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Spectral analysis of a transmission problem for the Helmholtz equation on the half-space

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Dedicated to Professor Gaetano Fichera on the occasion of his 70th anniversary.

RIASSUNTO: *Siano* κ_1, $\kappa_2 > 0$, *e* $\kappa : \mathbb{R} \to (0,\infty)$ *con*

$$\kappa(x) = \begin{cases} \kappa_1, & x \geq 0 \\ \kappa_2, & x < 0 \end{cases}$$

e $\lambda \in \mathbb{C} \setminus [0,\infty)$. *Si considera il seguente problema di trasmissione per l'equazione di Helmhotz:*

$$-\frac{1}{\kappa(x)}\Delta f(x,y) - \lambda f(x,y) = g(x,y)\,, \quad x \in \mathbb{R}\setminus\{0\}\,, \quad y > 0\,,$$
$$f(x,0) = 0\,, \qquad x \in \mathbb{R}\,,$$

dove g *è un'opportuna funzione sorgente.*

Si dimostra che l'operatore differenziale $-\frac{1}{\kappa(\cdot)}\Delta$ *con i dati omogenei di Dirichlet su* $y = 0$ *definisce un operatore auto-aggiunto* A *sullo spazio di Lebesgue con peso*

$$H := L_2\big(\mathbb{R} \times (0,\infty), \kappa(\cdot)\big)\,,$$

e ne viene data esplicitamente la famiglia spettrale.

ABSTRACT: *Let* κ_1, $\kappa_2 > 0$, *and* $\kappa : \mathbb{R} \to (0,\infty)$ *with*

$$\kappa(x) = \begin{cases} \kappa_1, & x \geq 0 \\ \kappa_2, & x < 0 \end{cases}$$

and $\lambda \in \mathbb{C} \setminus [0, \infty)$. *We consider the transmission problem for the Helmholtz equation on the two quadrants of the upper half-space given by*

$$-\frac{1}{\kappa(x)}\Delta f(x,y) - \lambda f(x,y) = g(x,y), \quad x \in \mathbb{R} \setminus \{0\}, \quad y > 0,$$
$$f(x,0) = 0, \qquad x \in \mathbb{R}.$$

Thereby g *is a suitable source-function.*

It is shown, that the differential operator $-\frac{1}{\kappa(\cdot)}\Delta$ *together with homogeneous Dirichlet data on* $y = 0$ *defines a selfadjoint operator* A *on the Lebesgue-space with weight*

$$H := L_2(\mathbb{R} \times (0, \infty), \kappa(\cdot)),$$

whose spectral family will be given explicitely.

KEY WORDS: *Spectral analysis – Helmholtz equation.*

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1 – Introduction and formulation of the problem

The propagation of electromagnetic waves in the atmosphere in the presence of a plane non-ideally conducting earth has been treated first by WEYL and SOMMERFELD ([12], [13]).

Physically this problem may be described as follows: Let the plane $t = 0$ of a cartesian (s, t, z)-coordinate system in $\mathbb{R}^3$ separate the earth and the atmosphere, the first one being located in the half-space $t > 0$, the second one in $t < 0$. Both electromagnetic media are supposed to be homogeneous with positive constants of the electrical conductivity. In the atmosphere and parallel to the z-axis an electromagnetic line source is located producing a time-harmonic primary field which is polarized parallel to the z-axis with respect to the electrical field vector. Then using Maxwell's theory of electromagnetic waves the problem to determine the diffracted field generated by the presence of the earth.

Mathematically this problem is a transmission problem for the homogeneous Helmholtz equation in the (s, t)-plane with different coefficients on $t > 0$ and on $t < 0$, respectively, and inhomogeneous transmission conditions along $t = 0$.

For spectral analysis reasons we do not consider this semi-homogeneous problem but another one given by an inhomogeneous Helmholtz equation and homogeneous transmission conditions.

Also the given configuration shows that the problem may be additively decomposed by symmetrization with respect to the t-axis. Thus, considering the odd part of the problem, a Dirichlet boundary condition arises by continuity arguments. Finally we rotate the (s,t)-plane positively by $\pi/2$ and introduce the canonical cartesian unity vectors e_1 and e_2 on the rotated $\mathbb{R}^2$. So we arrive at the following classical formulation of the odd part problem:

Let $\kappa_1,\ \kappa_2 > 0$ *and*

$$\kappa : \mathbb{R} \to \mathbb{R}_+ := (0,\infty) \quad \text{with}$$

$$\kappa(x) = \begin{cases} \kappa_1, & x \geq 0 \\ \kappa_2, & x < 0. \end{cases} \tag{1.1}$$

Let $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}_+}$, *i.e.* $o < \arg\lambda < 2\pi$, *and* $\mathbb{R}_- := (-\infty, 0)$.
Let

$$\begin{aligned} \Omega &:= \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{R}^2 \\ Q_1 &:= \mathbb{R}_+ \times \mathbb{R}_+ \subset \Omega \\ Q_2 &:= \mathbb{R}_- \times \mathbb{R}_+ \subset \Omega. \end{aligned} \tag{1.2}$$

Let $a > 0$ *and* $g : \Omega \to \mathbb{C}$ *be continuous and exponentially bounded on* $\overline{\Omega}$, *i.e.:*

$$\sup\left\{ e^{+a\sqrt{x^2+y^2}} / g(x,y) / : (x,y) \in \overline{\Omega} \right\} < \infty. \tag{1.3}$$

