (3.B).

THEOREM 4.1. *For* $\nu = 2$ *the three assertions* (4.A), (4.B) *and* (3.F) *are equivalent.*

The proof of this theorem uses the following two lemmas.

LEMMA 4.1. *The function* (4.1) *is a* $C^2[C^\infty]$*-solution on* $u \times u^1$ *of the double system of symmetric equations*

$$\partial Q^{a(A}/\partial \overset{0}{Z}_{B)} = 0 = \partial Q^{a(A}/\partial \overset{1}{Z}_{B)} \qquad (a, A, B = 1,2,3) \tag{4.3}$$

if and only if

$$\begin{aligned} Q^{aA} &= V^{aA} + \overset{[0]}{V}{}^{aAB}\overset{0}{Z}_B + \overset{[i]}{V}{}^{aAB}\overset{i}{Z}_B + \overset{[0i]}{V}{}^{aABC}\overset{0}{Z}_B\overset{i}{Z}_C + \\ &+ \overset{[i0]}{V}{}^{aABC}\overset{i}{Z}_B\overset{0}{Z}_C \qquad (a, A, B = 1,2,3) \end{aligned} \tag{4.4}$$

for some tensors $\mathbf{V} \in \mathcal{T}_\nu(\mathcal{V})$, $\overset{[0]}{\mathbf{V}}, \overset{[1]}{\mathbf{V}} \in \mathcal{T}^{\nu+1}(\mathcal{V})$ *and* $\overset{[01]}{\mathbf{V}}, \overset{[10]}{\mathbf{V}} \in \mathcal{T}^{\nu+2}(\mathcal{V})$ *that are totally skew-symmetric in their capital indexes.*

LEMMA 4.2. *The function* (4.1) *is a* $C^2[C^\infty]$*-solution on* $u \times u^1 \times \dots \times u^{P+R+2}$ *to system* $(4.2)_2$ *if and only if*

$$Q^{aA} = V^{aA} + \sum_{i=0}^{R} \overset{[i]}{V}{}^{aAD}\overset{i}{Z}_D + \sum_{ij=0}^{R} \overset{[ij]}{V}{}^{aADE}\overset{i}{Z}_D\overset{i}{Z}_E \quad (a, A = 1,2,3) \tag{4.5}$$

for some tensor $\mathbf{V} \in \mathcal{T}^\nu(\mathcal{V})$, $\overset{[i]}{\mathbf{V}} \in \mathcal{T}^{\nu+1}(\mathcal{V})$ *and* $\overset{[ij]}{\mathbf{V}} \in \mathcal{T}^{\nu+2}(\mathcal{V})$ *that are skew-symmetric in their capital indexes and such that*

$$\overset{[ij]}{\mathbf{V}} = -\overset{[ji]}{\mathbf{V}} \qquad (i, j = 1,2,\dots,R). \tag{4.6}$$

PROOF OF LEMMA 4.1. If $\widehat{\mathbf{Q}}$ solves system $(4.3)_1$ $[(4.3)_2]$, then the equalities

$$(4.7)[(4.8)] \qquad Q^{aA} = \varphi^{aA} + \psi^{aAB} \overset{0}{Z}_B \qquad [Q^{aA} = \eta^{aA} + \lambda^{aAB} \overset{1}{Z}_B]$$

hold for some tensors $\varphi \in \mathcal{T}^{\nu}(\mathcal{V})$ and $\psi \in \mathcal{T}^{\nu+1}(\mathcal{V})$ $[\eta \in \mathcal{T}^{\nu}(\mathcal{V})$ and $\lambda \in \mathcal{T}^{\nu+1}(\mathcal{V})]$, where tensors ψ and λ are skew-symmetric in their capital indexes – see equality [11, (3.1)] for $\tau = 2$ and $n = 3$. Hence if $\widehat{\mathbf{Q}}$ solves the double system (4.3), then both equalities (4.7) and (4.8) hold and hence equality (4.4) holds for some tensors $\mathbf{V} \in \mathcal{T}_{\nu}(\mathcal{V})$, $\overset{[0]}{\mathbf{V}}$, $\overset{[1]}{\mathbf{V}} \in \mathcal{T}_{\nu+1}(\mathcal{V})$ and $\overset{[01]}{\mathbf{V}}$, $\overset{[10]}{\mathbf{V}} \in \mathcal{T}_{\nu+2}(\mathcal{V})$ that are skew-symmetric in their capital indexes. Conversely, it is a trivial task to verify that (4.4) solves system (4.3). □

PROOF OF LEMMA 4.2. By iterating the method used in the above proof, as there are no tensors that are totally skew-symmetric in more than three indexes, we have that (4.5) holds for some tensors $\mathbf{V} \in \mathcal{T}_{\nu}(\mathcal{V})$, $\overset{[i]}{\mathbf{V}} \in \mathcal{T}_{\nu+1}(\mathcal{V})$ and $\overset{[ij]}{\mathbf{V}} \in \mathcal{T}_{\nu+2}(\mathcal{V})$ that are skew-symmetric in their capital indexes – see (4.4). Since $\overset{[ii]}{V}{}^{aADE} \overset{i}{Z}_E \overset{i}{Z}_D = 0$ one can choose $\overset{[ii]}{\mathbf{V}} = 0$; by the identity

$$\sum_{ij=0}^{R} \overset{[ij]}{V}{}^{aADE} \overset{i}{Z}_D \overset{i}{Z}_E = \sum_{ij=0}^{R} \overset{[ji]}{V}{}^{aAED} \overset{i}{Z}_E \overset{i}{Z}_D \,,$$

obtained by renaming the dummy indexes, by the skew-symmetry of $\overset{[ij]}{\mathbf{V}}$ in its capital indexes, we have the equality

$$(4.9) \qquad \sum_{ij=0}^{R} \overset{[ij]}{V}{}^{aADE} \overset{i}{Z}_D \overset{i}{Z}_E = - \sum_{ij=0}^{R} \overset{[ji]}{V}{}^{aADE} \overset{i'}{Z}_D \overset{i}{Z}_E \,.$$

Hence the arbitrariness of $\overset{i}{\mathbf{Z}}$ yields (4.6) for $i \neq j$. □

