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Some remarks on Hamilton-Jacobi equations and non convex minimization problems

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RIASSUNTO: *Si considerano problemi di minimizzazione, nella classe di Lipschitz, di funzionali integrali non convessi $F(u)$ che intervengono nel calcolo delle variazioni. Si dimostra che soluzioni generalizzate di alcune equazioni di Hamilton-Jacobi associate alla funzione integranda, minimizzano F. In particolare viene dimostrato un teorema di esistenza e data una rappresentazione esplicita della soluzione quando la funzione integranda ha crescita al più lineare rispetto al gradiente.*

ABSTRACT: *We consider the minimization problem, in the Lipschitz class, of non convex integral functionals $F(u)$ of the calculus of variations. We show that generalized solutions of some Hamilton-Jacobi equations associated to the integrand function, minimize F. We prove existence theorems and provide an explicit representation formula for the solution, when the integrand grows at most linearly with respect to the gradient variable.*

KEY WORDS: *Dirichlet problem for non convex functionals - Hamilton-Jacobi equations - Viscosity solutions.*

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– Introduction

This paper deals with minimization, in the Lipschitz class, of non convex integral functionals arising in the Calculus of Variations of the form

$$F(u) = \int_\Omega H(x, Du(x))\, dx\,,$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ sufficiently smooth and $H(x,p)$ is a real-valued continuous function on $\overline{\Omega} \times \mathbb{R}^N$.

Consider the problem:

$$\text{(P)} \qquad Min\{\ F(u) \ : u \in Lip(\Omega)\ ,\ u = u_o \text{ on } \partial\Omega\ \}$$

where $u_o \in C(\partial\Omega)$.

It is well known that the non convexity of the integrand, with respect to the gradient variable p, causes the lack of lower semicontinuity of $F(u)$ in $\star - W^{1,\infty}$ topology, making impossible the application of the Direct Method of the Calculus of Variations (see for example G.BUTTAZZO [1]) in order to obtain the existence of minima for $F(u)$.

Methods based on relaxation of $F(u)$ were considered, in the non convex case, by P.MARCELLINI (see [2],[3]) and by E.MASCOLO and R. SCHIANCHI (see [4],[5]). The results in ([4],[5]) apply when $H(x,p)$ is convex with respect to p, for large $|p|$.

In this paper Lipschitz-continuous solutions of some Hamilton-Jacobi equation associated to $H(x,p)$ are shown to minimize $F(u)$, under suitable assumptions on the hamiltonian H.

The results indicate that, if $H(x,p) \geq c(x)\cdot p + d(x)$, for some c and d, then no convexity for large $|p|$ is required. On the other hand we assume, roughly speaking, that the above inequality is strict for large $|p|$.

In particular, an existence theorem will be proved for integrands of the Isaacs' form:

$$H(x,p) = \inf_{a\in A}\ \sup_{b\in B}\{\ -f(x,a)b\cdot p - g(x,a) - h(x,a)\cdot b\ \}.$$

This approach allows to find an explicit representation formula for a solution of (P) in terms of the value funcion of the differential game associated to H. Moreover, thanks to a representation formula due to L.C. EVANS and P.E. SOUGANIDIS (see [6]), this result extends to general H growing at most linearly in p.

In the first section we explain the link between the minimization problem (P) and Hamilton-Jacobi equations and recall briefly some basic facts about viscosity solutions; the second section is devoted to the proof of existence theorems.

1 – Some relations between problem (P) and Hamilton-Jacobi equations

Let Ω be an open bounded subset of $\mathbb{R}^N$ ($N \geq 1$), with boundary $\partial\Omega$ sufficiently smooth. Let then $H \in C(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R})$ and $u_o \in C(\partial\Omega)$.

Consider the (non convex) minimization problem:

$$\text{(P)} \quad Min\left\{F(u) = \int_\Omega H(x, Du(x))dx : u \in Lip(\Omega), u = u_o \text{ on } \partial\Omega\right\}.$$

The next simple result relates the Dirichlet problem for the Hamilton-Jacobi equation

$$\text{(HJ)} \qquad \begin{cases} H(x,\, Du(x)) = c(x) \cdot Du(x) + d(x) & \text{in } \Omega \\ u = u_o & \text{on } \partial\Omega\,, \end{cases}$$

with the minimization problem (P).

Denote by S the (possibly empty) set of *generalized solutions* of (HJ), namely the set of $u \in Lip(\Omega) \cap C(\overline{\Omega})$ which satisfy the equation in (HJ) almost everywhere in Ω and the boundary condition pointwise.

PROPOSITION 1.1. *Assume $H(x,p)$ satisfies*

$$\text{(H1)} \qquad H(x,p) \geq c(x) \cdot p + d(x) \qquad \forall x \in \overline{\Omega}\,,\ \forall p \in \mathbb{R}^N,$$

for some $d \in C(\overline{\Omega})$ and $c \in Lip(\Omega; \mathbb{R}^N) \cap C(\overline{\Omega}; \mathbb{R}^N)$ such that $div\ c = 0$ in Ω.