Then $f : \Omega \to \mathbb{C}$ *is to be determined by the following conditions:*

i) $f \in C^2(Q_1) \cap C^2(Q_2)$ *and*

$$-\frac{1}{\kappa(x)} \Delta f(x,y) - \lambda f(x,y) = g(x,y), \quad (x,y) \in Q_1 \cup Q_2. \tag{1.4}$$

ii) $f \in C^1(\overline{Q}_1) \cap C^1(\overline{Q}_2)$ *and*

$$\begin{aligned} f(x,0) = 0, &\quad x \in \partial\Omega = \mathbb{R} \quad \textit{and} \\ f(0+0,y) = f(0-0,y), &\quad y > 0 \quad \textit{and} \\ D_1 f(0+0,y) = D_1 f(0-0,y), &\quad y > 0. \end{aligned} \tag{1.5}$$

iii) f *and* $\operatorname{grad} f$ *are exponentially bounded on* $\overline{\Omega}$, *i.e.: there exists* $b \in \mathbb{R}$ *with* $0 < b < a$, *such that it holds*:

$$\sup\left\{e^{+b\sqrt{x^2+y^2}}/h(x,y)/ : (x,y) \in \overline{\Omega}\right\} < \infty \quad \text{for} \quad h = f \quad \text{or} \quad h = \operatorname{grad} f . \tag{1.6}$$

For this problem a closed solution is formally derived in section two using the Fourier-sine-transformation.

In section three we study the differential operator L defined by

$$(L\bar{u})(x,y) := -\frac{1}{\kappa(x)}\Delta\bar{u}(x,y) \quad (x,y) \in Q_1 \cup Q_2 , \quad \bar{u} \in C^2(Q_1) \cap C^2(Q_2) . \tag{1.7}$$

It is shown, that this defines a selfadjoint operator A on the Lebesgue-space with weight

$$H := L_2(\Omega; \kappa(\cdot)) . \tag{1.8}$$

Thereby the inner products is given by

$$\rho(u,v) := (u,v) := \int_{\Omega} \overline{u(x,y)}v(x,y)\kappa(x)d(x,y) , \quad u,v \in H . \tag{1.9}$$

Section four is devoted to the construction of generalized eigenfunctions for the transmission problem of section one.

In section five the spectral analytical version of the formal solution of section two is presented. As it was shown in section two, after applying the Fourier-sine-transformation, the following operator is to be studied:

$$L_\tau w(x) := -\frac{1}{\kappa(x)}w''(x) - \tau^2 w(x) , \quad x \in \mathbb{R} \setminus \{0\} , \quad \tau \in \mathbb{R}_+ , \quad w \in C^2(\mathbb{R} \setminus \{0\}) \cap C^1(\mathbb{R}) . \tag{1.10}$$

This defines a selfadjoint operator A_τ on the Lebesgue-space with weight

$$H(\mathbb{R}) := L_2(\mathbb{R}, \kappa(\cdot)) . \tag{1.11}$$

Section six contains the main result of this paper, namely the normalization of the generalized eigenfunctions via the Stone-formula. For results based on this we refer to SCHULENBERGER and WILCOX ([11]), WILCOX ([14], [15]), LEIS ([5], [6]) and PICARD ([9]).

The analysis to be established can also be applied to half-space transmission problems of the Helmholtz equation on two quadrants involving "loaded" transmission conditions following LATZ and MEISTER ([4]) or involving third kind boundary conditions as it has been done in recent paper by MEISTER, SPECK and TEIXEIRA ([8]).

2 – Formal solution of the problem

Assume $f : \Omega \to \mathbb{C}$ to be a solution of the problem. Let the one-dimensional Fourier-sine-transformation of f with respect to the second variable y be defined as follows:

$$\hat{f} : \Omega \to \mathbb{C} \quad \text{with}$$

$$\hat{f}(x,\tau) := \sqrt{\frac{2}{\pi}} \int_0^\infty \sin \tau y \, f(x,y) dy\,, \quad (x,\tau) \in \Omega. \tag{2.1}$$

Applying this to the left-hand side and the right-hand side of (1.4) and using the boundary condition (1.5) ordinary differential equations result for $x > 0$ and $x < 0$ with right-hand sides given by

$$\hat{g} : \Omega \to \mathbb{C} \quad \text{with}$$

$$\hat{g}(x,\tau) := \sqrt{\frac{2}{\pi}} \int_0^\infty \sin \tau y \, g(x,y) dy\,. \tag{2.2}$$

In order to solve the transformed differential equations one has to fix a branch of the square root function:

$$\begin{aligned} &\gamma_j : \mathbb{R} \to \mathbb{C} \quad \text{with} \\ &\gamma_j(\tau) = \sqrt{\lambda\kappa_j - \tau^2}\,, \quad \tau \in \mathbb{R}\,, \\ &Jm\gamma_j(\tau) > 0\,, \quad \tau \in \mathbb{R}\,, \quad j = 1,2\,. \end{aligned} \tag{2.3}$$

Having in mind that $\hat{f}(\cdot,\tau) : \mathbb{R} \to \mathbb{C}$, $\tau > 0$, must vanish at infinity and must be continuously differentiable up to $x = 0$, it results after some elementary calculations:

$$\hat{f}(x,\tau) = \int_{\mathbb{R}} G_\lambda(x,t;\tau)\kappa(t)\hat{g}(t,\tau)dt\,, \quad (x,\tau) \in \Omega\,, \tag{2.4}$$

with

$$G_\lambda : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C} \quad \text{and}$$

