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Some properties of the sixth-order Legendre-type differential expression

W.N. EVERITT – L.L. LITTLEJOHN – S.M. LOVELAND

This paper is dedicated to Professor Gaetano Fichera in recognition of his manifold contributions to mathematics.

RIASSUNTO: *Il lavoro riguarda certe proprietà di una espressione differenziale del tipo di Legendre. Ci sono cinque diverse espressioni di questo tipo, normalizzate nell'intervallo* $[-1,+1]$ *dell'asse reale. La prima è del secondo ordine (la classica espressione di Legendre), tre espressioni sono del terzo ordine (quelle trovate da H.L. Krall nel 1938 e nel 1940). L'ultima espressione è del sesto ordine ed è stata trovata da L.L. Littlejohn nel 1981; essa ha un certo numero di proprietà interessanti, che sono esaminate in questo lavoro. Si determinano in particolare le proprietà di regolarità degli elementi del dominio massimale ed il dominio dell'operatore autoaggiunto. Questi risultati sono messi in relazione con i polinomi ortogonali generati in uno spazio di misura e con quelli generati dalla equazione differenziale spettrale del sesto ordine, associata alla espressione differenziale del tipo di Legendre.*

ABSTRACT: *This paper is concerned with certain properties of one of the Legendre-type differential expressions. After normalization to the compact interval* $[-1,1]$ *of the real line, there are five distinct such differential expressions. There is one of the second order (the classical Legendre differential expression), three expressions of the fourth order (discovered by H.L. Krall in 1938 and 1940), and one of the sixth order (discovered by Littlejohn in 1981). The sixth-order expression has a number of interesting properties when considered in the classical integrable-square space on* $(-1,1)$*, and in the relevant measure integrable-square space on* $[-1,1]$*. The paper discusses some of these properties and determines the smoothness conditions satisfied by elements of the maximal domain and the self-adjoint operator domain. These results are related to the orthogonal polynomials generated, firstly in the measure space and, secondly, by the* *sixth-order spectral differential equation linked to the Legendre-type differential expression.*

KEY WORDS: *Legendre-type differential expressions - Legendre-type orthogonal polynomials - Positive Borel measure - Legendre integral - Lagrange symmetric differential expression - Maximal domain - Self-adjoint operator.*

A.M.S. CLASSIFICATION: 33A65 - 34B20 - 41A10

1 – Introduction

The positive and non-negative integers are denoted by $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{N}_0 = \{0,1,2,\ldots\}$, and the real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$.

With M and N real, non-negative parameters let the monotonic, non-decreasing function $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\hat{\mu}(x) = \begin{cases} -1 - M & (x \in (-\infty, -1]) \\ x & (x \in (-1, 1)) \\ 1 + N & (x \in [1, \infty)) . \end{cases} \tag{1.1}$$

Let μ be the regular, non-negative measure generated by $\hat{\mu}$ on the Borel sets of $\mathbb{R}$, and let $L^2([-1,1];\mu)$ denote the integrable-square Hilbert space of equivalence classes of Borel measurable functions with norm and inner product given, respectively, by

$$\|f\|_\mu^2 := \int_{[-1,1]} |f(x)|^2 \, d\mu(x) \tag{1.2}$$

$$\begin{aligned} (f,g)_\mu &:= \int_{[-1,1]} f(x)\bar{g}(x) \, d\mu(x) \\ &= Mf(-1)\bar{g}(-1) + \int_{-1}^{1} f(x)\bar{g}(x)dx + Nf(1)\bar{g}(1) . \end{aligned} \tag{1.3}$$

The integral in (1.2) is a Lebesgue-Stieltjes integral and the integral in (1.3) is the standard Lebesgue integral.

The measure μ has finite moments in respect of the sequence of powers $\{x^n | n \in \mathbb{N}_0\}$; *i.e.*

$$x \mapsto x^n \in L^2([-1,1];\mu) \quad \text{or} \quad \int_{[-1,1]} |x^n|^2 d\mu(x) < \infty \quad (n \in \mathbb{N}_0) .$$

Furthermore, the set $\{x^n | n \in \mathbb{N}_0\}$ is linearly independent in $L^2([-1,1];\mu)$.

The Legendre-type polynomials is the orthogonal polynomial systems formed by applying the Gram-Schmidt orthogonalization process to the set $\{x^n | n \in \mathbb{N}_0\}$ in $L^2([-1,1];\mu)$. Five cases emerge from the measure μ:

$$\text{(1.4)} \qquad \begin{array}{ll} \text{(i)} & M = N = 0 \\ \text{(ii)} & M > 0, \quad N = 0 \\ \text{(iii)} & M = 0, \quad N > 0 \\ \text{(iv)} & M = N > 0 \\ \text{(v)} & M > 0, \quad N > 0, \quad M \neq N. \end{array}$$

Case (i) yields the classical Legendre polynomials; see CHIHARA [1, Chapter V] and SZEGÖ [17, Chapter IV]. The cases (ii), (iii) and (iv) were considered by H.L. KRALL [10], [11] and A.M. KRALL [9]. The final case (v), which is the subject of this paper, was developed by LITTLEJOHN in his thesis [12].

The orthogonal polynomials in all these five cases (1.4) are special examples of the general KOORNWINDER polynomials considered in [8]; see in particular [8; Sections 1-4] with $\alpha = \beta = 0$. The general Koornwinder notation of $\{P_n^{\alpha,\beta,M,N}(x) | x \in [-1,1]; n \in \mathbb{N}_0\}$ then reduces to

$$\text{(1.5)} \qquad \{P_n^{0,0,M,N}(x) | x \in [-1,1]; n \in \mathbb{N}_0\}$$

for the orthogonal polynomials considered in this paper.