PROOF OF THEOREM 4.1. If $\widehat{\mathbf{Q}}$ solves system $(4.2)_1$ and satisfies (3.A-B), then by Theorem 3.2 equalities (3.3) hold and thus it can be written as

$$(4.10) \qquad Q^{aA} = D^{abcABC} Y_{bB} Y_{cC} \qquad (a, A = 1, 2, 3) \,,$$

where the tensors $\mathbf{D} = \widehat{\mathbf{D}}\,(\overset{0}{\mathbf{Z}},\dots,\overset{R}{\mathbf{Z}}) \in \mathcal{T}_6(\mathcal{V})$ is totally skew-symmetric in a, b, c, A, B, C. If $\widehat{\mathbf{Q}}$ satisfies system $(4.2)_2$, then by Lemma 4.2

$$Q^{aA} = V^{aA} + \sum_{i=0}^{R} \overset{[i]}{V}{}^{aAD} \overset{i}{Z}_D + \sum_{ij=0}^{R} \overset{[ij]}{V}{}^{aADE} \overset{i}{Z}_D \overset{i}{Z}_E \tag{4.11}$$

for some tensors $\mathbf{V}$, $\overset{[i]}{\mathbf{V}}$ and $\overset{[ij]}{\mathbf{V}}$ that are totally skew-symmetric in their capital indexes. Hence if $\widehat{\mathbf{Q}}$ solves system (4.2), then (4.10) and (4.11) yield

$$\begin{aligned} Q^{aA} = \Bigg(& W^{abcABC} + \sum_{i=0}^{R} \overset{[i]}{W}{}^{abcABCD} \overset{i}{Z}_D + \\ & + \sum_{ij=0}^{R} \overset{[ij]}{W}{}^{abcABCDE} \overset{i}{Z}_D \overset{i}{Z}_E \Bigg) Y_{bB} Y_{cC} \end{aligned} \tag{4.12}$$

for some tensors $\mathbf{W}$, $\overset{i}{\mathbf{W}}$ and $\overset{[ij]}{\mathbf{W}}$ that are totally skew-symmetric in a, b, c. As $\overset{[i]}{\mathbf{W}}$ and $\overset{[ij]}{\mathbf{W}}$ are totally skew-symmetric in their capital indexes too, we have $\overset{[i]}{\mathbf{W}} = \overset{[ij]}{\mathbf{W}} = 0$; thus equality (3.3) holds for $\mathbf{W} = \mathbf{D}$.

[(3.F) $\Longrightarrow$ (4.A)] By Theorem 3.2 tensor (3.3) solves system $(4.2)_1$ and satisfies (3.A)-(3.B). It is a trivial task to verify that it solves system $(4.2)_2$ too. □

5 – Bodies of the differential type

From now onward we adopt the material description and we assume that:

$\mathcal{B}$ is a body; X is a material point; $\mathcal{K}$ is a reference configuration for $\mathcal{B}$;

$\mathcal{B}_\mathcal{K} = \mathcal{K}(\mathcal{B})$ is a closed regular region of $\mathbb{R}^3$(5);

$\mathbf{X} = \mathcal{K}(X)$ is the position of X in $\mathcal{B}_\mathcal{K}$ ($\mathbf{X}$ and X often will be identified);

(5) That is, $\mathcal{B}_\mathcal{K}$ is the closure of an open connected of the three-dimensional Euclidean space, identified with $\mathbb{R}^3$, and the boundary $\partial\mathcal{B}_\mathcal{K}$ of $\mathcal{B}_\mathcal{K}$ is piecewise smooth.

$\{x^a\}$ are spatial Euclidean coordinates in the inertial ambient space;

$\{X^A\}$ are material coordinates in the reference space;

$\rho_{\mathcal{K}} = \hat{\rho}_{\mathcal{K}}(\mathbf{X})$, with $\hat{\rho}_{\mathcal{K}} \in C^1(\mathcal{B}_{\mathcal{K}}, \mathbb{R}^+)$, is the mass-density;

$\mathbf{b} = \hat{\mathbf{b}}(\mathbf{X}, t)$ is the density per unit reference mass of the external and body forces and $r = \hat{r}(\mathbf{X}, t)$ is the density per unit reference mass of the heat supply[(6)];

$\hat{p} = (\hat{\mathbf{x}}, \widehat{\Theta})$, where $\hat{\mathbf{x}} : \mathcal{B}_{\mathcal{K}} \times \mathbb{R} \to \mathbb{R}^3$ and $\widehat{\Theta} : \mathcal{B}_{\mathcal{K}} \times \mathbb{R} \to \mathbb{R}^+$, denotes a (smooth enough) *physically possible* ($\mathcal{K}^-$) *process*[(7)], where

$$\text{(5.1)} \qquad \mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t) \quad \text{and} \quad \Theta = \widehat{\Theta}(\mathbf{X}, t)$$

denote the position and the temperature of $\mathbf{X}$ at time t, respectively.

As is customary,

$$\text{(5.2)} \qquad \mathbf{F} = \operatorname{grad} \hat{\mathbf{x}}(\mathbf{X}, t) \quad \text{and} \quad \mathbf{G} = \operatorname{grad} \widehat{\Theta}(\mathbf{X}, t)$$

denote the material position and temperature gradients, respectively, at $(\mathbf{X}, t)$.

Furthermore we put

$$\text{(5.3)} \qquad \begin{aligned} \overset{p}{\mathbf{F}} &= \frac{\partial^p}{(\partial t)^p} \operatorname{grad} \hat{\mathbf{x}}(\mathbf{X}, t), \qquad \overset{q}{\Theta} = \frac{\partial^q}{(\partial t)^q} \widehat{\Theta}(\mathbf{X}, t), \\ \overset{r}{\mathbf{G}} &= \frac{\partial^r}{(\partial t)^r} \operatorname{grad} \widehat{\Theta}(\mathbf{X}, t), \end{aligned}$$

for $p \in \{0, \dots, P\}$, $q \in \{0, \dots, Q\}$ and $r \in \{0, \dots, R\}$ ($\overset{0}{\mathbf{F}} = \mathbf{F}$, $\overset{0}{\Theta} = \Theta$, $\overset{0}{\mathbf{G}} = \mathbf{G}$).

(6) As is customary, here we regard the body force and the heat supply as "external" or "given" - see e.g. [17, p. 120] and [6, p. 143].

(7) As in [1] the concept of physical possibility is regarded here as primitive. By "$\hat{p}$ is a possible process" we mean "$\hat{p}$ could be carried out by ideal experimenters". The physical possibility of a process $\hat{p}$ is not equivalent to its compatibility with the axioms. The physically possible processes constitute a subclass of the processes compatible with axioms. The theorems which are proved here also hold in those theories $\mathcal{T}$ on continua which use conceivable processes instead, provided one postulates in $\mathcal{T}$ that, for any conceivable process p_c and material point $\mathbf{X}$, there is a physically possible process which locally coincides with p_c at $\mathbf{X}$. The results established in the present paper are reached by using physically possible processes, whose existence is assured by means of axioms of physical possibility - see Axiom 6.1 below - which only satisfy certain assigned local conditions, namely conditions at an arbitrary material point and at some time.