$$\begin{aligned}
G_\lambda(x,t;\tau) &= \frac{i}{\gamma_1(\tau)+\gamma_2(\tau)} e^{i\gamma_1(\tau)(x+t)} + \\
&+ \frac{i}{2\gamma_1(\tau)}\left[e^{+i\gamma_1(\tau)|x-t|} - e^{+i\gamma_1(\tau)(x+t)}\right], \quad x \geq 0\,, \quad t \geq 0\,,\\
G_\lambda(x,t;\tau) &= \frac{i}{\gamma_1(\tau)+\gamma_2(\tau)} e^{+i\left(\gamma_1(\tau)t-\gamma_2(\tau)x\right)}\,, \quad x < 0\,, \quad t \geq 0\,,\\
G_\lambda(x,t;\tau) &= \frac{i}{\gamma_1(\tau)+\gamma_2(\tau)} e^{-i\gamma_2(\tau)(x+t)} + \\
&+ \frac{i}{2\gamma_2(\tau)}\left[e^{+i\gamma_2(\tau)|x-t|} - e^{-i\gamma_2(\tau)(x+t)}\right], \quad x < 0\,, \quad t < 0\,,\\
G_\lambda(x,t;\tau) &= \frac{i}{\gamma_1(\tau)+\gamma_2(\tau)} e^{-i\left(\gamma_2(\tau)t-\gamma_1(\tau)x\right)}\,, \quad x \geq 0\,, \quad t < 0\,.
\end{aligned} \tag{2.5}$$

From (2.4) follows by the inverse Fourier-sine-transformation:

$$f : \Omega \to \mathbb{C} \qquad \text{with}$$

$$\begin{aligned}
f(x,y) &= \frac{2}{\pi}\int_{\Omega} \gamma_\lambda(x,y;t,\xi)\kappa(t)g(t,\xi)d(t,\xi) = \\
&=: (\theta g)(x,y) \quad (x,y) \in \Omega \quad \text{with}
\end{aligned} \tag{2.6}$$

$$\gamma_\lambda : \Omega \times \Omega \to \mathbb{C} \qquad \text{and}$$

$$\gamma_\lambda(x,y;t,\xi) := \int_0^\infty \sin(y\tau)G_\lambda(x,t;\tau)\sin\tau\xi\, d\tau\,. \tag{2.7}$$

This result may be summarized as follows:

Proposition 1. *The formal classical solution of the problem is given by the formulas* (2.5), (2.6) *and* (2.7).

$\theta : L_2(\Omega) \to L_2(\Omega)$ *is the Green-operator of the problem and the kernel of the defining integral operator is the Green-function of the problem.*

3 – Selfadjointness

Like in the paper [15] by Wilcox, functional analytic reasoning will now be introduced. It will be shown, that an operator A with domain $D(A) \subset H$ may be associated to the stated problem, where H is an appropriate Hilbert-space. Then it will be discussed how the properties of $A : D(A) \to H$ are connected with the original problem.

Let $C^\infty(\Omega)$ denote the linear space of functions which are infinitely differentiable on Ω.

Let $C_0^\infty(\Omega)$ be the subspace of $C^\infty(\Omega)$ with functions having compact support in Ω.

Let M be the linear space of equivalence classes of square integrable functions on Ω.

Let $(M, \sigma) = L_2(\Omega)$ and $(M, \rho) = L_2(\Omega; \kappa)$ be the Lebesgue-spaces constructed by endowing M with the inner products:

$$\begin{aligned} &\sigma(u,v) := (u,v)_0 := \int_\Omega \overline{u(x,y)}v(x,y)d(x,y) \quad \text{and} \\ &\rho(u,v) := (u,v) := \int_\Omega \overline{u(x,y)}v(x,y)\kappa(x)d(x,y)\,, \\ &u, v \in M\,. \end{aligned} \tag{3.1}$$

Then (M, σ) and (M, ρ) are equivalent Hilbert-spaces.

Let $\alpha = (\alpha_1, , \alpha_2) \in \mathbb{N}_0^2$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with $|\alpha| := \alpha_1 + \alpha_2$ and $u \in M$. Then u is said to have the distributional derivative $D^\alpha u = v \in M$, if it holds ([2]):

$$\begin{aligned} &\int_\Omega u(x,y)D^\alpha\varphi(x,y)d(x,y) = \\ &= (-1)^{|\alpha|}\int_\Omega v(x,y)\varphi(x,y)d(x,y)\,, \quad \varphi \in C_0^\infty(\Omega)\,. \end{aligned} \tag{3.2}$$

In a similar way the distributional Laplace-operator is introduced via test-functions from $C_0^\infty(\Omega)$, namely: for $u \in M$ suitable, $\Delta^* u = v \in M$ is defined by the Green-formula:

$$\begin{aligned} &\int_\Omega u(x,y)\Delta\varphi(x,y)d(x,y) = \\ &= \int_\Omega v(x,y)\varphi(x,y)d(x,y)\,, \quad \varphi \in C_0^\infty(\Omega)\,. \end{aligned} \tag{3.3}$$

Let $\Delta_2(\Omega) \subset M$ be defined by:

$$\Delta_2(\Omega) := \{u \in M \wedge \Delta^* u \in M\}\,. \tag{3.4}$$

In the sequel of this chapter all derivatives arising are understood in the distributional sense.

Let $m \in \mathbb{N}_0$ and $(M, \sigma_m) = L_2^m(\Omega)$ be the Sobolev-space of order m on Ω:

$$(M, \sigma_m) := L_2^m(\Omega) = \{u \in M \wedge D^\alpha u \in M \quad \text{for} \quad \alpha \in \mathbb{N}_0^2\,,\ |\alpha| \leq m\}$$

with

$$\begin{aligned} &\sigma_m(u,v) := (u,v)_m := \\ &= \int_\Omega \sum_{|\alpha|\leq m} \overline{D^\alpha u(x,y)}v(x,y)d(x,y) \quad \text{for} \quad u,v \in M\,. \end{aligned} \tag{3.5}$$

$(M, \sigma_m) = L_2^m(\Omega)$, $m \in \mathbb{N}_0$, are Hilbert-spaces with $L_2^0(\Omega) = L_2(\Omega)$ and $\sigma_0 = \sigma$.