Another significant unifying property of these five cases (1.4) of orthogonal polynomials is that each system is also generated by a formally symmetric spectral differential equation of the form

$$\text{(1.6)} \qquad \sum_{r=0}^{s} (-1)^r \left(q_r(x) y^{(r)}(x)\right)^{(r)} = \lambda y(x) \qquad (x \in (-1,1)),$$

where $s \in \mathbb{N}$, the spectral parameter $\lambda \in \mathbb{C}$, and the coefficients

$$\{q_r | r = 0, 1, \ldots s\}$$

are real-valued polynomials on $\mathbb{R}$ with degree $(q_r) = 2r$ $(r = 0, 1, \ldots s)$. The best possible (*i.e.* the smallest) integer s for which (1.6) is effective depends on the particular case determined by (1.4); the coefficients $\{q_r\}$ depend not only on the case in (1.4) but also on the parameters M and N, but not on the spectral parameter λ.

For case (i) of (1.4), we have $s = 1$, yielding the classical second-order Legendre differential equation (see [1] and [17]). For cases (ii), (iii) and (iv), the value of s is 2, yielding the fourth-order Legendre-type differential equations of H.L. KRALL [10], [11]. For case (v), $s = 3$, yielding the sixth-order Legendre-type differential equation studied by LITTLEJOHN [12].

Later work on these Legendre-type differential equations was undertaken by EVERITT, A.M. KRALL, LITTLEJOHN, LOVELAND, and MARIĆ; see [3], [4], [5], [7], and [9]. A detailed statement of some properties of all five Legendre-type differential equations (1.6) can be found in the research report of EVERITT, LITTLEJOHN, and LOVELAND [6, Sections 0,1, and 2]. The spectral theory of the self-adjoint differential operators generated by the differential equations (1.6) in the Hilbert spaces $L^2([-1,1];\mu)$ is considered in detail in the thesis of LOVELAND [13]; see also the forthcoming papers [14] and [15].

In this paper, we are concerned with properties of the operator domains arising in case (v) of (1.4) when the order of the Legendre-type differential equation is 6. We give below the explicit form of this differential equation, quoting from [6, Section 1, (1.22, 1.23, 1.24)]. For this purpose it is convenient to introduce the positive numbers A and B defined by (recall (v) of (1.4))

$$A = M^{-1} \qquad B = N^{-1}. \tag{1.7}$$

The differential equation then takes the form, hereby defining, in the notation of [6, (1.22)], the differential expression $M_k^{(3)}[\cdot]$

(1.8)
$$M_k^{(3)}[y](x) := -\left((1-x^2)^3 y^{(3)}(x)\right)^{(3)} + \left((1-x^2)(12+\alpha(1-x^2))y''(x)\right)'' - (\pi(x)y'(x))' + ky(x) = \lambda y(x) \qquad (x \in (-1,1)),$$

where, see (1.7),

$$\alpha := 3A + 3B + 6, \tag{1.9}$$

$$(1.10)\qquad \pi(x) := 6(2AB+3A+3B+4+2(A-B)x-(A+B+2AB)x^2),$$

and $k \geq 0$ is a translation parameter essential for the proof of certain spectral theoretic properties of the associated differential operators.

If, in the notation of Koornwinder, $\{P_n^{0,0,M,N}(x) | n \in \mathbb{N}_0\}$ denotes the system of orthogonal polynomials arising in case (v) of (1.4) and the corresponding measure μ, then it was established by LITTLEJOHN in [12] that $P_n^{0,0,M,N}(\cdot)$ is a solution of the differential equation (1.8) with $\lambda = \lambda_n(k)$ where (the set $\{\lambda_n(k) | n \in \mathbb{N}_0\}$ is called the set of eigenvalues of (1.8))

(1.11)
$$\begin{aligned}\lambda_n(k) = {} & (24AB + 12A + 12B)n + (12AB + 42A + 42B + 72)n(n-1)\\ & + (24A + 24B + 168)n(n-1)(n-2)\\ & + (3A + 3B + 96)n(n-1)(n-2)(n-3)\\ & + 18n(n-1)(n-2)(n-3)(n-4)\\ & + n(n-1)(n-2)(n-3)(n-4)(n-5) + k \qquad (n \in \mathbb{N}_0);\end{aligned}$$

this solution property of $P_n^{0,0,M,N}(\cdot)$ holds for all $n \in \mathbb{N}_0$. Furthermore the explicit form of these polynomials was obtained by LITTLEJOHN in [12], and is (recall (1.7))

(1.12)
$$P_n^{0,0,M,N}(x) = \sum_{r=0}^{n} \frac{(-1)^{[r/2]}(2n-r)!Q(n,r)x^{n-r}}{2^{n+1}(n-[(r+1)/2])![r/2]!(n-r)!(n^2+n+A+B)},$$

where

$$\begin{aligned}Q(n,r) = {} & \frac{1+(-1)^r}{2}\Big[(n^4 + (2A+2B-1)n^2 + 4AB)\\ & + 2r(n^2+n+A+B)\Big] + \frac{1-(-1)^r}{2}(4B-4A),\end{aligned}$$

and $[\cdot]$ denotes the greatest integer function. This definition of $P_n^{0,0,M,N}(\cdot)$ gives a normalization property of $P_n^{0,0,M,N}(1) = 1 (n \in \mathbb{N}_0)$.

In this paper it is convenient to take $k = 1$, and then with only one case of (1.4) under consideration, to define the differential expression $M[\cdot]$

by

$$M[\cdot] = M_1^{(3)}[\cdot]\,, \tag{1.13}$$

with eigenvalues $\{\lambda_n = \lambda_n(1) | n \in \mathbb{N}_0\}$.

The differential expression $M[\cdot]$ is Lagrange symmetric and of the general form considered in the now classic text of NAIMARK [16, Chapter V]. The domain D of $M[\cdot]$ is defined by

$$D := \left\{ f : (-1,1) \to \mathbb{C} | f \in AC_{\mathrm{loc}}^{(r)}(-1,1), r = 0,1,2,3,4,5 \right\}$$

and Green's formula takes the form, for all $f, g \in D$ and for all compact $[\alpha, \beta] \subset (-1,1)$,

$$\int_\alpha^\beta \{\bar{g}(x) \cdot M[f](x) - f(x) \cdot \overline{M[g]}(x)\} dx = [f,g](x) \Big|_\alpha^\beta \,. \tag{1.14}$$

Here the skew-symmetric bilinear form $[\cdot,\cdot](\cdot)$: $D \times D \times (-1,1) \to \mathbb{C}$, see [16, Section 15.3], is given explicitly in Section Six below, see (6.2).