In the two theorems below, in which the function $\hat{f}$ must be interpreted as a response function, recall that (i) $\mathbf{F}, \dot{\mathbf{F}}, \dots, \overset{P}{\mathbf{F}}$ are two-point tensors, (ii) $\Theta, \dot{\Theta}, \dots, \overset{Q}{\Theta}$ are scalars, and (iii) $\mathbf{G}, \dot{\mathbf{G}}, \dots, \overset{R}{\mathbf{G}}$ are vectors in the reference configuration; hence, under a Galilean [Euclidean] transformation of the ambient space, they transform as

$$(5.4)\qquad \overset{p}{\mathbf{F}} \longmapsto \mathbf{R}\overset{p}{\mathbf{F}} \qquad \left[\overset{p}{\mathbf{F}} \longmapsto D^p(\mathbf{RF}) := \sum_{h=0}^{p} \binom{p}{h} \overset{h}{\mathbf{F}} \overset{p-h}{\mathbf{R}}\right],$$

$(p = 0, \dots, P)$, $\overset{q}{\Theta} \longmapsto \overset{q}{\Theta}$ $(q = 0, \dots, Q)$, and $\overset{r}{\mathbf{G}} \longmapsto \overset{r}{\mathbf{G}}$ $(r = 0, \dots, R)$, respectively. For the sake of simplicity we put

$$(5.5)\qquad ((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}})) = (\mathbf{F}, \dot{\mathbf{F}}, \dots, \overset{P}{\mathbf{F}}, \Theta, \dot{\Theta}, \dots, \overset{Q}{\Theta}, \mathbf{G}, \dot{\mathbf{G}}, \dots, \overset{R}{\mathbf{G}}).$$

As is customary Lin [Orth$^+$] denotes the set of second tensors [proper rotations] on $\mathcal{V}$. Let u be an open connected subset of Lin $^+$; for $p = 1, \dots, P$ let u^p be an open connected subset of Lin ; let v be an open connected subset of $\mathbb{R}^+$; for $q = 1, \dots, Q$ let v^q be an open connected subset of $\mathbb{R}$; and for $r = 0, \dots, R$ let $\mathbf{v}^r$ be an open connected subset of $\mathbb{R}^3$; let $f = \hat{f}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}))$ be a function with domain

$$D = u \times u^1 \times \dots \times u^P \times v \times v^1 \times \dots \times v^Q \times \mathbf{v} \times \mathbf{v}^1 \times \dots \times \\ \times \mathbf{v}^R \subset \mathrm{Lin}^{\,P+1} \times \mathbb{R}^{Q+1} \times (\mathbb{R}^3)^{R+1},$$

and codomain Lin [$\mathbb{R}$ or $\mathbb{R}^3$].

THEOREM 5.1. *The above function* $f = \hat{f}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}))$ *of codomain* Lin [$\mathbb{R}$ *or* $\mathbb{R}^3$] *is Galilean invariant if* $\hat{f}(\mathbf{R}(\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}})) = \mathbf{R}\hat{f}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}))$ $\left[\hat{f}(\mathbf{R}(\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}})) = \hat{f}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}))\right]$ *for each* $\mathbf{R} \in \mathrm{Orth}^+$ *and* $((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}})) \in D$.

THEOREM 5.2. *The above function* $f = \hat{f}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}))$ *of codomain* Lin [$\mathbb{R}$ *or* $\mathbb{R}^3$] *is Euclidean invariant if* $\hat{f}((D^p(\mathbf{RF})), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}})) = \mathbf{R}\hat{f}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}))$ $\left[\hat{f}((D^p(\mathbf{RF})), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}})) = \hat{f}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}))\right]$ *for each*

$\widehat{\mathbf{R}} \in C^P([-\varepsilon,0],\mathrm{Orth}^+)$, $((\overset{p}{\mathbf{F}}),(\overset{q}{\Theta}),(\overset{r}{\mathbf{G}})) \in D$, *and* $\widehat{\mathbf{F}} \in C^P([-\varepsilon,0],u)$ *such that* $\Big(\frac{d}{dt}\mathbf{F}(t),\dots,\frac{d^P}{(dt)^P}\mathbf{F}(t)\Big) \in u^1 \times \dots \times u^P$ $\forall t \in [-\varepsilon,0]$, *where in* (5.4) $\overset{i}{\mathbf{R}} = \frac{d^i\widehat{\mathbf{R}}}{(dt)^i}(0)$, $\overset{i}{\mathbf{F}} = \frac{d^i\widehat{\mathbf{F}}}{(dt)^i}(0)$ *for* $i = 0,\dots,P$, *and* $\varepsilon > 0$.

Recall that, obviously, Euclidean invariance implies Galilean invariance.

6 – Bodies of the differential type and arbitrary complexity

For any $(P,Q,R) \in \{0,1,2,\dots\}^3$ a body $\mathcal{B}_{PQR}$ whose constitutive equations depend on the variables $\mathbf{F},\dot{\mathbf{F}},\dots,\overset{P}{\mathbf{F}}$, $\Theta,\dot{\Theta},\dots,\overset{Q}{\Theta}$, and $\mathbf{G},\dot{\mathbf{G}},\dots,\overset{R}{\mathbf{G}}$ is said to be a *body of the differential type and complexity* (P,Q,R). Thus thermoelastic bodies have complexity $(0,0,0)$. Bodies of the differential type and complexity $(1,0,0)$ are considered in [2] and in [3]. Bodies of the differential type and complexity $(P,0,0)$ are considered by NOLL in [16].

Following [1], [8] and [9] one can rigorously define generalized and ordinary systems of response functions, for these bodies, from the Mach-Painlevé point of view. Read the rigorous Definitions 3.1 and 3.2 written in [14]. Recall that possible indeterminations in the response functions are allowed by the aforementioned definitions – see [1], [8] and [9]. From now onward we assume that the response functions are Galilean invariant; the stronger condition of Euclidean invariance is used in Theorem 9.1.

Let

$\mathbf{P}_\mathcal{K}$	represent the *first Piola stress tensor*,
$\mathbf{q}_\mathcal{K}$	represent the *material heat flux vector*,
$e_\mathcal{K}$	represent the *internal energy per unit mass in the reference configuration*,
$\eta_\mathcal{K}$	represent the *entropy per unit mass in the reference configuration*,
$\mathbf{q}_\mathcal{K}\cdot\mathbf{N}$	represent the *rate of flux of heat energy by conduction from* $\mathcal{P}_\mathcal{K}$ *to its exterior across the boundary* $\partial\mathcal{P}_\mathcal{K}$ *at its material point* $\mathbf{X}$.

The global balance laws imply the following local laws

$$\rho_{\mathcal{K}}\dot{\mathbf{v}} = \rho_{\mathcal{K}}\mathbf{b} + \mathrm{Div}\,\mathbf{P}_{\mathcal{K}}\,, \tag{6.1}$$

$$\mathbf{P}_{\mathcal{K}}\mathbf{F}^{\mathsf{T}} = (\mathbf{P}_{\mathcal{K}}\mathbf{F}^{\mathsf{T}})^{\mathsf{T}}\,, \tag{6.2}$$

$$\rho_{\mathcal{K}}\dot{e}_{\mathcal{K}} = \mathbf{P}_{\mathcal{K}}\cdot\dot{\mathbf{F}} + \rho_{\mathcal{K}}r - \mathrm{Div}\,\mathbf{q}\,, \tag{6.3}$$

$$\rho_{\mathcal{K}}\dot{\eta}_{\mathcal{K}} \geq \rho_{\mathcal{K}}\Theta^{-1}r - \mathrm{Div}(\Theta^{-1}\mathbf{q}_{\mathcal{K}})\,. \tag{6.4}$$

With the help of (6.1)-(6.3), inequality (6.4) yields the reduced dissipation inequality

$$\rho_{\mathcal{K}}(\dot{\psi}_{\mathcal{K}} + \dot{\Theta}\eta_{\mathcal{K}}) - \mathbf{P}_{\mathcal{K}}\cdot\dot{\mathbf{F}} + \Theta^{-1}\mathbf{q}_{\mathcal{K}}\cdot\mathbf{G} \leq 0\,, \tag{6.5}$$

where $\psi_{\mathcal{K}} = e_{\mathcal{K}} - \Theta\eta_{\mathcal{K}}$ is the free energy.