The linear space $C_0^\infty(\Omega)$ is a subspace of $L_2^m(\Omega)$ for all $m \in \mathbb{N}_0$. In the sense of this inclusion we define

$$L_2^{1,0}(\Omega) := Cl C_0^\infty(\Omega) \subset L_2^1(\Omega) = (M, \sigma_1)\,. \tag{3.6}$$

This definition includes the generalized homogeneous Dirichlet condition of the problem and was pointed out and discussed e.g. in [7] and in [15].

Now we shall construct an operator A on an appropriate space H associated to the problem formulated in section one and formally solved in section two:

$$H := (M, \rho) = L_2(\Omega, \kappa)$$

$$D(A) := \Delta_2(\Omega) \cap L_2^{1,0}(\Omega) \subset (M, \rho) \tag{3.7}$$

$$A := -\frac{1}{\kappa(\cdot)}\Delta^* : D(A) \to H\,.$$

Like in the paper [15] by WILCOX it will be shown that A is a self-adjoint operator on H.

This property is shown by several lemmas whose proofs may be immediately copied from WILCOX's paper.

LEMMA 1. *The following equations hold:*

$$(Au, v) = -\int_\Omega \Delta^* \bar{u}(x,y) v(x,y) d(x,y) \tag{3.8}$$

and

$$\begin{aligned}(Au, v) = \int_\Omega [D_1\bar{u}(x,y)D_1 v(x,y) + \\ + D_2\bar{u}(x,y)D_2 v(x,y)]d(x,y)\,, \\ u \in D(A)\,, \qquad v \in L_2^{1,0}(\Omega)\,.\end{aligned} \tag{3.9}$$

LEMMA 2.

$$D(A) \subset H \quad \textit{is dense} \tag{3.10}$$

and

$$(Au, v) = (u, Av)\,, \quad u, v \in D(A)\,. \tag{3.11}$$

LEMMA 3. *Let* $I : D(A) \to D(A) \subset H$ *be the identity operator on* $D(A)$. *Then*

$$(I + A) : D(A) \to H \quad \textit{is surjective}. \tag{3.12}$$

From the last two lemmas it follows

PROPOSITION 2. *$A : D(A) \to H$ being defined by* (3.7) *is selfadjoint on H.*

REMARK. A direct consequence of Lemma 1 is that A is nonnegative, from what follows ([3]) that the spectrum $\Sigma(A)$ of A is contained in $\overline{\mathbb{R}}_+ = [0, \infty)$.

As announced at the beginning some results are now stated connecting the abstract version of the problem in this chapter to the original one.

In section one a solution of the problem was required to have the property $f \in C^2(Q_1) \cap C^2(Q_2)$. This is partly contained in the following

LEMMA 4. *The following direct sum decomposition holds*

$$L_2(\Omega) = L_2(Q_1) \oplus L_2(Q_2)\,. \tag{3.13}$$

Like in [15] this follows immediately from regularity properties of elliptic operators ([7]).

The homogeneous boundary condition of the problem in section one has the following representation in $D(A)$:

LEMMA 5. *It holds:*

$$\text{“}v(\cdot,0) = 0 \quad \textit{in} \quad L_2(\mathbb{R})\,, \quad v \in D(A)\text{”} : \Longleftrightarrow$$

$$v(\cdot, y) \in L_2(\mathbb{R})\,, \quad v \in D(A) \wedge y \in \mathbb{R}_+\,, \tag{3.14}$$

and

$$\lim_{y \to 0+0} \|v(\cdot, y)\|_2 = 0\,, \quad v \in D(A)\,. \tag{3.15}$$

Concerning the first transmission condition (1.5) of the problem we arrive at:

LEMMA 6. *It holds:*

$$\text{“}v(0+0,\cdot) = v(0-0,\cdot) \ \textit{in}\ L_2(\mathbb{R}_+)\,, \quad v \in D(A)\text{”} : \Longleftrightarrow$$

$$v(x,\cdot) \in L_2(\mathbb{R}_+)\,, \quad v \in D(A) \wedge x \in \mathbb{R} \tag{3.16}$$

and

$$\lim_{x \to 0 \pm 0} \|v(x,\cdot) - v(0,\cdot)\|_2 = 0\,, \quad v \in D(A)\,. \tag{3.17}$$

For the second transmission condition (1.5) we get analogously:

LEMMA 7. *It holds:*

$$\text{“}D_1 v(0+0,\cdot) = D_1 v(0-0,\cdot) \quad \textit{in} \quad L_2(\mathbb{R}_+)\,, \quad v \in D(A)\text{”} : \Longleftrightarrow$$

$$D_1 v(x,\cdot) \in L_2(\mathbb{R}_+)\,, \quad v \in D(A) \wedge x \in \mathbb{R} \tag{3.18}$$

and

$$\lim_{x \to \pm 0} \|D_1 v(x,\cdot) - D_1 v(0,\cdot)\|_2 = 0\,, \quad v \in D(A)\,. \tag{3.19}$$

The proof of the above lemmas is essentially the same as in [15].
From the lemmas 4 to 7 follows

LEMMA 8. *It holds:*

$$\begin{aligned} D(A) = {} & L_2^1(\Omega) \wedge [L_2^2(Q_1) \oplus L_2^2(Q_2)] \wedge \\ & \Big\{ v \in M : [v(\cdot,0) = 0 \quad \textit{in} \quad L_2(\mathbb{R})] \wedge \\ & [v(0+0,\cdot) = v(0-0,\cdot) \quad \textit{in} \quad L_2(\mathbb{R}_+)] \wedge \\ & [D_1 v(0+0,\cdot) = D_1 v(0-0,\cdot) \quad \textit{in} \quad L_2(\mathbb{R}_+)] \Big\}\,. \end{aligned} \tag{3.20}$$

The above results may be summarized as follows:

PROPOSITION 3. *The operator A has the following properties:*
1. *A is nonnegative and the spectrum is contained in* $[0,\infty)$.
2. *$D(A) \subset H$ is characterized by formula* (3.20).