The maximal domain of $M[\cdot]$ in $L^2(-1,1)$ is defined by (note Δ here is $\Delta_1^{(3)}$ of [6, Section 1])

$$\Delta := \{ f : (-1,1) \to \mathbb{C} | f \in D; f, M[f] \in L^2(-1,1)\}\,. \tag{1.15}$$

From (1.14) it follows that the limits

$$\lim_{x \to \pm 1} [f,g](x) := [f,g](\pm 1) \tag{1.16}$$

exist and are finite in $\mathbb{C}$, for all $f, g \in \Delta$.

The classical theory of the determination of all self-adjoint operators generated by $M[\cdot]$ in $L^2(-1,1)$ is given in [16, Section 18]. In this space $L^2(-1,1)$, the domains of all self-adjoint operators with *separated* boundary conditions applied at the endpoints ± 1, are found by applying a well-determined number of boundary conditions to elements of the maximal domain Δ, of the form

$$[f, w_-](-1) = 0 \qquad [f, w_+](1) = 0\,; \tag{1.17}$$

here w_- and w_+ are chosen in a prescribed way from Δ. For details of this method see [16, Section 18], and for the application in the case of $M[\cdot]$ in $L^2(-1,1)$ full details are given in [13, Section 7.6].

In this paper we are concerned with the domain $D(T)$ of the self-adjoint operator T in the Hilbert space $L^2([-1,1];\mu)$, with measure μ determined by (v) (1.4), such that:

(i) the spectrum of T is discrete with eigenvalue $\{\lambda_n | n \in \mathbb{N}_0\}$ given by (1.11) with $k = 1$,

(ii) the corresponding eigenvectors of T are the orthogonal polynomials $\{P_n^{0,0,M,N} | n \in \mathbb{N}_0\}$.

This required operator T was first defined in [12] and later considered in [13]. From these works we take the following definitions:

$$\text{(1.18)} \qquad \text{(i)} \quad D(T) := \{f \in \Delta | [f, \psi_-](-1) = [f, \psi_+](1) = 0\},$$

where $\psi_\pm \in \Delta$ and are defined on $[-1,1]$ by

$$\text{(1.19)} \qquad \begin{aligned} \psi_-(x) &= 4(1-x^2) + (B+2)(1-x^2)^2 \\ \psi_+(x) &= 4(1-x^2) + (A+2)(1-x^2)^2, \end{aligned}$$

$$\text{(1.20)} \quad \text{(i)} \ (Tf)(x) = \begin{cases} 24A[f''(-1) - (B+1)f'(-1)] + f(-1) & x = -1 \\ M[f](x) \qquad \text{almost all} & x \in (-1,1) \\ 24B[f''(1) + (A+1)f'(1)] + f(1) & x = +1. \end{cases}$$

The spectral properties of T are established in [13, Chapter VII].

It will not escape notice that the definition of the operator T requires information about $f \in D(T)$ at the singular endpoints ± 1 of the interval $(-1,1)$. The purpose of this paper is to prove the following Theorem which distinguishes between the properties of the maximal domain Δ of $M[\cdot]$ in $L^2(-1,1)$, and of the operator domain $D(T)$ in $L^2([-1,1];\mu)$. The properties of the domain $D(T)$ justify the explicit definition of the operator T as given in (1.20).

THEOREM 1.1. *Let the Lagrange symmetric differential expression $M[\cdot]$ be defined by* (1.8) *and* (1.13)*; let the maximal domain Δ be defined by* (1.15)*; let the operator domain $D(T)$ be defined by* (1.18) *and* (1.19). *Then the following properties hold:*

(i) *if* $f \in \Delta$, *then* $f' \in L^2(-1,1)$, $f \in AC[-1,1]$, *and*

$$f'(x) = O(|\ln(1-x^2)|) \qquad (x \to \pm 1), \tag{1.21}$$

(ii) *if* $f \in D(T)$, *then* $f^{(3)} \in L^2(-1,1)$ *and* $f, f', f'' \in AC[-1,1]$.

The result stated in (i) *and* (ii) *are best possible in the following sense:*

(i)* *there exists* $g \in \Delta$ *such that* $g'(x) \sim \ln(1-x^2)$ $(x \to \pm 1)$

(ii)* *there exists* $g \in D(T)$ *such that* $g^{(3)} \notin L^p(-1,1)$ *for any index* $p > 2$; *here* g *is independent of* p.

PROOF. The proof of the statements (i) and (i)* is given in Sections Three and and Four below. The proof of the statements (ii) and (ii)* is given in Sections Five, Six and Seven below.

REMARKS. 1. Even though the differential expression $M[\cdot]$ has singularities at the endpoints ± 1, in that the leading coefficient $x \mapsto (1-x^2)^3$ has zeros of order 3 at both ± 1, nevertheless all functions in the maximal domain Δ are continuous on the closed interval $[-1,1]$; this is in marked contrast to the general behaviour at finite singular endpoints of functions in the maximal domain of differential expressions, when singular behaviour is to be expected.

2. Even more striking is the degree of smoothness of all elements in the operator domain $D(T)$; here all functions have a continuous second derivative on the closed interval $[-1,1]$.

3. The smoothness results for all elements of the operator domain $D(T)$ justify the form of the definition of the operator T given in (1.20).

4. The methods of proof of these results owe much to earlier work in this area; in particular, to the results obtained for the classical Legendre equation (case (i) of (1.4)) in [7], and to the results obtained for the fourth-order Legendre-type expression determined by case (iv) as given in [3] and [4].