Then, every $u \in S$ is a solution of (P). *Conversely, if $S \neq \emptyset$, then any solution of* (P) *belongs to S.*

PROOF. Let $u \in S$ and let v any other Lipschitz-continuous function which attains the same boundary value u_o. (H1) yields:

$$\int_\Omega H(x,\, Dv(x))\, dx \geq \int_\Omega c(x) \cdot Dv(x)\, dx + \int_\Omega d(x)\, dx$$

and from the divergence theorem and the fact that $div\, c = 0$, it follows:

$$\text{(1.1)} \qquad \int_\Omega c(x) \cdot Dv(x)\, dx = \int_\Omega c(x) \cdot Du(x)\, dx.$$

Hence,

$$\int_\Omega H(x, Dv(x))\,dx \geq \int_\Omega H(x, Du(x))\,dx$$

and the first part of the claim is proved.

Now, if $u \in Lip(\Omega)$ is a solution of (P), then it coincides with u_o on $\partial\Omega$. Moreover,

$$H(x, Du(x)) = c(x) \cdot Du(x) + d(x) \quad \text{a.e. in } \Omega.$$

In fact, let us suppose by contradiction, using (H1), that there exists a Lebesgue-measurable subset A of Ω with positive measure, such that:

$$H(x, Du(x)) > c(x) \cdot Du(x) + d(x) \quad \text{a.e. in } A.$$

Then, for any $v \in S\,(\neq \emptyset)$,

$$\int_\Omega H(x, Dv(x))\,dx \geq \int_\Omega H(x, Du(x)) > \int_\Omega c(x) \cdot Du(x) + d(x)\,dx$$

so, by (1.1),

$$\int_\Omega [H(x, Dv(x)) - c(x) \cdot Dv(x) - d(x)]\,dx > 0$$

and this provides, using (H1), a contradiction because v is assumed to satisfy the Hamilton-Jacobi equation almost everywhere in Ω. □

Let us observe that (P) may have solutions which doesn't belong to S but, in this case, S has to be empty. Consider, for istance, the one-dimensional minimization problem with integrand $H(x, v'(x)) = |v'(x)|$ in $\Omega = (a, b)$, with $u_o(a) \neq u_o(b)$. This verifies (H1) with $c = d = 0$. The unique solution of (P) is the affine function $u(x) = u_o(a) + \dfrac{u_o(b) - u_o(a)}{(b - a)}(x - a)$, but u doesn't belong to S (which in fact is empty).

Let us observe also that, from (H1), it follows that $u \in Lip(\Omega)$, $u = u_o$ on $\partial\Omega$, is a solution of (HJ) (and consequently of (P)), if and only if $H(x, Du(x)) \leq c(x) \cdot Du(x) + d(x)$ almost everywhere in Ω.

REMARK 1. A different connection between the minimization problem (P) and Hamilton-Jacobi equations was exploited by E. MASCOLO–R. SCHIANCHI (see [4],[5]). Namely, they considered the Dirichlet problem:

$$\text{(1.2)} \qquad \begin{cases} 1_{K(x)}(Du(x)) = 0 & \text{in } \Omega \\ u = u_o & \text{on } \partial\Omega, \end{cases}$$

where

$$K(x) = \left\{ p \in \mathbb{R}^N \ : \ H(x,p) > H^{**}(x,p) \right\} \qquad \text{a.e. in } \Omega$$

and $H^{**}(x,p)$ denotes the bipolar function of H, namely the lower convex envelope of H with respect to p. They proved that any solution of (1.2) is a minimum for (P), provided $\cup_{x\in\Omega} K(x)$ is an open bounded set. Let us observe that this assumption fails, in general, if H grows at most linearly.

In the next section we will discuss the existence of generalized solution of (HJ). This is more easily done by proving first the existence of weak solutions in the viscosity sense and then their lipschitz continuity.

Let us recall that $u \in C(\Omega)$ is a viscosity solution of (HJ) (see for example [7] for more details) if

$$\text{(SUB)} \qquad H(x,p) \leq c(x) \cdot p + d(x) \qquad \forall p \in D^+u(x)$$

and

$$\text{(SUP)} \qquad H(x,p) \geq c(x) \cdot p + d(x) \qquad \forall p \in D^-u(x)$$

where

$$D^+u(x) = \left\{ p \in \mathbb{R}^N : \limsup_{y\to x, y\in\Omega} \frac{u(y) - u(x) - p\cdot(y-x)}{|y-x|} \leq 0 \right\}$$

and

$$D^-u(x) = \left\{ p \in \mathbb{R}^N : \liminf_{y\to x, y\in\Omega} \frac{u(y) - u(x) - p\cdot(y-x)}{|y-x|} \geq 0 \right\} .$$

It is easy to check that any viscosity solution of (HJ), which is in $Lip(\Omega)$, satisfies the equation almost everywhere in Ω. If H is convex in p, then generalized solutions of (HJ) are also viscosity solutions; but this is false, in general, for H non convex (see e.g. [8]).