4 – Generalized eigenfunctions

The problem of section one is a semi-homogeneous one in the sense that the boundary-conditions are homogeneous and the differential equations inhomogeneous. Moreover $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ is assumed.

In this section we consider the corresponding eigenvalue problem: *Find functions $\varphi : \Omega \to \mathbb{C}$ with the following properties*:

(i) $\varphi \in C^2(Q_1) \cap C^2(Q_2)$ *and*

$$-\frac{1}{\kappa(x)}\Delta\varphi(x,y) = \mu\varphi(x,y)\,, \quad (x,y) \in \Omega\,, \quad \mu \geq 0\,, \tag{4.1}$$

(ii) $\varphi \in C^1(\overline{\Omega})$ *and*

$$\varphi(x,0) = 0\,, \quad x \in \partial\Omega = \mathbb{R}\,, \tag{4.2}$$

and

(iii)

$$\sup\Big\{|\varphi(x,y)| : (x,y) \in \Omega\Big\} < \infty\,. \tag{4.3}$$

Like in paper [15] by WILCOX we call solutions to this problem *generalized or improper eigenfunctions* to the problem of section one since they are not square integrable over Ω in general.

In the case $\mu = 0$ the improper eigenfunctions are the constant functions on Ω:

$$\begin{aligned} &\varphi(\cdot;0) : \Omega \to \mathbb{C} \qquad \text{with} \\ &\varphi(x,y;0) = \omega_0\,, \quad (x,y) \in \Omega\,, \quad \omega_0 \in \mathbb{C} \setminus \{0\}\,. \end{aligned}$$

They may be omitted in the sequel.

The construction of eigenfunctions in the case of $\mu > 0$ works with the help of the following ansatz:

$$\varphi(x,y) = \sin \tau y \chi(x)\,, \quad (x,y) \in \Omega\,, \quad \tau > 0\,. \tag{4.4}$$

This ansatz with $\chi : \mathbb{R} \to \mathbb{C}$ fulfills the Dirichlet boundary condition of the eigenvalue problem. The other conditions are reflected by the following reduced eigenvalue problem.

Determine functions $\chi : \mathbb{R} \to \mathbb{C}$ *with the following properties*:

(iv) $\chi \in C^2(\mathbb{R}_+) \cap C^2(\mathbb{R}_-)$ *and*

(4.5)
$$-\frac{1}{\kappa(x)}[D^2\chi(x) - \tau^2\chi(x)] = \mu\chi(x)\,, \quad x \in \mathbb{R}_- \cup \mathbb{R}_+\,, \quad \mu > 0\,, \quad \tau > 0$$

(v) $\chi \in C^1(\mathbb{R})$ *and*

(vi)

$$\sup\Big\{|\chi(x)| : x \in \mathbb{R}\Big\} < \infty\,. \tag{4.6}$$

Like in paper [15] functions $\chi : \mathbb{R} \to \mathbb{C}$ satisfying the above conditions will be called *reduced generalized or improper eigenfunctions* related to the problem of section one.

In order to analyse this problem we define:

$$\begin{aligned} &w_j : \mathbb{R}_+ \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+ \qquad \text{by} \\ &w_j(\tau,\mu) := \sqrt{|-\tau^2 + \mu\kappa_j|} = \\ &= \begin{cases} \sqrt{\mu\kappa_j - \tau^2}\,, & \text{if} \quad 0 < \tau^2 \le \mu\kappa_j \\ \sqrt{+\tau^2 - \mu\kappa_j}\,, & \text{if} \quad 0 < \mu\kappa_j < \tau^2\,, \end{cases} \\ &\tau > 0\,, \quad \mu > 0\,, \quad j = 1,2\,. \end{aligned} \tag{4.7}$$

We start then with the simplest case, namely

$$\begin{aligned} \kappa_2 &= \kappa_1 =: \kappa_0 \\ w_2 &= w_1 =: w_0\,. \end{aligned} \tag{4.8}$$

Assume $\tau > 0$ with $\mu\kappa_0 < \tau^2$. All solutions to (4.5) in this case are unbounded on $\mathbb{R}$.

Therefore there exists no reduced improper eigenfunctions.

Assume further $\tau > 0$ with $0 < \tau^2 < \mu\kappa_0$. Even and odd solutions to the reduced problem in this case are given by:

$$\chi_1^{(0)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \qquad \text{with}$$

$$\chi_1^{(0)}(x;\tau,\mu) := \eta_1(\tau,\mu) \cdot \cos\left[w_0(\tau,\mu)x\right], \tag{4.9}$$

$$x \in \mathbb{R}, \quad \eta_1(\tau,\mu) \in \mathbb{C} \setminus \{0\}, \quad \tau > 0, \quad \mu > 0, \quad 0 < \tau^2 \leq \mu\kappa_0$$

and

$$\chi_2^{(0)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \qquad \text{with}$$

$$\chi_2^{(0)}(x;\tau,\mu) := \eta_2(\tau,\mu) \cdot \sin\left[w_0(\tau,\mu)x\right], \tag{4.10}$$

$$x \in \mathbb{R}, \quad \eta_2(\tau,\mu) \in \mathbb{C} \setminus \{0\}, \quad \tau > 0, \quad \mu > 0, \quad 0 < \tau^2 \leq \mu\kappa_0 .$$

The next case of our interest is $\kappa_2 < \kappa_1$.