5. It is to be noted that in the proof of Theorem 1.1 no use is made of the required $M \neq N$ as given in (v) of (1.4). However this condition is required to establish the existence of the sixth-order differential equation (1.8), as was shown in the thesis [12]. If in (1.8) we put $M = N$, *i.e.*

$A = B$, then all the results in this paper and in the report [6] remain valid. It is interesting to note that when $M = N$ the orthogonal polynomials $\{P_n^{0,0,N,N} | n \in \mathbb{N}_0\}$, as given in (1.12), identify with the orthogonal polynomials which arise in case (iv) of (1.4). Thus the set $\{P_n^{0,0,N,N} | n \in \mathbb{N}_0\}$ is generated from both a fourth-order and a sixth-order differential equation. Moreover, it can be shown that these two differential equations are essentially distinct.

The contents of this paper are as follows. Section Two contains the statement of technical lemma (due to CHISHOLM and EVERITT [2]); the results of this lemma are essential to the proof of Theorem 1.1. Section Three indicates how to reduce the proof of Theorem 1.1 to the simplest possible form of elements in Δ and $D(T)$. Section Four gives a proof of properties (i) and (i)* for Δ. Sections Five, Six and Seven give a proof of properties (ii) and (ii)* for the operator domain $D(T)$.

2 – A boundedness result in $L^2(-1,1)$

The following result is essential for our proof of Theorem 1.1. The proof of Theorem 2.1 may be found in [2, Section 2].

THEOREM 2.1 (Chisholm-Everitt). *Let $[a,b]$ be a compact interval of $\mathbb{R}$ and suppose $\lambda, \nu : [a,b] \to \mathbb{C}$ satisfy*

$$\lambda \in L^2_{loc}[a,b)\,, \qquad \nu \in L^2_{loc}(a,b]\,.$$

Define the two operators $A, B : L^2(a,b) \to L^2_{loc}(a,b)$ by

$$(Af)(x) := \nu(x) \int_a^x \lambda(t) f(t) dt\,, \qquad (x \in (a,b))$$

$$(Bf)(x) := \lambda(x) \int_x^b \nu(t) f(t) dt\,, \qquad (x \in (a,b))\,,$$

for all $f \in L^2(a,b)$. Then a necessary and sufficient condition for both A and B to map $L^2(a,b)$ into $L^2(a,b)$ is that there exists a positive number K such that

$$\int_a^x |\lambda(t)|^2 dt \int_x^b |\nu(t)|^2 dt \le K\,, \qquad (x \in (a,b))\,. \qquad \square$$

3 – Preliminaries

We remind the reader that a right-definite spectral analysis of the sixth-order expression $M_k^{(3)}[\cdot]$, defined in (1.8), may be found in the thesis [13] of LOVELAND. Our objective here is to confine consideration to the proof of Theorem 1.1 given in Section One.

In Section Four, we prove part (i) of Theorem 1.1 and show that this results is best possible. The proof of part (ii) of Theorem 1.1 is more detailed and lengthy than that of part (i); we give this proof in Sections Five and Six. In Section Seven, we show that the result in part (ii) of Theorem 1.1 is best possible in the sense of (ii)* of Theorem 1.1.

We recall the definitions of $M[\cdot]$, Δ, and $D(T)$; see (1.13), (1.15) and (1.18), respectively.

In the proof we can restrict attention to the endpoint +1 since the results for the endpoint −1 can be obtained by similar methods.

Now let $f \in \Delta$. In proving the required results we can take, without loss of generality

(i) f to be real-valued on $(-1,1)$,

(ii) f to be identically zero in the interval $\left[-1,\frac{1}{2}\right]$ by using the fundamental result in NAIMARK [14, Section 17.3, Lemma 2], in order to simplify the analysis at the endpoint +1.

To summarize, we can take $f \in \Delta$ with the properties

$$(3.1) \qquad f:(-1,1)\to \mathbb{R} \qquad f(x)=0 \qquad \left(x\in\left[-1,\frac{1}{2}\right]\right).$$

4 – The maximal domain Δ

We first establish the results given in parts (i) and (i)* of Theorem 1.1.

With $M[f]$ given in (1.8) and (1.13), integrate twice over the interval $[0,x]$ with $x \in \left(\frac{1}{2},1\right)$ to obtain, and *hereby define the mapping* Λ : $[0,1]x\Delta \to \mathbb{R}$

$$(4.1)\quad \Lambda(x;f) := -((1-x^2)^3 f^{(3)}(x))' + (1-x^2)(12+\alpha(1-x^2))f''(x) =$$

$$= \int_0^x \left\{ \int_0^t (M[f](s) - f(x))ds \right\} dt + \int_0^x \pi(t) f'(t)dt =$$

$$(4.2)\qquad := \int_0^x \left\{ \int_0^t (M[f](s) - f(x))ds \right\} dt + \pi(x)f(x) - \int_0^x \pi'(t)f(t)dt,$$

on integrating by parts to give (4.2). In all that follows we use either definition (4.1) or (4.2) as required.

We note from (4.2) that

$$(4.3)\qquad \Lambda(\cdot;f) \in L^2[0,1) \quad \text{and} \quad \Lambda(\cdot;f) \in AC_{loc}[0,1).$$

Define the second-order, Lagrange symmetric differential expression $N[\cdot]$ by, here $g : [0,1) \to \mathbb{R}$ and $g, g' \in AC_{loc}[0,1)$,

$$(4.4)\qquad N[g](x) := -((1-x^2)^3 g'(x))' + (1-x^2)(12+\alpha(1-x^2))g(x) \qquad (x \in [0,1)).$$

Now rewrite the definition (4.1) in the form

$$(4.5)\qquad N[f''](x) = \Lambda(x;f) \qquad (x \in [0,1))$$

regarding this result as a functional identity for f with Λ defined by (4.2). This suggest that we study the non-homogeneous, second-order differential equation

$$(4.6)\qquad N[y](x) = \Lambda(x;f) \qquad (x \in [0,1))$$

which requires consideration of the solutions of the homogeneous equation

$$(4.7)\qquad N[y](x) \equiv -((1-x^2)^3 y'(x)) + (1-x^2)(12+\alpha(1-x^2))y(x) = 0 \qquad (x \in [0,1)).$$

If we consider this equation, with its analytic coefficients, in the complex plane then the point $+1$ is a regular singularity of (4.7) for which the indicial roots are 1 and -3. From the Frobenius theory of series solutions the equation (4.7) has a solution φ of the form

$$\varphi(x) = (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n \qquad (a_0 \neq 0) \tag{4.8}$$

with convergence for $|x-1| < 2$. Clearly this solution can have only a finite number of zeros in $[0,1)$, since $\lim_{x\to 1}\varphi'(x) \neq 0$; hence for some $\xi \in [0,1)$ we have $\varphi(x) \neq 0$ $(x \in [\xi,1))$.