REMARK 2. Note that, if (H1) holds, then the supersolution condition (SUP) is always trivially satisfied.

Moreover, let us point out that in many respects, is useful to consider viscosity solutions because they are stables with respect to uniform convergence. So, they may be approximated by solutions of elliptic problems, with the vanishing viscosity method (see [9]) or of Hamilton Jacobi problems like

$$
(\mathrm{HJ})_n \qquad \begin{cases} H_n(x, Du_n(x)) = c(x) \cdot Du_n(x) + d(x) & \text{in } \Omega \\ u_n = u_o & \text{on } \partial\Omega\,, \end{cases}
$$

with $H_n \to H$ locally uniformly in $\Omega \times \mathbb{R}^N$. In fact, holds that, if u_n is a viscosity solution of $(HJ)_n$ and if $u_n \to u$ locally uniformly in Ω, then u is a viscosity solution of (HJ). Furthermore, let us remark that in some case, the viscosity solution of (HJ) is known to be unique (see [10]). All this fails if one consider only generalized solutions.

2 – Existence of minima for (P)

In this section we prove some existence results for problem (P). The idea is to show that, under some suitable assumptions, equation (HJ) has a viscosity solution $u \in Lip(\Omega) \cap C(\overline{\Omega})$ and to apply then proposition 1.1. The main difficulty is related to the compatibility conditions that the boundary data u_o must satisfy. The next result provides such conditions in a one-dimensional case, when H depends only on p.

PROPOSITION 2.1. *Let $H \in C(\mathbb{R})$ satisfy* (H1) *in $\Omega = (a,b) \subset \mathbb{R}$, with c and d independent of x.*

Then (HJ) *has a viscosity solution $u \in Lip(a,b) \cap C([a,b])$, if and only if, either $\overline{p} = \frac{u_o(b) - u_o(a)}{b-a}$ solves $H(\overline{p}) = c\overline{p} + d$, or there exist $p_i \in \mathbb{R}$ (i=1,2) such that*

$$
(2.1) \qquad p_1 < \overline{p} < p_2 \quad \text{and} \quad H(p_i) = cp_i + d.
$$

PROOF. If $H(\overline{p}) = c\overline{p} + d$, by an affine extension of u_o, (that we still denote by u_o), we obtain trivially a solution of (HJ). On the other hand, if (2.1) holds, a lipschitz continuous viscosity solution of (HJ) may be obtained by putting

$$u(x) = \begin{cases} u_o(a) + p_1(x-a) & a \leq x < \overline{x} \\ u_o(a) + p_2(x-b) & \overline{x} \leq x \leq b \end{cases}$$

with $\overline{x} = \dfrac{u_o(b) - u_o(a) + p_1 a - p_2 b}{p_1 - p_2}$, as it is easy to check, taking into account Remark 2 and that u is differentiable in $\Omega \setminus \{\overline{x}\}$ and $D^+u(\overline{x}) = \emptyset$.

Conversely, assume by contradiction there exists a function $v \in Lip(a,b) \cap C([a,b])$ viscosity solution of (HJ) and that

$$\text{(2.2)} \qquad H(p) > c \cdot p + d\,, \qquad \text{for all} \quad p \geq \overline{p}\,.$$

So, v satisfies (HJ) almost everywhere in (a,b), as observed above and, by (2.2), $v' \neq u_o'$ in a subset of positive measure. Hence $\|v'\|_{L^1(a,b)} > |\overline{p}|(b-a)$ and consequently $\|v'\|_{L^\infty(a,b)} > |\overline{p}|$. Thus, we may choose $(\alpha,\beta) \subset (a,b)$, with $|\beta - \alpha| > 0$, such that

$$|v'(x)| > |\overline{p}| \qquad \text{a.e. in } (\alpha,\beta)$$

and, assuming $\overline{p} > 0$ (which is not restrictive), and denoting by $I_1 = \{x \in (\alpha,\beta) : v'(x) > \overline{p}\}$ and $I_2 = \{x \in (\alpha,\beta) : v'(x) < -\overline{p}\}$, we may decompose (α,β) in $(\alpha,\beta) = I_1 \cup I_2 \cup N$, where N is a negligeable subset where v' doesn't eventually exist. Since $|\beta - \alpha| > 0$, at least one of these I_i $(i = 1,2)$ need to have positive measure. If $|I_1| > 0$, v cannot be a solution of (HJ) since, by (2.2), it does not verify the equation in I_1, so we get a contradiction. If $|I_1| = 0$, $v'(x) < -\overline{p}$ almost everywhere in (α,β) and, consequently, $v'(x) < \overline{p}$ almost everywhere in (a,b), in view of the definition of (α,β). So,

$$v(b) - v(a) < \overline{p}(b-a) = u(b) - u(a)$$

and the boundary condition fails.