Experiments of contact between two different bodies can be easily realized. There is no reason not to postulate their physical possibility. For instance MÜLLER [15] uses this kind of experiments[(8)].

Hence we assume the natural axiom of contact Axiom 3.1 in [14], which postulates the possibility to put any material point on $\mathcal{B}$'s boundary in a suitable contact with a certain thermoelastic body $\mathcal{B}_{te}$. Recall the definition [14, Definition 3.2] of *ordinary system of response functions* for a differential body (see also the informal definition given here in § 1). By the uniqueness properties of the response functions of $\mathcal{B}_{te}$, proved in [8] and [9], in [14] we prove that

if $\hat{\sigma}_{\mathcal{K}} = (\widehat{\mathbf{P}}_{\mathcal{K}}, \hat{\mathbf{q}}_{\mathcal{K}}, \hat{e}_{\mathcal{K}}, \hat{\eta}_{\mathcal{K}})$ *and* $\tilde{\sigma}_{\mathcal{K}} = (\widetilde{\mathbf{P}}_{\mathcal{K}}, \tilde{\mathbf{q}}_{\mathcal{K}}, \tilde{e}_{\mathcal{K}}, \tilde{\eta}_{\mathcal{K}})$ *are two ordinary systems of response functions for the differential body* $\mathcal{B}$, *then the transformation* $\hat{\sigma}_{\mathcal{K}} \rightarrow \tilde{\sigma}_{\mathcal{K}}$ *does not affect any initial-boundary value problem for* $\mathcal{B}$ *(that is* $\hat{\sigma}_{\mathcal{K}}$ *and* $\tilde{\sigma}_{\mathcal{K}}$ *are physically equivalent in a satisfactory sense).*

We also assume the physical possibility Axiom 3.2 in [14], which postulates the possibility to attain any admissible assignment of values for the local state variables, at each internal material point of the body, with a large degree of arbitrariness (a) in their time derivatives, and (b) in their material gradients.

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(8) "I consider a wall between two bodies of different material I and II which has the property that the temperature is continuous across this wall for all processes in the bodies I and II, ... Therefore, by considering an ideal wall between a viscous heat-conducting fluid and an isotropic thermo-elastic solid, ... we may conclude, ..." – see [15, p. 237].

The following assertion will be used in the statements below.

(6.A). *Both* $\hat{\sigma}_{\mathcal{K}} = (\widehat{\mathbf{P}}_{\mathcal{K}}, \hat{\mathbf{q}}_{\mathcal{K}}, \hat{e}_{\mathcal{K}}, \hat{\eta}_{\mathcal{K}})$ *and* $\tilde{\sigma}_{\mathcal{K}} = (\widetilde{\mathbf{P}}_{\mathcal{K}}, \tilde{\mathbf{q}}_{\mathcal{K}}, \tilde{e}_{\mathcal{K}}, \tilde{\eta}_{\mathcal{K}})$ *are general systems of (Galilean invariant) response functions for the body* $\mathcal{B}$ *of the differential type and complexity* $(P, Q, R) \in \{0, 1, 2, \dots\}^3$, *referred to the configuration* $\mathcal{K}$; *and let*

$$
\begin{aligned}
E = \widehat{E}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), X) &= \tilde{e}_{\mathcal{K}}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), X) + \\
&\quad - \hat{e}_{\mathcal{K}}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X}), \\
S = \widehat{S}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X}) &= \tilde{P}_{\mathcal{K}}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X}) + \\
&\quad - \widehat{P}_{\mathcal{K}}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X}), \\
Q = \widehat{Q}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X}) &= \tilde{q}_{\mathcal{K}}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X}) + \\
&\quad - \hat{q}_{\mathcal{K}}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X}).
\end{aligned}
\tag{6.6}
$$

LEMMA 6.1. *Assume* (6.A). *Then equality* (6.8) *below holds along any smooth enough process that the body can undergo.*

PROOF. Assume (6.A); by subtracting the energy balance law (6.3) from the same written in $\tilde{\sigma}_{\mathcal{K}}$, we find

$$
\rho_{\mathcal{K}} \dot{E} = \mathbf{S} \cdot \dot{\mathbf{F}} - \operatorname{Div} \mathbf{Q}, \tag{6.7}
$$

which by (6.1) and (6.7) at any $((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), (\overset{r}{\mathbf{G}}), \mathbf{X})$ can be written as

$$
\begin{aligned}
&\rho_{\mathcal{K}} \left(\sum_{p=0}^{P} \frac{\partial E}{\partial \overset{p}{\mathbf{F}}} \cdot \overset{p+1}{\mathbf{F}} + \sum_{q=0}^{Q} \frac{\partial E}{\partial \overset{q}{\Theta}} \overset{q+1}{\Theta} + \sum_{r=0}^{R} \frac{\partial E}{\partial \overset{r+1}{\mathbf{G}}} \cdot \overset{r+1}{\mathbf{G}} \right) - \mathbf{S} \cdot \dot{\mathbf{F}} + \\
&+ \sum_{p=0}^{P} \frac{\partial Q^H}{\partial \overset{p}{F}{}^i_L} \overset{p}{F}{}^i_{LH} + \sum_{q=0}^{Q} \frac{\partial Q^H}{\partial \overset{q}{\Theta}} \overset{q}{\Theta}_H + \sum_{r=0}^{R} \frac{\partial Q^H}{\partial \overset{r}{G}_L} \overset{r}{G}_{LH} + \frac{\partial Q^H}{\partial X^H} = 0 \\
&\qquad \left(f_H = \frac{\partial f}{\partial X^H}, \qquad \Theta_H = G_H \right).
\end{aligned}
\tag{6.8}
$$

□

7 – On the response function for the stress

Exactly as in the thermoelastic case the next theorem states that the difference between any two generalized response functions for the Cauchy stress is a constant Euclidean pressure.

THEOREM 7.1. *Assume* (6.A). *Then*

(7.A) *the difference* $\widehat{\mathbf{S}}$ *between the generalized response functions for the Piola-stress* $\widehat{\mathbf{P}}_{\mathcal{K}}$ *and* $\widetilde{\mathbf{P}}_{\mathcal{K}}$ *is given by*

$$\mathbf{S} = \widehat{\mathbf{S}}(F) = J\omega\mathbf{F}^{-\mathrm{T}} \qquad (J = \det\mathbf{F}) \tag{7.1}$$

for some $\omega \in \mathbb{R}$.

Note that $\mathbf{S}$ is the Piola-transform of the Euclidean pressure $\omega\mathbf{I}$ defined in the actual configuration.