A second, linearly independent solution to the equation (4.7) is given by

$$\psi(x) := \varphi(x)\int_{\xi}^{x} \frac{dt}{(1-t^2)^3(\varphi(t))^2} \quad (x \in [\xi,1)). \tag{4.9}$$

As initial values for ψ at ξ, we see that

$$\psi(\xi) = 0 \quad \text{and} \quad \psi'(\xi) = \left(\varphi(\xi)(1-\xi^2)^3\right)^{-1},$$

and so for the Wronskian of the pair of solutions φ and ψ, we have

$$W(\varphi,\psi)(x) = (1-x^2)^3(\varphi(x)\psi'(x) - \varphi'(x)\psi(x)) = 1 \qquad (x \in [\xi,1)). \tag{4.10}$$

The asymptotic form of these solutions φ and ψ near 1 can then be obtained from (4.8) and (4.9) (we omit the details for ψ) to give, as $x \to 1$,

$$\varphi(x) = a_0(x-1) + 0(|x-1|^2) \quad \varphi'(x) = a_0 + 0(|x-1|) \tag{4.11}$$

$$\psi(x) = \frac{b_0}{(x-1)^3} + 0\left(\frac{1}{|x-1|^2}\right) \quad \psi'(x) = \frac{-3b_0}{(x-1)^4} + 0\left(\frac{1}{|x-1|^3}\right), \tag{4.12}$$

where we note that $a_0 \neq 0$ and $b_0 \neq 0$.

Now define the function $\Psi : [\xi, 1] \times \Delta \to \mathbb{R}$ by

$$\Psi(x; f) := \varphi(x) \int_{\xi}^{x} \psi(t)\Lambda(t; f)dt + \psi(x) \int_{x}^{1} \varphi(t)\Lambda(t; f)dt, \tag{4.13}$$

noting that Ψ is well-defined on using (4.11), (4.12) and the properties (4.3) of $\Lambda(x; f)$. We have $\Psi(\cdot\,; f) \in C[\xi, 1]$ but in general, in view of (4.12), $\Psi(\cdot; f)$ will be singular at 1.

By direct differentiation and use of φ, ψ as solutions of (4.7) satisfying (4.10) it may be seen that

$$N[\Psi(x; f)] = \Lambda(x; f) \qquad (x \in [\xi, 1)), \tag{4.14}$$

so that $\Psi(\cdot; f)$ is a particular solution of the equation (4.6) on $[\xi, 1)$. Hence the general solution of (4.6) on $[\xi, 1)$ is of the form, where $\alpha, \beta \in \mathbb{R}$,

$$y(x) = \alpha\varphi(x) + \beta\psi(x) + \Psi(x; f) \quad (x \in [\xi, 1)).$$

Returning now to (4.1), if we write $y(x) = f''(x)$ $(x \in [\xi, 1))$ then (4.6) becomes (4.5) and so, for some unique $\alpha, \beta \in \mathbb{R}$, we obtain the representation

$$f''(x) = \alpha\varphi(x) + \beta\psi(x) + \Psi(x; f) \quad (x \in [\xi, 1)). \tag{4.15}$$

It should be noted that this is a functional identity for f from which we draw information about the form of $f \in \Delta$ near the singular end-point 1.

Since, in this part (i) of Theorem 1.1, we require information on f' we integrate (4.15) over the interval $[\xi, x]$ to obtain

$$f'(x) = f'(\xi) + \alpha \int_{\xi}^{x} \varphi(t)dt + \beta \int_{\xi}^{x} \psi(t)dt + \int_{\xi}^{x} \Psi(t; f)dt \qquad (x \in [\xi, 1)). \tag{4.16}$$

We have to examine in detail the last term on the right-hand side of (4.16); *i.e.*

$$\int_\xi^x \Psi(t;f)dt = \int_\xi^x \varphi(t)\left\{\int_\xi^t \psi(s)\Lambda(s;f)ds\right\}dt + \tag{4.17}$$
$$+\int_\xi^x \psi(t)\left\{\int_t^1 \varphi(s)\Lambda(s;f)ds\right\}dt.$$

Consider the first term on the right-hand side of (4.17); integration by parts yields

$$\int_\xi^x \varphi(t)\left\{\int_\xi^t \psi(s)\Lambda(s;f)ds\right\}dt = -\int_x^1 \varphi(s)ds\int_\xi^x \psi(s)\Lambda(s;f)ds +$$
$$+\int_\xi^x \left\{\int_t^1 \varphi(s)ds\right\}\psi(t)\Lambda(t;f)dt;$$

i.e. for all $x \in [\xi, 1)$,

$$\left|\int_\xi^x \varphi(t)\left\{\int_\xi^t \psi(s)\Lambda(s;f)ds\right\}dt\right| \le \int_x^1 |\varphi(s)|ds\int_\xi^x |\psi(s)\Lambda(s;f)|ds + \tag{4.17a}$$
$$+\int_\xi^x \left\{\int_t^1 |\varphi(s)|ds\right\}|\psi(t)\Lambda(t;f)|dt.$$