Analogously, if

$$\text{(2.3)} \qquad H(p) > c \cdot p + d\,, \qquad \text{for all} \quad p \leq \overline{p}\,,$$

and $|I_2| > 0$, the contradiction follows from the same argument; while, if it is negligeable, in order to satisfy the boundary condition, there should exist $(\sigma,\delta) \subset (a,b)$ with $|\delta-\sigma| > 0$ and $(\sigma,\delta)\cap(\alpha,\beta) = \emptyset$, where $-\overline{p} < v'(x) < \overline{p}$. Also in this case, we get a contradiction because in (σ,δ) v does not verify the equation. □

REMARK 3. MARCELLINI (see [2]) proved a similar result by a different method, under the assumption of superlinear growth.

The following corollary is a straightforward consequence of propositions 1.1 and 2.1

COROLLARY 2.2. *Assume $H \in C(\mathbb{R})$ satisfy* (H1) *in $\Omega = (a,b) \subset \mathbb{R}$, with c and d independent of x and either $\overline{p} = \frac{u_0(b)-u_0(a)}{b-a}$ solves $H(\overline{p}) = c\overline{p} + d$, or there exist $p_i \in \mathbb{R}$ (i=1,2) such that (2.1) holds.*

Then (P) has a solution. □

Let us consider now the minimization problem (P) with integrand $H(x,p)$ of the following type:

$$H(x,p) = \inf_{a\in A}\sup_{b\in B}\{\, -f(x,a)b\cdot p - g(x,a) - h(x,a)\cdot b\,\}\ , \tag{2.4}$$

with A and B compact subsets of $\mathbb{R}^N$, $f \in C(\overline{\Omega}\times A\,;\mathbb{R}^{N^2})$, $g \in C(\overline{\Omega}\times A\,;\mathbb{R})$, and $h \in C(\overline{\Omega}\times A\,;\mathbb{R}^N)$ satisfying, for some $L > 0$,

$$\left.\begin{aligned} |f(x,a)-f(y,a)| &\le L|x-y| \\ |g(x,a)-g(y,a)| &\le L|x-y| \\ |h(x,a)-h(y,a)| &\le L|x-y| \end{aligned}\right.\qquad \forall x,y\in\overline{\Omega};\quad \forall a\in A. \tag{2.5}$$

Assume, moreover, that

$$\forall x\in\overline{\Omega}, \forall a\in A, \quad \exists b\in B\ :\ h(x,a)\cdot b \le 0, \tag{2.6}$$

and that

$$\begin{aligned} &\forall x\in\overline{\Omega} \text{ and } \forall a\in A,\ \exists\mu=\mu(x,a)>0 \text{ such that} \\ &f(x,a)(B) \supseteq \mu\partial B(0,1), \end{aligned} \tag{2.7}$$

where $\partial B(0,1)$ is the unit sphere of $\mathbb{R}^N$ centered at zero. Notice that the continuity of f and the compactness of A and $\overline{\Omega}$ imply that $\inf_{x\in\overline{\Omega};a\in A} \mu(x,a) = \bar{\mu} > 0$, so that (2.7) is equivalent to

$$\exists\bar{\mu} > 0 \text{ such that } f(x,a)(B) \supseteq \bar{\mu}\partial B(0,1) \ \forall x \in \overline{\Omega},\ \forall a \in A. \tag{2.8}$$

Integrands H of the form (2.4) are special but important cases of Isaacs' functions and are related to the theory of two players, zero-sum differential games (see [11]).

For this kind of integrands, under suitable assumptions, one may give an existence theorem for the solution of (P) and a representation formula, as we will see below.

At this purpose, let us denote by $\mathcal{A}$ and $\mathcal{B}$, respectively, the function spaces $L^\infty([0,+\infty)\,;\,A)$ and $L^\infty([0,+\infty)\,;\,B)$; set

$$t_x(a,b) = \inf\{\ t\ :\ y_x(t) \in \partial\Omega\ \},$$

where $y_x(t)$ is the solution of the ordinary differential equation

$$\begin{cases} \dot{y}_x(t) = f(y_x(t)\,,\,a(t))b(t) & \text{for } t > 0 \\ y_x(0) = x\ \in\ \overline{\Omega}\,, \end{cases} \tag{2.9}$$

for fixed $a \in \mathcal{A}$ and $b \in \mathcal{B}$; denote by

$$\begin{aligned}\Gamma = \{\beta : \mathcal{A} \to \mathcal{B}\ \text{ s.t. } a(t) = a'(t) \text{ for a.e. } t \le t' \\ \text{implies } \beta[a](t) = \beta[a'](t) \ \text{ for a.e. } t \le t'\ \}\end{aligned}$$

and, for $x \in \overline{\Omega}$,

$$\Gamma_x = \{\beta \in \Gamma\ :\ t_x(a,\beta(a)) < +\infty,\quad \forall a \in \mathcal{A}\}. \tag{2.10}$$