PROOF. Assume (6.A); by subtracting the linear momentum law (6.1) written in $\hat{\sigma}_{\mathcal{K}}$ with the same written in $\tilde{\sigma}_{\mathcal{K}}$ we find

$$\operatorname{Div}\mathbf{S} = \mathbf{0}\,, \tag{7.2}$$

and thus by $(6.6)_1$ we have

$$\sum_{p=0}^{P} \frac{\partial S^{aA}}{\partial \overset{p}{F}{}^{b}_{B}} \overset{p}{F}{}^{b}_{bA} + \sum_{q=0}^{Q} \frac{\partial S^{aA}}{\partial \overset{q}{\Theta}} \overset{q}{G}_{A} + \sum_{r=0}^{R} \frac{\partial S^{aA}}{\overset{r}{G}_{B}} \overset{r}{G}_{BA} + \frac{\partial S^{aA}}{\partial X^{A}} = 0\,. \tag{7.3}$$

Now let $\mathbf{X} \in \overset{0}{\mathcal{B}}_{\mathcal{K}}$, $((\overset{p}{\mathbf{F}}),(\overset{q}{\Theta}),(\overset{r}{\mathbf{G}})) \in \mathcal{D}_{\mathcal{K}}(\mathbf{X})$ – see [14, Definition 3.1] – $A, b, B \in \{1,2,3\}$ and $(p,r) \in \{0,\dots,P\} \times \{0,\dots,R\}$; by choosing $u_p = 0 = v_r$ – see [14, Axiom 3.2] – equality (7.3) yields

$$\sum_{q=0}^{Q} \frac{\partial S^{aA}}{\partial \overset{q}{\Theta}} \overset{q}{G}_{A} + \frac{\partial S^{aA}}{\partial X^{A}} = 0 \qquad (a = 1,2,3)\,; \tag{7.4}$$

hence equalities (7.3)-(7.4) yield

$$\sum_{i=0}^{P} \frac{\partial S^{aA}}{\partial \overset{i}{F}{}^{b}_{B}} \overset{i}{F}{}^{b}_{BA} + \sum_{j=0}^{R} \frac{\partial S^{aA}}{\overset{j}{G}} \overset{j}{G}_{BA} = 0\,. \tag{7.5}$$

By [14, Axiom 3.2] again, for $\varepsilon > 0$, $p = 0, 1, \ldots, P$, and $q = 0, 1, \ldots, Q$, one can choose

$$(u_i, v_j) = (\delta_{ip}\varepsilon, 0) \quad \text{and} \quad (u_i, v_j) = (0, \delta_{jr}\varepsilon);$$

hence (7.5) yields

$$\text{(7.6)} \qquad \begin{array}{l} \partial S^{a(A}/\partial \overset{p}{F}{}^{b}_{B)} = 0 \quad \text{and} \quad \partial S^{a(A}/\partial \overset{r}{G}_{B)} = 0 \\ (p = 0, 1, \ldots, P, \quad q = 0, 1, \ldots, Q) . \end{array}$$

Therefore Theorem 4.1 yields

$$\text{(7.7)} \qquad S^{aA} = \omega \varepsilon^{abc} \varepsilon^{ABC} F_{bB} F_{cC} \quad \left(\omega = \hat{\omega}((\overset{q}{\Theta}), X)\right),$$

that is, equality (7.1) holds for some C^2-function $\hat{\omega}$. Now (7.4) and (7.7) yield $\left(\sum_{q=0}^{Q} \frac{\partial \omega}{\partial \overset{q}{\Theta}} \overset{q}{G}_A + \frac{\partial \omega}{\partial X^A}\right) \varepsilon^{abc} \varepsilon^{ABC} F_{bB} F_{cC} = 0$, that is

$$\text{(7.8)} \qquad \sum_{q=0}^{Q} \frac{\partial \omega}{\partial \overset{q}{\Theta}} \overset{q}{G}_A + \frac{\partial \omega}{\partial X^A} = 0 \quad \left(\omega = \hat{\omega}((\overset{q}{\Theta}), X)\right).$$

By $(7.8)_2$ and the arbitrariness of $\overset{q}{G}_A$ equality $(7.8)_1$ yields $\partial \omega / \partial \overset{q}{\Theta} = 0$ $(q = 0, \ldots, Q)$ and $\partial \omega / \partial X^A = 0$ $(A = 1, 2, 3)$; that is $\omega \in \mathbb{R}$. □

8 – Uniqueness of the ordinary response function for the stress

From [14, § 3] we recall the notion of ordinary system of response functions for the differential body $\mathcal{B}_{PQR}$ of any given complexity $(P, Q, R) \in \{0, 1, 2, \ldots\}^3$.

Let $\mathcal{B}_{te}$ be a thermoelastic body. The generalized system of response functions $\bar{\sigma}_{te} = (\overline{\mathbf{P}}_{te}, \bar{\mathfrak{q}}_{te}, \bar{e}_{te}, \bar{\eta}_{te})$ for $\mathcal{B}_{te}$ is said to be *boundary-ordinary* if $\overline{\mathbf{P}}_{te}$ and $\bar{\mathfrak{q}}_{te}$ give rise to zero normal stress and heat fluxes, respectively, on any portion of $\mathcal{B}_{te}$'s boundary, whenever this portion is put in contact with vacuum – see [14, Definition 3.2 a].

We say that the generalized system of response functions $\hat{\sigma}_{\mathcal{K}} = (\hat{\mathbf{P}}_{\mathcal{K}}, \hat{\mathbf{q}}_{\mathcal{K}}, \hat{e}_{\mathcal{K}}, \hat{\eta}_{\mathcal{K}})$ for the differential body $\mathcal{B}$ is *ordinary* if for any point $\mathbf{X} \in \partial\mathcal{B}_{\mathcal{K}}$ the conditions

$$\hat{\mathbf{P}}_{\mathcal{K}}\mathbf{N} = \overline{\mathbf{P}}_{te}\mathbf{N} \quad \text{and} \quad \hat{\mathbf{q}}_{\mathcal{K}} \cdot \mathbf{N} = \bar{\mathbf{q}}_{te} \cdot \mathbf{N}$$

hold on some surface neighbourhood σ ($\subset \partial\mathcal{B}_{\|}$) of $\mathbf{X}$ in case $\mathcal{B}$ is put in contact through σ with a given thermoelastic body $\mathcal{B}_{te}$ in a certain way (rigorously specified by Axiom 3.1 in [14]), and furthermore $\overline{\mathbf{P}}_{te}$ and $\bar{\mathbf{q}}_{te}$ belong to a boundary-ordinary system of response functions for $\mathcal{B}_{te}$ – see [14, Definition 3.2 b].

Obviously one assumes that (i) for any thermoelastic body there is a boundary-ordinary system of response functions and (ii) for any differential body there is an ordinary system of response functions.

Lastly, a response function, e.g. for the stress, is said to be generalized [ordinary] if it belongs to a generalized [ordinary] system of response functions for the body.

By theorem 7.1 the difference between any two generalized response functions for the Piola stress is the Piola transform of an Euclidean pressure defined in the actual configuration. Hence by the aforementioned contact axiom ([14, Axiom 3.1]), one can easily prove the strict uniqueness for the ordinary response function of the stress in any differential body.