Now from (4.11) and (4.12), with K a positive number, not necessarily the same number in subsequent lines,

$$\int_x^1 |\varphi(s)|ds \le K\int_x^1 (1-s)ds \le K(1-x)^2 \quad (x \in [\xi, 1))$$

$$|\psi(s)| \le K(1-s)^{-3} \qquad (s \in [\xi, 1));$$

and so, for all $x \in [\xi, 1)$,

$$2\int_x^1 |\varphi(s)|ds \int_\xi^x |\psi(s)\Lambda(s;f)|ds \le K(1-x^2)\int_\xi^x (1-t)^{-3}|\Lambda(t;f)|dt.$$

To the right-hand side of this last result, we apply Theorem 2.1 above with

$$a=\xi, \quad b=1, \quad \nu(x)=(1-x)^2, \quad \lambda(x)=(1-x)^{-3} \quad (x\in[\xi,1))$$

for which we have

$$\int_\xi^x |\lambda(t)|^2 dt \int_x^1 |\nu(t)|^2 dt \le K \quad (x\in[\xi,1)).$$

There is a similar result for the second term of (4.17a).
Since $\Lambda(\cdot;f)\in L^2[\xi,1)$ (see (4.3)), Theorem 2.1 gives

$$x \mapsto \int_x^1 \varphi(s)ds \int_\xi^x \psi(s)\Lambda(s;f)ds \in L^2[\xi,1).$$

The second term in (4.17), on integration by parts, firstly reduces to consideration of

$$\int_\xi^x |\psi(s)|ds \int_x^1 |\varphi(s)\Lambda(s;f)|ds \le K(1-x)^{-2}\int_x^1 (1-t)|\Lambda(t;f)|dt$$

on $[\xi,1)$; this yields a term in $L^2[\xi,1)$ on use of Theorem 2.1 with $a=\xi$, $b=1$, $\lambda(x)=(1-x)^{-2}$, $\nu(x)=(1-x)$; *i.e.*

$$x \mapsto \int_\xi^x \psi(t)\left\{\int_t^1 \varphi(s)\Lambda(s;f)ds\right\}ds \in L^2[\xi,1).$$

Secondly to a similar term with also lies in $L^2[\xi,1)$.

Taken together these results give

$$x \mapsto \int_{\xi}^{x} \Psi(t; f)dt \in L^2[\xi, 1). \tag{4.18}$$

Returning now to (4.16) we integrate again over $[\xi, x]$ to give, for all $x \in [\xi, 1)$,

$$f(x) = f(\xi) + (x-\xi)f'(\xi) + \int_{\xi}^{x} \left\{ \int_{\xi}^{t} [\alpha\varphi(s) + \beta\psi(s) + \Psi(s; f)]\, ds \right\} dt. \tag{4.19}$$

From the asymptotic form of ψ given in (4.12) we obtain

$$\beta \int_{\xi}^{x} \left\{ \int_{\xi}^{t} \psi(s)ds \right\} dt \sim \frac{\beta b_0}{2(x-1)} \quad (x \to 1)$$

which shows that this term in (4.19) is not in $L^2[\xi, 1)$ if $\beta \neq 0$. From (4.11) and (4.18) all other terms in (4.19) are in $L^2[\xi, 1)$; thus we must have $\beta = 0$ and the representation (4.15) for f'' reduces to

$$f''(x) = \alpha\varphi(x) + \Psi(x; f) \qquad (x \in [\xi, 1)), \tag{4.20}$$

and (4.16) for f' reduces to

$$f'(x) = f'(\xi) + \alpha \int_{\xi}^{x} \varphi(t)dt + \int_{\xi}^{x} \Psi(t; f)dt \quad (x \in [\xi, 1)). \tag{4.21}$$

We have seen that, in particular using (4.18), all terms on the right-hand side of (4.21) are in $L^2[\xi, 1)$, and hence we deduce that $f' \in L^2[0, 1)$. The result $f' \in L^2(-1, 0]$ follows from a similar argument and together we obtain the result

$$f \in \Delta \quad \text{implies} \quad f' \in L^2(-1, 1) \tag{4.22}$$

which gives in turn

$$f \in \Delta \quad \text{implies} \quad f \in AC[-1, 1], \tag{4.23}$$

provided we define, as we now do, $f(\pm 1) := \lim_{x\to\pm 1} f(x)$.

It now follows from (4.2) that

$$(4.24)\ f\in\Delta \quad \text{implies} \quad \Lambda'(\cdot;f)\in L^2(-1,1) \quad \text{and} \quad \Lambda(\cdot;f)\in AC[-1,1].$$

Returning to (4.21) we note the first two terms in the rigid-hand side are bounded near 1^-. From (4.24) we obtain the result that $\Lambda(\cdot;f)$ is bounded on $[-1,1]$ and so from (4.17),

$$\left|\int_\xi^x \Psi(t;f)dt\right| \le K\int_\xi^x |\varphi(t)|\left\{\int_\xi^t |\psi(s)|ds\right\}dt + K\int_\xi^x |\psi(t)|\left\{\int_t^1 |\varphi(s)|ds\right\}dt \le$$

$$\le K\int_\xi^x (1-t)(1-t)^{-2}dt + K\int_\xi^x (1-t)^{-3}(1-t)^2 dt \le$$

$$\le K|\ln(1-x)| \qquad (x\in[\xi,1)).$$

With a similar inequality holding near the endpoint -1 it now follows from (4.21) that

$$(4.25) \qquad f'(x) = 0\Big(|\ln(1-x^2)|\Big) \qquad (x\to 1);$$

in particular, $f'\in L^2[0,1)$. This complete the proof of part (i) of Theorem 1.1.