Define

$$d(x) = \inf_{a\in A}\{-g(x,a)\} \tag{2.11}$$

and

$$T(x) = \inf_{\beta\in\Gamma_x}\ \sup_{a\in\mathcal{A}} t_x(a,\beta(a)). \tag{2.12}$$

As a consequence of (2.6) and (2.8), the following hold:

(2.13) $\forall x \in \Omega\,,\ \exists b_x \in B$ such that $t_x(a, b_x) < \infty\ \forall a \in \mathcal{A}\,;$

(2.14) $\forall x \in \partial\Omega,\quad \exists b_x \in B$ such that $f(x,a)b_x \cdot n(x) > 0 \quad \forall a \in \mathcal{A}\,,$

where $n(x)$ is the exterior normal to $\partial\Omega$ at x; and for any fixed $p \in \mathbb{R}^N \setminus \{0\}$, there exists $b \in B$ such that

(2.15) $$f(x,a)b \cdot p \leq -\gamma < 0 \qquad \forall x \in \overline{\Omega}\,, \forall a \in A\,,$$

where γ is a positive constant.

In particular, (2.13) means that $\Gamma_x \neq \emptyset$ for any $x \in \overline{\Omega}$ while (2.13) and (2.14) imply the continuity of $T(x)$ in $\overline{\Omega}$, (see [11]).

For the minimization problem (P) associated to H the following result holds:

THEOREM 2.3. *Let Ω be convex. Assume* (2.5), (2.6), (2.8) *and that*

$\forall x \in \overline{\Omega}$, *and* $\forall \beta \in \Gamma_x$, $\exists \hat{a} \in \mathcal{A}$ *such that:*

(2.16) $$g(y_x(t, \hat{a}(t), \beta(\hat{a})(t)), \hat{a}(t)) = \max_{a \in \mathcal{A}}\{g(y_x(t, a(t), \beta(a)(t)), a(t))\}$$

and

$$h(y_x(t), \hat{a}(t)) \cdot \beta(\hat{a})(t) \geq h_o \qquad \forall t > 0,$$

where h_o is a non negative constant. Then

$$u(x) = \inf_{\beta \in \Gamma_x} \sup_{a \in \mathcal{A}} \Big\{ \int_0^{t_x(a,\beta(a))} g(y_x(t, a, \beta(a)), a) + h(y_x(t, a, \beta(a)), a) \cdot \beta(a)$$

(2.17) $$+ d(y_x(t, a, \beta(a)))dt + u_o(y_x(t_x(a(t_x), \beta(a)(t_x)))) \Big\}$$

is a solution of (P) provided the following compatibility condition holds:

(2.18) $$\exists \delta > 0 \text{ such that } u_o(x) \leq u(x) \qquad \forall x \in \Omega_\delta$$

where $\Omega_\delta = \{x \in \Omega \ :\ dist(x, \partial\Omega) \leq \delta\}$ and u_o is any lower semicontinuous extension to Ω_δ of the boundary datum.

PROOF.

1° *part:* H *verifies* (H1) *with* $c(x) = 0$ *and* $d(x)$ *defined in* (2.11). Indeed, by (2.6) and (2.8), $\forall x \in \overline{\Omega}$, $\forall a \in A$ and $\forall p \in \mathbb{R}^N$, there exists $b \in B$ such that $-f(x,a)b \cdot p - h(x,a) \cdot b \geq 0$. Hence, for any $x \in \overline{\Omega}$,

$$\inf_{a\in A} \sup_{b\in B}\{ -f(x,a)b \cdot p - g(x,a) - h(x,a) \cdot b \} \geq \inf_{a\in A}\{ -g(x,a) \} = d(x).$$

2° *part:* $u(x)$ *defined by* (2.17) *is bounded in* $\overline{\Omega}$ *and verifies*

$$u(x) \leq u_o(y_x(T(x))) + KT(x) \qquad \text{in } \overline{\Omega}\,, \tag{2.19}$$

with $K > 0$ *and* $T(x)$ *defined in* (2.12). By (2.16) we get

$$\begin{aligned} u(x) &\geq \inf_{\beta\in\Gamma_x} \{h_o t_x(\hat{a}, \beta(\hat{a})) + u_o(y_x(t_x(\hat{a}(t_x), \beta(\hat{a})(t_x))))\} \\ &\geq -\|u_o\|_{L^\infty(\partial\Omega)} \end{aligned} \tag{2.20}$$

so that u is bounded from below even if the integrand in (2.17) is not strictly positive. On the other hand, by definition of $d(x)$ and the compactness of A and B, we obtain from (2.5) that there exists $K > 0$ such that

$$\forall a \in \mathcal{A},\ \forall \beta \in \Gamma_x,$$
$$|g(y_x(t,a,\beta(a)),a) + h(y_x(t,a,\beta(a)),a) \cdot \beta(a) + d(y_x(t,a,\beta(a)))| \leq K.$$

Hence, $u(x) \leq KT(x) + \|u_o\|_{L^\infty(\partial\Omega)}$ which is bounded in $\overline{\Omega}$, as observed above.