Assume $\mu > 0$, then it follows $0 < \mu\kappa_2 < \mu\kappa_1$.

Let $\tau > 0$ and $\mu\kappa_1 \leq \tau^2$. Then it follows $\tau^2 - \mu\kappa_1 \geq 0$ and $\tau^2 - \mu\kappa_2 > 0$. Therefore in this case no bounded solutions exist.

Let $\tau > 0$ and $\mu\kappa_2 < \tau^2 < \mu\kappa_1$, then solutions to the reduced eigenvalue problem are given by:

$$\chi_1^{(1)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \qquad \text{with}$$

$$\chi_1^{(1)}(x;\tau,\mu) := \alpha_1(\tau,\mu)\Big\{ \cos\left[w_1(\tau,\mu)x\right] + \tag{4.11}$$

$$+ \frac{w_2(\tau,\mu)}{w_1(\tau,\mu)} \sin\left[w_1(\tau,\mu)\right]\Big\}, \quad x \geq 0$$

and

$$\chi_1^{(1)}(x;\tau,\mu) := \alpha_1(\tau,\mu) e^{w_2(\tau,\mu)x}, \quad x < 0,$$

$$\alpha_1(\tau,\mu) \in \mathbb{C} \setminus \{0\}, \quad \tau > 0, \quad \mu > 0, \quad \mu\kappa_2 < \tau^2 < \mu\kappa_1 .$$

Let $\tau > 0$ and $\tau^2 = \mu\kappa_2$, then no solutions to the reduced eigenvalue problem exist.

Let $\tau > 0$ and $0 < \tau^2 < \mu\kappa_2$ hold, then, by *a* standard argument as before, reduced improper eigenfunctions are given by the following two formulas:

$$\chi_2^{(1)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \quad \text{with}$$

$$\chi_2^{(1)}(x;\tau,\mu) := \begin{cases} \alpha_2(\tau,\mu)\cos[w_1(\tau,\mu)x], & x \geq 0 \\ \alpha_2(\tau,\mu)\cos[w_2(\tau,\mu)x], & x < 0, \end{cases} \tag{4.12}$$

$$\alpha_2(\tau,\mu) \in \mathbb{C}\setminus\{0\}, \quad \tau > 0, \quad \mu > 0, \quad 0 < \tau^2 < \mu\kappa_2$$

and

$$\chi_3^{(1)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \quad \text{with}$$

$$\chi_3^{(1)}(x;\tau,\mu) := \begin{cases} \dfrac{\alpha_3(\tau,\mu)}{w_1(\tau,\mu)}\sin[w_1(\tau,\mu)x], & x \geq 0 \\ \dfrac{\alpha_3(\tau,\mu)}{w_2(\tau,\mu)}\sin[w_2(\tau,\mu)x], & x < 0, \end{cases} \tag{4.13}$$

$$\alpha_3(\tau,\mu) \in \mathbb{C}\setminus\{0\}, \quad \tau > 0, \quad \mu > 0, \quad 0 < \tau^2 < \mu\kappa_2 .$$

We have finally to discuss the case $\kappa_1 < \kappa_2$.

Let $\mu > 0$ and $\tau > 0$ and $\mu\kappa_2 < \tau^2$, then, like before, it holds $\tau^2 - \mu\kappa_2 \geq 0$ and $\tau^2 - \mu\kappa_1 \geq 0$, so that in this case no reduced improper eigenfunctions exist.

Let $\tau > 0$ and $\mu\kappa_1 < \tau^2 < \mu\kappa_2$ hold, then solutions to the reduced eigenvalue problem in this case are given by:

$$\chi_1^{(2)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \quad \text{with}$$
$$\chi_1^{(2)}(x;\tau,\mu) := \beta_1(\tau,\mu)e^{-w_1(\tau,\mu)x}, \quad x \geq 0$$

and

$$\chi_1^{(2)}(x;\tau,\mu) := \beta_1(\tau,\mu)\Big\{\cos[w_2(\tau,\mu)x] + \\ - \frac{w_1(\tau,\mu)}{w_2(\tau,\mu)}\sin[w_2(\tau,\mu)x]\Big\}, \quad x < 0, \tag{4.14}$$

$$\beta_1(\tau,\mu) \in \mathbb{C}\setminus\{0\}, \quad \tau > 0, \quad \mu > 0, \quad \mu\kappa_1 < \tau^2 < \mu\kappa_2 .$$

Let $\tau > 0$ and $\tau_2 = \mu\kappa_1$, then, like before, no solutions to the reduced eigenvalue problem exist.

Let $\tau > 0$ and $0 < \tau^2 < \mu\kappa_1$ hold, then, like before, reduced improper eigenfunctions are given by the following two formulas:

$$\chi_2^{(2)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \qquad \text{with}$$

$$\chi_2^{(2)}(x;\tau,\mu) := \begin{cases} \beta_2(\tau,\mu)\cos\left[w_1(\tau,\mu)x\right], & x \geq 0 \\ \beta_2(\tau,\mu)\cos\left[w_2(\tau,\mu)x\right], & x < 0, \end{cases} \tag{4.15}$$

$$\beta_2(\tau,\mu) \in \mathbb{C}\setminus\{0\}, \quad \tau > 0, \quad \mu > 0, \quad 0 < \tau^2 < \mu\kappa_1$$

and

$$\chi_3^{(2)}(\cdot;\tau,\mu) : \mathbb{R} \to \mathbb{C} \qquad \text{with}$$

$$\chi_3^{(2)}(x;\tau,\mu) := \begin{cases} \dfrac{\beta_3(\tau,\mu)}{w_1(\tau,\mu)}\sin\left[w_1(\tau,\mu)x\right], & x \geq 0 \\ \dfrac{\beta_3(\tau,\mu)}{w_2(\tau,\mu)}\sin\left[w_2(\tau,\mu)x\right], & x < 0, \end{cases} \tag{4.16}$$

$$\beta_3(\tau,\mu) \in \mathbb{C}\setminus\{0\}, \quad \tau > 0, \quad \mu > 0, \quad 0 < \tau^2 < \mu\kappa_1 .$$