This last result (4.25) is best possible for the maximal domain Δ; this is best seen from the detailed Frobenius analysis of the solutions of the homogeneous differential equation $M[y]=0$ on $(-1,1)$. This analysis is carried out in [13, pages 181-184] where it is shown that this equation has a solution $\hat{\varphi}_1$ with a series representation

$$\hat{\varphi}_1(x) = 3\ln(|1-x|)(x-1)\sum_{n=0}^{\infty} c_n(x-1)^n + (x-1)\sum_{n=0}^{\infty} d_n(x-1)^n,$$

with $c_0\neq 0$ and $d_0\neq 0$, and this representation is valid for $(-1,1)$. Since $\hat{\varphi}_1\in L^2[0,1)$ it follows that $\hat{\varphi}_1\in\Delta$, at least on $[0,1)$, and so

$$\hat{\varphi}_1'\in L^2[0,1) \quad \text{and} \quad \hat{\varphi}_1\in AC[0,1].$$

However

$$\hat{\varphi}_1'(x) \sim 3c_0 \ln(|1-x|) \quad (x \to 1)$$

which implies that, in general, (4.25) is best possible. This establishes part (i)* of Theorem 1.1 on choosing $g = \hat{\varphi}_1$.

5 – The operator domain $D(T)$: Part 1

We begin our proof of part (ii) of Theorem 1.1. Here again we argue only in the neighbourhood of the endpoint +1 and again take f to have the properties (3.1).

From (4.24) we can write

$$\Lambda(t;f) = -\int_t^1 \Lambda'(s;f)ds + \Lambda(1;f) \quad (t \in [-1,1]). \tag{5.1}$$

We now state

LEMMA 5.1. *Let $f \in \Delta$; then*

$$f'' \in L^2(-1,1) \quad \textit{and} \quad f, f' \in AC[-1,1] \quad \textit{if and only if} \quad \Lambda(\pm 1; f) = 0. \tag{5.2}$$

PROOF. Consider the endpoint +1. From (4.21) and then (4.13) we obtain, for all $x \in [\xi, 1)$,

$$\begin{aligned} f''(x) &= \alpha\varphi(x) + \Psi(x;f) = \\ &= \alpha\varphi(x) + \varphi(x)\int_\xi^x \psi(t)\Lambda(t;f)dt + \psi(x)\int_x^1 \varphi(t)\Lambda(t;f)dt = \\ &= \alpha\varphi(x) + \Lambda(1;f)\left[\varphi(x)\int_\xi^x \psi(t)dt + \psi(x)\int_x^1 \varphi(t)dt\right] + \\ &\quad - \varphi(x)\int_\xi^x \psi(t)\left(\int_t^1 \Lambda'(s;f)ds\right)dt - \psi(x)\int_x^1 \varphi(t)\left(\int_t^1 \Lambda'(s;f)ds\right)dt \end{aligned} \tag{5.3}$$

on using also (5.1). We claim that the last two terms in (5.3) are in $L^2[\xi, 1)$.

For integrating by parts,

$$\left|\varphi(x)\int_\xi^x \psi(t)\Big(\int_t^1 \Lambda'(s;f)ds\Big)dt\right| \le \left|\varphi(x)\int_\xi^x \psi(t)dt\int_x^1 \Lambda'(s;f)ds\right| +$$

$$(5.4)\quad + \left|\varphi(x)\int_\xi^x \Big(\int_\xi^t \psi(s)ds\Big)\Lambda'(t;f)dt\right| \le$$

$$\le K(1-x)^{-1}\int_x^1 1\cdot|\Lambda'(s;f)|ds + K(1-x)\int_\xi^x (1-t)^{-2}|\Lambda'(t;f)|dt$$

where we use the asymptotic forms (4.11) and (4.12). An application of Theorem 2.1, following the applications given above in Section Four, yields the result that both terms of the rigid-hand side of (5.4) are in $L^2[\xi, 1)$. There is a similar argument for the last term of (5.3), which is also in $L^2[\xi, 1)$.

Looking now at the second term on the rigid-hand side of (5.3), the asymptotic form (4.11) and (4.12) of φ and ψ show that

$$(5.5)\quad \varphi(x)\int_\xi^x \psi(t)dt + \psi(x)\int_x^1 \varphi(t)dt = -a_0b_0(x-1)^{-1} + 0(1) \quad (x \to 1)$$

and we recall $a_0 \neq 0$, $b_0 \neq 0$.

Returning now to (5.3) it follows that $f'' \in L^2[0,1)$ if and only if $\Lambda(1;f) = 0$. There is a similar result for the endpoint -1; *i.e.* $f'' \in L^2(-1,0]$ if and only if $\Lambda(-1;f) = 0$.

This complete the proof of Lemma 5.1. □

It follows from (4.2), on differentiating, twice, that $\Lambda''(\cdot;f) \in L^2(-1,1)$ if and only if $f'' \in L^2(-1,1)$. Thus, we have, recall (4.24),

$$(5.6)\quad \begin{aligned} f \in \Delta \quad \text{and} \quad \Lambda(\pm 1;f) = 0 \quad \text{implies} \quad \Lambda''(\cdot;f) \in L^2(-1,1) \\ \text{and} \quad \Lambda'(\cdot;f) \in AC[-1,1] \end{aligned}$$

in which circumstances we have for all $x \in [\xi, 1)$

$$\Lambda(x; f) = (x-1)\Lambda'(1; f) + \int_x^1 \left(\int_t^1 \Lambda''(s; f) ds \right) dt. \tag{5.7}$$

We can now state

LEMMA 5.2. *Let $f \in \Delta$ then*

$$\begin{aligned} &f^{(3)} \in L^2(-1,1) \quad \text{and} \quad f, f', f'' \in AC[-1,1] \quad \text{in and only if} \\ &\Lambda(\pm 1; f) = 0. \end{aligned} \tag{5.8}$$

PROOF. If $\Lambda(1; f) \neq 0$ then $f'' \notin L^2[0,1)$ and this implies that $f^{(3)} \notin L^2[0,1)$.

Suppose then $\Lambda(1; f) = 0$.