In order to check (2.19), let us observe that $\forall \varepsilon > 0$, $\exists \beta_\varepsilon \in \Gamma_x$ such that:

$$T(x) \leq t_x(a, \beta_\varepsilon(a)) \leq T(x) + \varepsilon \qquad \forall a \in \mathcal{A}. \tag{2.21}$$

On the other hand,

$$u(x) \leq \sup_{a\in\mathcal{A}}\{Kt_x(a,\beta_\varepsilon(a)) + u_o(y_x(t_x(a,\beta_\varepsilon(a))))\};$$

therefore, there exists $a_\varepsilon \in \mathcal{A}$ such that:

$$u(x) \leq Kt_x(a_\varepsilon, \beta_\varepsilon(a_\varepsilon)) + u_o(y_x(t_x(a_\varepsilon, \beta_\varepsilon(a_\varepsilon)))) + \varepsilon.$$

Thanks to (2.13) and (2.14), $T(x)$ is finite everywhere in $\overline{\Omega}$, so we may subtract $u_o(y_x(T(x)))$ and obtain

$$\begin{aligned} u(x) - u_o(y_x(T(x))) \leq \omega_{u_o}(|y_x(T(x)) - y_x(t_x(a_\varepsilon, \beta_\varepsilon(a_\varepsilon)))|) \\ + Kt_x(a_\varepsilon, \beta_\varepsilon(a_\varepsilon)) + \varepsilon, \end{aligned}$$

where ω_{u_o} denotes the modulus of continuity of u_o on $\partial\Omega$. So, taking into account (2.21) and the lipschitz continuity of $t \to y_x(t)$, as a consequence of the continuity of f and the compactness of A and B, we get

$$u(x) \leq u_o(y_x(T(x))) + KT(x) + \varepsilon,$$

and letting ε go to zero, the claim.

3° *part: $u(x)$ defined by* (2.17) *is continuous in* Ω. In fact, fix $x_o \in \Omega$, $\eta > 0$ such that $B(x_o, \eta) \subset \Omega$ and let $x \in B(x_o, \eta)$. Assume first that $u(x) \geq u(x_o)$. For any $\epsilon > 0$, let $\tilde\beta \in \Gamma_{x_o}$ such that

$$\begin{aligned} &\sup_{a\in\mathcal{A}} \Big\{ \int_0^{t_{x_o}(a,\tilde\beta(a))} g(y_{x_o}(t,a,\tilde\beta(a)),a) + h(y_{x_o}(t,a,\tilde\beta(a)),a)\cdot\tilde\beta(a) + \\ &+ d(y_{x_o}(t,a,\tilde\beta(a)))dt + u_o(y_{x_o}(t_{x_o}(a(t_{x_o}),\tilde\beta(a)(t_{x_o})))) \Big\} \qquad (2.22)\\ &\leq u(x_o) + \epsilon, \end{aligned}$$

and choose $\bar a \in \mathcal{A}$ verifying:

$$\begin{aligned} u(x) \leq \int_0^{t_x(\bar a,\beta(\bar a))} & g(y_x(t,\bar a,\beta(\bar a)),\bar a) + h(y_x(t,\bar a,\beta(\bar a)),\bar a)\cdot\beta(\bar a) \\ & + d(y_x(t,\bar a,\beta(\bar a)))dt + u_o(y_x(t_x(\bar a(t_x),\beta(\bar a)(t_x)))) + \epsilon, \end{aligned} \qquad (2.23)$$

for any $\beta \in \Gamma_x$.

By (2.8), there exists $\bar b \in B$ such that:

$$f(y_x(t),\bar a(t))\bar b = \bar\mu\frac{x_o - x}{|x_o - x|}$$

for $t < \bar{t} = \frac{1}{\bar{\mu}}|x_o - x|$, the first time where $y_x(t, \bar{a}(t), \bar{b}) = x_o$.