THEOREM 8.1. *Assume* (6.A), *where* $\hat{\mathbf{P}}_{\mathcal{K}}$ *and* $\tilde{\mathbf{P}}_{\mathcal{K}}$ *are ordinary response functions for the Piola stress. Then* $\hat{\mathbf{P}}_{\mathcal{K}} = \tilde{\mathbf{P}}_{\mathcal{K}}$.

9 – On the uniqueness of the response function for the internal energy

Next we prove the uniqueness property for the response function of the internal energy in any body of the differential type and complexity $(P, 0, 0)$, with $P = 0, 1, 2, \ldots$ [(9)].

Let $\hat{E}$ be the difference between any two response functions for the internal energy and let $\hat{\mathbf{Q}}$ be the difference between any two response

[(9)]In order to prove the uniqueness of the internal energy, the response functions are assumed to be Euclidean invariant.

functions for the heat flux in a body $\mathcal{B}_{PQ0}$ with $P, Q = 0, 1, 2, \ldots$. Then, under a certain condition regarding either $\widehat{E}$ or $\widehat{\mathbf{Q}}$ (i.e. either $(9.2)_1$ or $(9.2)_2$), we prove that $\widehat{E}$ depends on the material point only – see Theorem 9.1 (b). To prove these properties we use Theorems 7.1 - 8.1 and the equality $\operatorname{Div} \mathbf{Q} = 0$, which is proved in [14, Lemma 6.2].

THEOREM 9.1 (a). *Assume* (6.A) *where* $Q = 0 = R$, *the response functions are Euclidean invariant, and both* $\widehat{\mathbf{P}}_{\mathcal{K}}$ *and* $\widetilde{\mathbf{P}}_{\mathcal{K}}$ *are ordinary response functions for the Piola stress. Then*

$$E = \widehat{E}(\mathbf{X}). \tag{9.1}$$

(b) *Under the assumptions above, where* $Q \geq 1$ *and* $R = 0$, *the two equalities below are equivalent and imply* (9.1).

$$\frac{\partial \widehat{E}}{\partial \mathbf{G}} \equiv \mathbf{0}, \qquad \frac{\partial \widehat{\mathbf{Q}}}{\partial \dot{\Theta}} \equiv \mathbf{0}. \tag{9.2}$$

PROOF OF (a) Under the assumptions in (a), by Theorem 8.1 we have $\widehat{\mathbf{S}} \equiv 0$; hence equality (6.8) becomes

$$\rho_{\mathcal{K}} \dot{E} = -\operatorname{Div} \mathbf{Q}. \tag{9.3}$$

By the hypothesis of Euclidean invariance, adopted in [14], we can use [14, Lemma 6.3]; hence we have $\operatorname{Div} \mathbf{Q} = 0$; thus (6.8) yields $\dot{E} = 0$, that is

$$\sum_{p=0}^{P} \frac{\partial E}{\partial \overset{p}{\mathbf{F}}} \cdot \overset{p+1}{\mathbf{F}} + \frac{\partial E}{\partial \Theta} \dot{\Theta} + \frac{\partial E}{\partial \mathbf{G}} \cdot \dot{\mathbf{G}} = 0 \tag{9.4}$$

– see $(6.6)_1$.

By the arbitrariness of $\overset{P+1}{\mathbf{F}}$, $\dot{\Theta}$ and $\dot{\mathbf{G}}$ equality (9.4) yields

$$\frac{\partial E}{\partial \overset{P}{\mathbf{F}}} = \mathbf{0}, \quad \frac{\partial E}{\partial \Theta} = 0, \quad \frac{\partial E}{\partial \mathbf{G}} = \mathbf{0}. \tag{9.5}$$

Now the arbitrariness of $\overset{P}{\mathbf{F}}$ in equality (9.4) yields $\partial E / \partial \overset{P-1}{\mathbf{F}} = 0$. By repeating the last step $P - 1$ times yields equality (9.1). □

PROOF OF (b) Under the assumption in (b) Theorem 8.1 yields $\hat{\mathbf{S}} \equiv \mathbf{0}$; hence equality (6.8) becomes (9.3). By the hypothesis of Euclidean invariance, adopted in [14], and Lemmas 6.2-6.3 in [14], both equality $\operatorname{Div} \mathbf{Q} = 0$ and the equivalence between equalities $(9.2)_1$ and $(9.2)_2$ follow; hence (6.8) yields

$$\dot{E} = \sum_{p=0}^{P} \frac{\partial E}{\partial \overset{p}{\mathbf{F}}} \cdot \overset{p+1}{\mathbf{F}} + \sum_{q=0}^{Q} \frac{\partial E}{\partial \overset{q}{\Theta}} \overset{q+1}{\Theta} + \frac{\partial E}{\partial \mathbf{G}} \cdot \dot{\mathbf{G}} = 0 \tag{9.6}$$

– see $(6.6)_1$.

By the arbitrariness of $\overset{P+1}{\mathbf{F}}$, $\overset{Q+1}{\Theta}$ and $\dot{\mathbf{G}}$, equality (9.6) yields

$$\frac{\partial E}{\partial \overset{P}{\mathbf{F}}} = \mathbf{0}\,, \quad \frac{\partial E}{\partial \overset{Q}{\Theta}} = 0\,, \quad \frac{\partial E}{\partial \mathbf{G}} = \mathbf{0}\,; \tag{9.7}$$

now by the arbitrariness of $\overset{P}{\mathbf{F}}$ and $\overset{Q}{\Theta}$ equalities (9.6)-(9.7) yield

$$\frac{\partial E}{\partial \overset{P-1}{\mathbf{F}}} = \mathbf{0}\,, \quad \frac{\partial E}{\partial \overset{Q-1}{\Theta}} = 0\,. \tag{9.8}$$

By iterating the last step we find (9.1). □

Note that if the response functions are not Euclidean invariant, but rather Galilean invariant only, then Lemma 6.3 in [14] cannot be used, and thus the equalities $\operatorname{Div} \mathbf{Q} = 0 = \dot{E}$ (see (9.3) and (9.6)) do not follow. Hence the thesis of the theorem does not follow.

10 – Indeterminateness of the constitutive functions

Next we consider a body $\mathcal{B}$ of the differential type and complexity $(P, Q, 0)$, with $Q \geq 1$. By Theorem 9.1 (b) we have that a necessary and sufficient condition that the difference between any two response functions for the internal energy in $\mathcal{B}$ be a function of the material point only, is that either equality $(9.2)_1$ or $(9.2)_2$ be true. We show that if equalities (9.2) do not hold for $\mathcal{B}$, then there is a threefold indetermination in its constitutive equations which cannot be detected by usual experiments. Indeed we prove the following assertion.