From the above improper eigenfunctions result by means of (4.4) generalized eigenfunctions of A. Like in WILCOX' paper they are related with help of the arguments of the section before by their property of being locally in $D(A)$, i.e.

$$\varphi \cdot \chi \in D(A) \quad \text{for all} \quad \chi \in C_0^\infty(\Omega) . \tag{4.17}$$

The main result of this section are now collected in

PROPOSITION 4. 1. *If $\kappa_2 = \kappa_1 = \kappa_0$ holds, generalized eigenfunctions exist, namely*

$$\varphi_1^{(0)}(\cdot;\tau,\mu) : \Omega \to \mathbb{C} \qquad \textit{with}$$

$$\varphi_1^{(0)}(x,y;\tau,\mu) := \sqrt{\frac{2}{\pi}}\sin\tau y \cdot \chi_1^{(0)}(x;\tau,\mu) , \tag{4.18}$$

$$(x,y) \in \Omega , \quad \tau > 0, \quad \mu > 0, \quad 0 < \tau^2 \leq \mu\kappa_0 ,$$

and

$$\varphi_2^{(0)}(\cdot;\tau,\mu):\Omega\to\mathbb{C}\qquad \textit{with}$$

(4.19)
$$\varphi_2^{(0)}(x,y;\tau,\mu):=\sqrt{\frac{2}{\pi}}\sin\tau y\cdot\chi_2^{(0)}(x;\tau,\mu)\,,$$
$$(x,y)\in\Omega\,,\quad \tau>0\,,\quad \mu>0\,,\quad 0<\tau^2<\mu\kappa_0\,.$$

Thereby the reduced generalized eigenfunctions $\chi_j^{(0)}(\cdot;\tau,\mu)$, $j=1,2$, *are given by the formulas* (4.9) *and* (4.10).

2. *If* $\kappa_2<\kappa_1$ *holds, generalized eigenfunctions exist, given by*

$$\varphi_1^{(1)}(\cdot;\tau,\mu):\Omega\to\mathbb{C}\qquad \textit{with}$$

(4.20)
$$\varphi_1^{(1)}(x,y;\tau,\mu):=\sqrt{\frac{2}{\pi}}\sin\tau y\cdot\chi_1^{(1)}(x;\tau,\mu)\,,$$
$$(x,y)\in\Omega\,,\quad \tau>0\,,\quad \mu>0\,,\quad \mu\kappa_2<\tau^2<\mu\kappa_1$$

and

$$\varphi_j^{(1)}(\cdot;\tau,\mu):\Omega\to\mathbb{C}\qquad \textit{with}$$

(4.21)
$$\varphi_j^{(1)}(x,y;\tau,\mu):=\sqrt{\frac{2}{\pi}}\sin\tau y\chi_j^{(1)}(x;\tau,\mu)\,,$$
$$(x,y)\in\Omega\,,\quad \tau>0\,,\quad \mu>0\,,\quad 0<\tau^2<\mu\kappa_2\,,\quad j=2,3\,.$$

Thereby the reduced generalized eigenfunctions $\chi_j^{(1)}(\cdot;\tau,\mu)$, $j=1,2,3$, *are given by the formulas* (4.11), (4.12) *and* (4.13).

3. *If* $\kappa_1<\kappa_2$ *holds, generalized eigenfunctions exist, given by*

$$\varphi_1^{(2)}(\cdot;\tau,\mu):\Omega\to\mathbb{C}\qquad \textit{with}$$

(4.22)
$$\varphi_1^{(2)}(x,y;\tau,\mu):=\sqrt{\frac{2}{\pi}}\sin\tau y\cdot\chi_1^{(2)}(x;\tau,\mu)\,,$$
$$(x,y)\in\Omega\,,\quad \tau>0\,,\quad \mu>0\,,\mu\kappa_1<\tau^2<\mu\kappa_2$$

and

$$\varphi_j^{(2)}(\cdot;\tau,\mu):\Omega\to\mathbb{C} \quad \textit{with}$$

$$\varphi_j^{(2)}(x,y;\tau,\mu):=\sqrt{\frac{2}{\pi}}\sin\tau y\cdot\chi_j^{(2)}(x;\tau,\mu)\,, \tag{4.23}$$

$$(x,y)\in\Omega\,,\quad \tau>0\,,\quad \mu>0\,,\quad 0<\tau^2<\mu\kappa_1\,,\quad j=2,3\,.$$

Thereby the reduced improper eigenfunctions $\chi_j^{(2)}(\cdot;\tau,\mu)$, $j=1,2,3$, *are given by the formulas* (4.14), (4.15), (4.16).

4. *The generalized eigenfunctions are locally in* $D(A)$.

5 – Fourier-sine-transformation

In section two a formal solution to the problem of section one was derived using the Fourier-sine-transformation.

In this section the functional analytical version of this procedure is presented and consequences are stated.

We start with the following classical result:

LEMMA 9. 1. $(F_s u)(x,\tau):=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin\tau y\cdot u(x,y)dy$ *is defined for almost every* $(x,\tau)\in\Omega$ *for all* $u\in H:=L_2(\Omega;\kappa)$.