Hence, if we choose $\tilde{a}(t) = \bar{a}(t + \bar{t})$ in (2.22) and

$$\beta(\bar{a}(t)) = \begin{cases} \bar{b} & t < \bar{t} \\ \tilde{\beta}(\tilde{a}(t)) & t \geq \bar{t}, \end{cases}$$

we get $\beta \in \Gamma_x$ and, by (2.23),

$$\begin{aligned} u(x) \leq & \int_0^{\bar{t}} g(y_x(t, \bar{a}(t), \bar{b}), \bar{a}(t)) + h(y_x(t, \bar{a}(t), \bar{b}), \bar{a}(t)) \cdot \bar{b} + \\ & + d(y_x(t, \bar{a}(t), \bar{b}))dt + \int_0^{t_{x_o}(\tilde{a}, \tilde{\beta}(\tilde{a}))} g(y_{x_o}(t, \tilde{a}(t), \tilde{\beta}(\tilde{a})(t)), \tilde{a}(t)) + \\ & + h(y_{x_o}(t, \tilde{a}(t), \tilde{\beta}(\tilde{a})(t)), \tilde{a}(t)) \cdot \tilde{\beta}(\tilde{a})(t) \\ & + d(y_{x_o}(t, \tilde{a}(t), \tilde{\beta}(\tilde{a})(t)))dt + u_o(y_{x_o}(t_{x_o}(\tilde{a}(t_{x_o}), \tilde{\beta}(\tilde{a})(t_{x_o})))) + \epsilon \\ \leq & \int_0^{\bar{t}} g(y_x(t, \bar{a}(t), \bar{b}), \bar{a}(t)) + h(y_x(t, \bar{a}(t), \bar{b}), \bar{a}(t)) \cdot \bar{b} + \\ & + d(y_x(t, \bar{a}(t), \bar{b}))dt + u(x_o) + 2\epsilon \leq K\bar{t} + u(x_o) + 2\epsilon . \end{aligned} \tag{2.24}$$

Therefore,

$$0 \leq u(x) - u(x_o) \leq K\bar{t} + 2\epsilon = \frac{K}{\bar{\mu}}|x - x_o| + 2\epsilon.$$

The same argument applies if $u(x) \leq u(x_o)$, changing the role of x and x_o. Letting $\epsilon \to 0$, we obtain the continuity of u in Ω.

4° *part: If* (2.18) *holds, u is continuous also on $\partial\Omega$ and satisfies (HJ) in the viscosity sense.* In fact, let $z \in \partial\Omega$. Then

$$\begin{aligned} u_o(z) \leq \liminf_{\xi \to z, \xi \in \Omega_\delta} u_o(\xi) \leq \liminf_{\xi \to z, \xi \in \Omega_\delta} u(\xi) \leq \limsup_{\xi \to z, \xi \in \Omega_\delta} u(\xi) \\ \leq \lim_{\xi \to z} (u_o(y_\xi(T(\xi))) + KT(\xi)) = u_o(z), \end{aligned}$$

since $u_o(y_x(T(x)))$ is continuous in $\overline{\Omega}$ and $T(z) = 0$ for each $z \in \partial\Omega$. Hence, u is continuous on $\partial\Omega$.

Moreover, u satisfies the Dynamic Programming Principle:

$$u(x) = \inf_{\beta\in\Gamma_x} \sup_{a\in\mathcal{A}} \left\{ \int_0^{\tau\wedge t_x(a,\beta(a))} g(y_x(t,a,\beta(a)),a) + h(y_x(t,a,\beta(a)),a)\cdot\beta(a) \right.$$
$$\left. + d(y_x(t,a,\beta(a)))dt + u(y_x(\tau\wedge t_x(a(t_x),\beta(a)(t_x)))) \right\};$$

so it is easy to verify that u is a viscosity solution of the equation (HJ) (see [12]).

5° *part: u is a solution of* (P). By (2.15), it follows that for any $G > 0$ and for any $p \in \mathbb{R}^N$ with $|p| > G$, there exists $b_p \in B$ such that:

$$\inf_{a\in A}\sup_{b\in B}\{ -f(x,a)b\cdot p - h(x,a)\cdot b - g(x,a) \} \geq \gamma G + \inf_{a\in A}\{-g(x,a) - h(x,a)\cdot b_p\}.$$

Thus, if $G > C/\gamma$, where C is an upper bound for $|h(x,a)\cdot b_p|$, $\forall x \in \overline{\Omega}$, then $H(x,p) - d(x) > 0$ uniformly in Ω. Therefore,

$$\text{(2.25)} \qquad \liminf_{|p|\to\infty} H(x,p) - d(x) > 0 \qquad \text{uniformly in } \Omega,$$

so, choosing $u_o(y_x(T(x))) + KT(x)$ as supersolution of (HJ) (see Remark 2) greater than $u(x)$, the global lipschitz continuity of u follows from the same arguments as used by ISHII (see [13]), taking into account the convexity of Ω. Hence, applying proposition 1.1, we get the claim. □

REMARK 4. We observe that condition (2.25), which is used in the last step of the proof, is similar to one made by P. MARCELLINI (see [3]) in order to prove the existence of solution of (P), when H depends only on the absolute value of Du and u_o is constant on $\partial\Omega$.