(10.A). *Let $\mathcal{B}$ be a body of the differential type and complexity* $(P, Q, 0)$, *with* $Q \geq 1$, *and let* $(\widehat{\mathbf{P}}_\mathcal{K}, \hat{\mathbf{q}}_\mathcal{K}, \hat{e}_\mathcal{K}, \hat{\eta}_\mathcal{K})$ *be either a generalized or an ordinary system of response functions for it. There are Euclidean invariant functions* $\widehat{E}$, $\widehat{\mathbf{Q}}$, *and* $\widehat{\mathbf{N}}$ *such that* $(\widehat{\mathbf{P}}_\mathcal{K}, \hat{\mathbf{q}}_\mathcal{K} + \widehat{\mathbf{Q}}, \hat{e}_\mathcal{K} + \widehat{E}, \hat{\eta}_\mathcal{K} + \widehat{\mathbf{N}})$ *is another such a system for $\mathcal{B}$ which does not satisfy both conditions* (9.1) *and* (9.2).

To prove assertion (10.A) firstly we seek three Euclidean invariant functions $E = \widehat{E}((\overset{q}{\Theta}), \mathbf{G}, \mathbf{X})$, $\mathbf{Q} = \widehat{\mathbf{Q}}((\overset{q}{\Theta}), \mathbf{G}, \mathbf{X})$ and $N = \widehat{N}((\overset{p}{\mathbf{F}}), (\overset{q}{\Theta}), \mathbf{G}, \mathbf{X})$ – see (5.4) – which identically satisfy the two equalities

$$\rho_\mathcal{K} \dot{E} = -\operatorname{Div} \mathbf{Q}, \quad \rho_\mathcal{K} \dot{N} = \operatorname{Div}(\Theta^{-1}\mathbf{Q}), \tag{10.1}$$

along any process of the body. Equality $(10.1)_1$ rewrites as

$$\begin{aligned} &\rho_\mathcal{K}\Big(\frac{\partial E}{\partial \Theta}\dot{\Theta} + \frac{\partial E}{\partial \overset{1}{\Theta}}\overset{2}{\Theta} + \ldots + \frac{\partial E}{\partial \overset{Q}{\Theta}}\overset{Q+1}{\Theta} + \frac{\partial E}{\partial G_A}\dot{G}_A\Big) = \\ &= -\Big(\frac{\partial Q^A}{\partial \Theta}G_A + \frac{\partial Q^A}{\partial \overset{1}{\Theta}}\overset{1}{G}_A + \ldots + \frac{\partial Q^A}{\partial \overset{Q}{\Theta}}\overset{Q}{G}_A + \\ &+ \frac{\partial Q^A}{\partial G_B}G_{BA} + \frac{\partial Q^A}{\partial X^A}\Big). \end{aligned} \tag{10.2}$$

As both $\widehat{E}$ and $\widehat{\mathbf{Q}}$ do not depend on $\overset{Q+1}{\Theta}$, equality (10.2) yields

$$\partial E / \partial \overset{Q}{\Theta} = 0. \tag{10.3}$$

Similarly, since $\widehat{E}$ and $\widehat{\mathbf{Q}}$ do not depend on $\overset{1}{\mathbf{G}}, \ldots, \overset{Q}{\mathbf{G}}$, we have

$$\partial Q^A / \partial \overset{q}{\Theta} = 0 \quad \text{for} \quad q \in \{2, \ldots, Q\} \tag{10.4}$$

and

$$\rho_\mathcal{K} \frac{\partial E}{\partial G_A} = -\frac{\partial Q^A}{\partial \dot{\Theta}}. \tag{10.5}$$

The left-hand side of equality (10.2) does not depend on $\overset{2}{\dot\Theta},\dots,\overset{Q}{\dot\Theta}$ and G_{AB}; thus (10.2-3) imply

$$\partial E/\partial \overset{q}{\dot\Theta} = 0 \quad \text{for} \quad q \in \{1,\dots,Q\}, \tag{10.6}$$

and

$$\partial Q^{(A}/\partial G_{B)} = 0 \qquad (A, B = 1,2,3). \tag{10.7}$$

Hence equalities (10.3-4-6) yield

$$E = \widehat{E}(\Theta, \mathbf{G}, \mathbf{X}), \qquad \mathbf{Q} = \widehat{\mathbf{Q}}(\Theta, \dot\Theta, \mathbf{G}, \mathbf{X}).$$

By Theorems 6.2 and 4.2* in [13] the equalities (10.7) yield

$$Q^A = W^{AB}(\Theta, \dot\Theta, \mathbf{X}) G_B + V^A(\Theta, \dot\Theta, \mathbf{X}) \tag{10.8}$$

for some vector $\mathbf{V}$ and some skew-symmetric tensor $\mathbf{W}$. By (10.8) taking the derivatives of both the sides of equality (10.5) yields

$$\rho_K \frac{\partial^2 E}{\partial G_A \partial G_B} = -\frac{\partial W^{AB}}{\partial \dot\Theta}. \tag{10.9}$$

Hence by the symmetry of the left-hand side and by the skew-symmetry of the right-hand side of equality (10.9) we have

$$\frac{\partial^2 E}{\partial G_A \partial G_B} = 0, \qquad \frac{\partial W^{AB}}{\partial \dot\Theta} = 0. \tag{10.10}$$

By $(10.10)_1$ the function $\widehat{E}$ is linear in G; hence we have

$$\rho_K \widehat{E} = f^A(\Theta, \mathbf{X}) G_A + g(\Theta, \mathbf{X}) \tag{10.11}$$

for some scalar functions $g(.,.)$ and $f^A(.,.)$ $(A = 1,2,3)$. Thus by (10.11) and $(10.10)_2$ equality (10.5) implies

$$Q^A = W^{AB}(\Theta, \mathbf{X}) G_B - \dot\Theta f^A(\Theta, \mathbf{X}) \quad (W^{AB} + W^{BA} = 0). \tag{10.12}$$

By (10.11-12) equality $(10.1)_1$ yields

$$\left(\frac{\partial f^A}{\partial \Theta} G_A + \frac{\partial g}{\partial \Theta}\right)\dot{\Theta} = -\left(\frac{\partial W^{AB}}{\partial \Theta} G_B - \dot{\Theta}\frac{\partial f^A}{\partial \Theta}\right) G_A + - \frac{\partial W^{AB}}{\partial X_A} G_B + \frac{\partial f^A}{\partial X_A}\dot{\Theta}, \tag{10.13}$$

hence

$$\frac{\partial g}{\partial \Theta} = \frac{\partial f^A}{\partial X_A} \quad \text{and} \quad \frac{\partial W^{AB}}{\partial X_A} = 0. \tag{10.14}$$

Equality $(10.12)_2$ yields $W^{AB} = \varepsilon^{ABC} W^C$ for some vector function $\mathbf{w} = \hat{\mathbf{w}}(\Theta, \mathbf{X})$ and by $(10.14)_2$ we have $0 = \partial W^{AB}/\partial X_A = \varepsilon^{ABC} w^C_{/A}$, i.e. $\mathrm{CURL}\,\mathbf{w} = 0$; hence, for some C^2-function $\varphi = \hat{\varphi}(\Theta, \mathbf{X})$ we have

$$W^{AB} = \varepsilon^{ABC} \frac{\partial \varphi}{\partial X_C}. \tag{10.15}$$