2. $F_s u=:\hat{u}\in H$, $u\in H$.

3. $F_s:H\to H$ *is an unitary operator.*

Since $D(A)\subset H$ and $A:D(A)\to H$ and $F_s:H\to H$ is bijective, we may define:

$$\begin{aligned} &D(\widehat{A}):=F_s(D(A))\subset H \quad \text{and}\\ &\widehat{A}:=F_s A F_s^{-1}:D(\widehat{A})\to H\,. \end{aligned} \tag{5.1}$$

We call $\widehat{A}$ *the Fourier-sine-transformation of* A.

Like in the paper [15] by WILCOX, there exists a characterization of $D(A)$ and $D(\widehat{A})$, which is omitted in this place. We confine ourselves to the following result:

PROPOSITION 5. *It holds*

$$\hat{A}\hat{v}(x,\tau) = -\frac{1}{\kappa(x)}[D_1^2\hat{v}(x,\tau) - \tau^2\hat{v}(x,\tau)] \tag{5.2}$$

for almost all $(x,\tau) \in \Omega$ *and for all* $\hat{v} \in D(\hat{A})$.

It was shown in section two that the Fourier-sine-transformation reduces the original problem to a one-dimensional one for almost all $\tau \in \mathbb{R}_+$.

For this situation the arguments of section three to prove the selfadjointness are now repeated.

Let $C_0^\infty(\mathbb{R})$ be the linear space of infinitely differentiable functions on $\mathbb{R}$ with compact support.

Let $M(\mathbb{R})$ be the linear space of equivalence classes of square integrable functions on $\mathbb{R}$.

Let as a distributional derivative be defined:

$$\Delta_2(\mathbb{R}) := \{p \in M(\mathbb{R}) : D^2p \in M(\mathbb{R})\}. \tag{5.3}$$

Let $L_2^1(\mathbb{R})$ be the Sobolev space of order one on $\mathbb{R}$ and

$$L_2^{1,0}(\mathbb{R}) := Cl C_0^\infty(\mathbb{R}) \subset L_2^1(\mathbb{R}). \tag{5.4}$$

Then we can define in analogy to section three to $\tau \in \mathbb{R}_+$:

$$\begin{aligned} &H(\mathbb{R}) := L_2(\mathbb{R};\kappa),\\ &D(A_\tau) := \Delta_2(\mathbb{R}) \cap L_2^{1,0}(\mathbb{R}) \subset H(\mathbb{R}),\\ &A_\tau : D(A_\tau) \to H(\mathbb{R}) \qquad \text{with}\\ &A_\tau p(x) := -\frac{1}{\kappa(x)}[D^2p(x) - \tau^2 p(x)]\\ &\text{for almost } x \in \mathbb{R} \text{ and all } p \in D(A_\tau). \end{aligned} \tag{5.5}$$

As in WILCOX's paper [15] we then have in analogy to proposition 2:

PROPOSITION 6. *It holds for* $\tau \in \mathbb{R}_+$:

$$A_\tau \quad \text{is selfadjoint,}$$

$$(A_\tau \nu, \nu) \geq 0\,, \quad v \in D(A_\tau)\,,$$

$$\Sigma(A_\tau) \subset [0, \infty)\,,$$

$$\Sigma(A_\tau) \quad \text{contains no eigenvalues.}$$

6 – Application of Stone's formula

We are now in the position to apply the spectral theoretical apparatus. Especially we can apply the Stone's formula ([14, [15], [5], [6], [9]) to calculate explicitely the spectral measure, which means to normalize the generalized eigenfunctions of section four. Since this procedure is quite technical, we will confine ourselves to giving the results.

In the case $\kappa_2 = \kappa_1 = \kappa_0$ the following normalization factors appear in formulas (4.9) and (4.10):

$$\eta_1(\tau, \mu) = \eta_2(\tau, \mu) = \sqrt{\frac{1}{2\pi w_0(\tau, \mu)}}\,. \tag{6.1}$$

In the case $\kappa_2 < \kappa_1$, the resulting spectral measure related to the generalized eigenfunctions under (4.11), (4.12) and (4.13) is given by

$$\alpha_1(\tau, \mu) = \sqrt{\frac{w_1(\tau, \mu)}{\pi\mu(\kappa_1 - \kappa_2)}}\,, \tag{6.2}$$

$$\alpha_2(\tau, \mu) = \sqrt{\frac{1}{\pi[w_1(\tau, \mu) - w_2(\tau, \mu)]}} \tag{6.3}$$

$$\alpha_3(\tau, \mu) = \sqrt{\frac{w_1(\tau, \mu) \cdot w_2(\tau, \mu)}{\pi[w_1(\tau, \mu) - w_2(\tau, \mu)]}}\,. \tag{6.4}$$

Finally the third case $\kappa_1 < \kappa_2$, normalization in (4.14), (4.15) and

(4.16) is achieved by

$$\beta_1(\tau,\mu) = \sqrt{\frac{w_2(\tau,\mu)}{\pi\mu(\kappa_2-\kappa_1)}}\,, \tag{6.5}$$

$$\beta_2(\tau,\mu) = \alpha_2(\tau,\mu)\,, \tag{6.6}$$

$$\beta_3(\tau,\mu) = \alpha_3(\tau,\mu)\,. \tag{6.7}$$

The main result of our paper may be summarized as follows.

PROPOSITION 7. *The generalized eigenfunctions of the transmission problem constructed in section four are normalized by spectral measures given by the formulas* (6.1) *to* (6.7).

From this result spectral theorems may be established following the basic paper [15] by WILCOX and related to the reduced one-dimensional problem and the original two-dimensional transmission problem.

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