REMARK 5. The compatibility condition (2.18) plays in the proof, the same role as the bounded slope condition on u_o considered by MASCOLO-SCHIANCHI (see [4], [5]).

Thanks to assumptions (2.6) and (2.16), this condition holds if, for example, u_o can be extended in Ω_δ in such a way that

$$\text{(2.26)} \qquad |u_o(x) - u_o(y)| \leq \frac{h_o}{M}|x-y| \qquad \forall x, y \in \Omega_\delta,$$

where $M = \sup_{x\in\Omega_\delta;a\in A}|\mu(x,a)|$.

In fact, by (2.20), as

$$t_x(\hat{a},\beta(\hat{a})) \geq \frac{1}{M}|x - y_x(t_x(\hat{a},\beta(\hat{a})))| \qquad \forall\beta\in\Gamma_x,$$

we get, using (2.26),

$$u(x) \geq \inf_{\beta\in\Gamma_x}\{\frac{h_o}{M}|x - y_x(t_x(\hat{a},\beta(\hat{a})))| + u_o(y_x(t_x(\hat{a},\beta(\hat{a}))))\} \geq u_o(x)$$

and therefore (2.18).

Let us remark that (2.18) is a generalization of the standard condition which in the convex case, is necessary and sufficient to assure the continuity, up to the boundary, of solutions of Hamilton Jacobi equations, see ([9]).

REMARK 6. The result of theorem 2.3 applies to non convex integrand $H(x,p)$, bounded from below with respect to p and satisfying the structure conditions:

(2.27)

$$|H(x,0)| \leq C$$
$$|H(x,p) - H(y,q)| \leq C(|x-y| + |p-q|) \qquad \forall x,y\in\overline{\Omega};\forall p,q\in\mathbb{R}^N,$$

for some $C > 0$, and

$$\liminf_{|p|\to\infty} H(x,p) > \inf_{p\in\mathbb{R}^N} H(x,p) \qquad \forall x\in\overline{\Omega}. \tag{2.28}$$

In particular, to H which grows at infinity as $|p|^\alpha$, for some $\alpha\in(0,1]$, or which has an oblique asymptote with respect to p.

In fact, in this case, (H1) is satisfied with $c(x)\equiv 0$ and $d(x) = \inf_{p\in\mathbb{R}^N} H(x,p)$. Moreover, in view of (2.28), it is clearly not restrictive to look at $H(x,p)$ for $p\in B(0,R)$, where $B(0,R)$ is a ball of $\mathbb{R}^N$ centered at zero and with radius R, which contains all points of minimum of $p\to H(x,p)$, for any $x\in\overline{\Omega}$. Furthermore, as observed by EVANS and SOUGANIDIS (see [6]), under assumption (2.27), for any $\rho > 0$ and $|p|\leq\rho$, $H(x,p)$ can be represented as inf-sup of affine functions in the following way:

$$H(x,p) = \inf_{a\in B(0,\rho)}\sup_{b\in\partial B(0,1)}\{-Cb\cdot p + H(x,a) + Ca\cdot b\}. \tag{2.29}$$

Therefore, if we choose $\rho = R$, as (2.29) is a special case of (2.4), and (2.5), (2.6), (2.8), (2.16) are clearly satisfied, we can apply theorem 2.3 provided (2.18) holds. Moreover notice that, in the present case, $f(x,a)b = Cb$. Hence, $y_x(t)$ and, consequently, t_x depend only on b.

REMARK 7. Let us mention that, if $H \geq 0$ is a lipschitz function which depends only on $|p|$ and satisfies:

$$|H(0)| \leq C$$
$$|H(p) - H(q)| \leq C|p - q| \qquad \forall p, q \in \mathbb{R}^N,$$

and

$$\liminf_{|p|\to\infty} H(|p|) > 0,$$

then (2.18) is verified if, for example, u_o has a lipschitz extension on a neighbourhood Ω_δ of the boundary $\partial\Omega$, with constant less than $L_o = \max\{|p| : H(|p|) = 0\}$. Observe that, in this case, $M = C$ while $h_o = L_oC$, as it is easy to check, writing H as (2.29), with $\rho = R > L_o$. Therefore, all assumtions of theorem 2.3 are satisfied and u given by (2.17) is a minimum for (P). Furthermore the particular form of f, independent of a, implies that u is given by

$$u(x) = \inf_{\beta\in\Gamma_x} \{L_oCt_x(\beta) + u_o(y_x(t_x(\beta)))\},$$

as an easy calculation shows. In particular notice that, if u_o is constant, $\inf_{\beta\in\Gamma_x} t_x(\beta) = C^{-1}dist(x, \partial\Omega)$.

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