By (10.11), (10.12) and $(10.14)_2$ equality $(10.1)_2$ yields

$$\begin{aligned} &\rho_{\mathcal{K}}\left(\sum_{q=0}^{Q} \frac{\partial N}{\partial \overset{q}{\Theta}} \overset{q+1}{\Theta} + \sum_{p=0}^{P} \frac{\partial N}{\partial \overset{p}{\mathbf{F}}} \overset{p+1}{\mathbf{F}} + \frac{\partial N}{\partial G_A}\dot{G}_A\right) = \\ &= -\Theta^{-2}(W^{AB} G_B - \dot{\Theta} f^A) G_A + \\ &- \Theta^{-1}\left(f^A \dot{G}_A + \frac{\partial f^A}{\partial X_A}\dot{\Theta} + \dot{\Theta}\frac{\partial f^A}{\partial \Theta} G_A\right), \end{aligned} \tag{10.16}$$

which implies

$$\begin{aligned} &\frac{\partial N}{\partial \overset{q}{\Theta}} = 0 \quad \text{for} \quad q \in \{1, \dots, Q\}, \\ &\frac{\partial N}{\partial \overset{p}{\mathbf{F}}} = 0 \quad \text{for} \quad p \in \{0, \dots, P\}, \end{aligned} \tag{10.17}$$

and

$$\rho_{\mathcal{K}} \frac{\partial N}{\partial G_A} = -\Theta^{-1} f^A, \quad \rho_{\mathcal{K}} \frac{\partial N}{\partial \Theta} = +\Theta^{-2} f^A - \Theta^{-1}\left(\frac{\partial f^A}{\partial X_A} + \frac{\partial f^A}{\partial \Theta} G_A\right). \tag{10.18}$$

Hence $N = \widehat{N}(\Theta, \mathbf{G}, \mathbf{X})$; taking the derivatives of both the sides of equalities $(10.18)_{1,2}$ with respect to Θ and G_A, respectively, yields

$$\frac{\partial^2 N}{\partial G_A \partial \Theta} = \frac{\partial^2 N}{\partial \Theta \partial G_A};$$

this integrability condition implies the existence of a C^2-function $N = \widehat{N}(\Theta, \mathbf{G}, \mathbf{X})$ which solves $(10.1)_2$.

Lastly we explicitly show some functions E, $\mathbf{Q}$, N which solves equalities (10.1): for any arbitrary choice of the nine smooth functions

$$\varphi = \varphi(\Theta, \mathbf{X}), \quad k^A = k^A(\mathbf{X}), \quad s^A = s^A(\mathbf{X}), \quad t = t(\mathbf{X}) \quad \text{and}$$
$$h = h(\mathbf{X}), \qquad (A = 1, 2, 3),$$

we put

$$(10.19) \qquad \rho_K E = [k^A(\mathbf{X})\Theta + s^A(\mathbf{X})]G_A + \frac{1}{2}\frac{\partial k^A}{\partial X_A}\Theta^2 + \frac{\partial s^A}{\partial X_A}\Theta + t(\mathbf{X}),$$

$$(10.20) \qquad Q^A = W^{AB}(\Theta, \mathbf{X})G_B - \dot{\Theta}[k^A(\mathbf{X})\Theta + s^A(\mathbf{X})] \quad (A = 1, 2, 3)$$

– see (10.15) – and

$$(10.21) \quad \rho_K N = [k^A(\mathbf{X}) + s^A(\mathbf{X})\Theta^{-1}]G_A + \Theta\frac{\partial k^A}{\partial X_A} + ln(\Theta)\frac{\partial s^A}{\partial X_A} + h(\mathbf{X}).$$

Obviously the functions (10.19)-(10.21) are Euclidean invariant; furthermore they satisfy equalities (10.1) and the conditions

$$\frac{\partial E}{\partial G_A} \equiv -\frac{\partial Q^A}{\partial \dot{\Theta}} \not\equiv 0$$

along any smooth enough process of the body. Now, if $\hat{\sigma} = (\widehat{\mathbf{P}}_K, \hat{\mathbf{q}}_K, \hat{e}_K, \hat{\eta}_K)$ is a generalized system of response functions for $\mathcal{B}$, then

$$\tilde{\sigma} = (\widehat{\mathbf{P}}_K, \hat{\mathbf{q}}_K + \widehat{\mathbf{Q}}, \hat{e}_K + \widehat{E}, \hat{\eta}_K + \widehat{N})$$

is another such a system, because the energy law and the Clausius-Duhem inequality, written using the system $\tilde{\sigma}$, yield

$$(10.22) \qquad \rho_K(\dot{e}_K + \dot{E}) = \mathbf{P}_K \cdot \dot{\mathbf{F}} + \rho_K r - \operatorname{Div}\mathbf{q}_K - \operatorname{Div}\mathbf{Q}$$

and

$$\rho_{\mathcal{K}}(\dot{\eta}_{\mathcal{K}} + \dot{N}) \geq \rho_{\mathcal{K}}\Theta^{-1}r - \mathrm{Div}(\Theta^{-1}\mathbf{q}_{\mathcal{K}}) - \mathrm{Div}(\Theta^{-1}\mathbf{Q})\,, \tag{10.23}$$

respectively. But by (10.1) equalities (10.22)-(10.23) are equivalent to

$$\rho_{\mathcal{K}}\dot{e}_{\mathcal{K}} = \mathbf{P}_{\mathcal{K}} \cdot \dot{\mathbf{F}} + \rho_{\mathcal{K}}r - \mathrm{Div}\,\mathbf{q}_{\mathcal{K}} \tag{10.24}$$

and

$$\rho_{\mathcal{K}}\dot{\eta}_{\mathcal{K}} \geq \rho_{\mathcal{K}}\Theta^{-1}r - \mathrm{Div}(\Theta^{-1}\mathbf{q}_{\mathcal{K}})\,. \tag{10.25}$$

Hence the transformation $\hat{\sigma} \longrightarrow \tilde{\sigma}$ of response functions does not affect both the energy law and the Clausius-Duhem inequality. Furthermore, assume that the functions $W^{AB}(\Theta, \mathbf{X})$, $k^A(\mathbf{X})$, and $s^A(\mathbf{X})$ $(A, B = 1, 2, 3)$ vanish on the boundary of the body; then also Q - see (10.20) - vanishes on the boundary of $\mathcal{B}$; hence the response functions $\hat{\mathbf{q}}_{\mathcal{K}}$ and $\hat{\mathbf{q}}_{\mathcal{K}} + \widehat{\mathbf{Q}}$ give rise to the same normal heat flux on the boundary of $\mathcal{B}$; that is, the transformation above does not affect the boundary conditions. Hence it cannot be detected by usual experiments (processes) of the (whole) body, that is, solutions of an initial-boundary value problem for $\mathcal{B}$. Note that these experiments do not involve cuts and contacts with other bodies. It is a purely technical matter to show that this indetermination really disappears by using certain experiments of cut and contact, just as is made in [9, § 8].

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Lavoro pervenuto alla redazione il 9 giugno 1993
ed accettato per la pubblicazione il 29 settembre 1993

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