Differentiate (4.20), which is valid for all $f \in \Delta$, to obtain

$$f^{(3)}(x) = \alpha\varphi'(x) + \Psi'(x; f) \quad (x \in [\xi, 1)). \tag{5.9}$$

Clearly $\varphi' \in L^2[\xi, 1)$; for Ψ' we have, from (4.13), (5.6) and (5.7)

$$\begin{aligned} \Psi'(x; f) &= \varphi'(x) \int_\xi^x \psi(t)\Lambda(t; f) dt + \psi'(x) \int_x^1 \varphi(t)\Lambda(t; f) dt = \\ &= \Lambda'(1; f) \left(\varphi'(x) \int_\xi^x (t-1)\psi(t) dt + \psi'(x) \int_x^1 (t-1)\varphi(t) dt \right) + \\ &+ \varphi'(x) \int_\xi^x \psi(t) \left\{ \int_t^1 \left(\int_s^1 \Lambda''(u; f) du \right) ds \right\} dt + \\ &+ \psi'(x) \int_x^1 \varphi(t) \left\{ \int_t^1 \left(\int_s^1 \Lambda''(u; f) du \right) ds \right\} dt. \end{aligned} \tag{5.10}$$

On using the asymptotic forms (4.11) and (4.12) the first term on the right-hand side of (5.10) becomes

$$\Lambda'(1;f)\left(a_0(-b_0(x-1)^{-1})+\left(-3b_0(x-1)^{-4}\left[\frac{1}{4}a_0(x-1)^3\right]+0(1)\right)\right)=$$

$$=0(1) \qquad (x\to 1)$$

since the principal terms in $(x-1)^{-1}$ cancel out.

For the second term in (5.10) we note that following appropriate applications of Theorem 2.1 we obtain, using (5.6)

$$s :\mapsto \frac{1}{s-1}\int_s^1 \Lambda''(u;f)du \in L^2[\xi,1) \tag{5.11}$$

$$t :\mapsto \frac{1}{(t-1)^2}\int_t^1 (s-1)\left(\frac{1}{s-1}\int_s^1 \Lambda''(u;f)du\right)ds \in L^2[\xi,1) \tag{5.12}$$

and

$$x :\mapsto 1\int_x^1 (t-1)^2\psi(t)\left\{\frac{1}{(t-1)^2}\int_t^2 (s-1)\left(\frac{1}{s-1}\int_s^1 \Lambda''(u;f)du\right)ds\right\}dt \in L^2[\xi,1). \tag{5.13}$$

This last expression is the second term in (5.10) except for multiplication by $\varphi'(x)\sim a_0\neq 0$ $(x\to 1^-)$ and so the whole term is in $L^2[\xi,1)$.

For the third term in (5.10) we note that following appropriate applications of Theorem 2.1 we obtain, again using (5.6), that (5.11) and (5.12) hold together with

$$x\mapsto \frac{1}{(x-1)^4}\int_x^1 (t-1)^2\varphi(t)\left\{\frac{1}{(t-1)^2}\int_t^1 (s-1)\left(\frac{1}{s-1}\int_s^1 \Lambda''(u;f)du\right)ds\right\}dt \in L^2[\xi,1)$$

(recall $\varphi(t) = a_0(t-1) + 0(|t-1|^2)$. This last expression is the third term except for the factor $-3b_0$ in ψ', and so the whole term is in $L^2[\xi, 1)$.

Taking these assessments of each of the three terms on the right-hand side of (5.10) it follows that, subject to the condition $\Lambda(1; f) = 0$, that $\Psi'(\cdot; f) \in L^2[0, 1)$ and then from (5.9) that $f^{(3)} \in L^2[0, 1)$.

A similar argument at the endpoint -1 shows, subject to the condition $\Lambda(-1; f) = 0$, that $\Psi'(\cdot; f) \in L^2(-1, 0]$ and then again from (5.9) that $f^{(3)} \in L^2(-1, 0]$. Finally provided that we make the following definition of function values at ± 1

$$f^{(r)}(\pm 1) := \lim_{x \to \pm 1} f^{(r)}(x) \quad (r = 0, 1, 2), \tag{5.14}$$

it also follows that f, f' and $f'' \in AC[-1, 1]$.

This complete the proof of Lemma 5.2. □

6 – The operator domain $D(T)$: Part 2

The final stage in the proof of part (ii) of Theorem 1.1 is to establish

LEMMA 6.1. *Let the operator domain $D(T)$ be defined as in* (1.18) *and* (1.19); *i.e.*

$$D(T) := \{f \in \Delta | [f, \psi_+](+1) = [f, \psi_-](-1) = 0\}$$

then

$$D(T) = \{f \in \Delta | \Lambda(1; f) = \Lambda(-1; f) = 0\}. \tag{6.1}$$

PROOF. We recall that $[\cdot, \cdot]$ is the skew-symmetric form associated with the differential expression $M[\cdot]$; this form has the property $[\cdot, \cdot]$: $\Delta \times \Delta \to \mathbb{C}$. The explicit representation of this form, in terms of f,

$g \in \Delta$ is from [13, page 168],

$$
\begin{aligned}
[f,g](x) = \Big\{ & -((1-x^2)^3 f^{(3)}(x))'' + ((1-x^2)(12+\\
& + \alpha(1-x^2)) f''(x))' - \pi(x) f'(x)\Big\}\bar{g}(x)+\\
& - \Big\{ -((1-x^2)^3 \bar{g}^{(3)}(x))'' + ((1-x^2)(12+\\
& + \alpha(1-x^2))\bar{g}(x))' - \pi(x)\bar{g}'(x)\Big\} f(x)+\\
& - \Big\{ -((1-x^2)^3 f^{(3)}(x))' + (1-x^2)(12+\\
& + \alpha(1-x^2)) f''(x)\Big\}\bar{g}'(x)+\\
& + \Big\{ -((1-x^2)^3 \bar{g}^{(3)}(x))' + (1-x^2)(12+\\
& + \alpha(1-x^2))\bar{g}''(x)\Big\} f'(x)+\\
& - (1-x^2)^3\{f^{(3)}(x)\bar{g}''(x) - \bar{g}^{(3)}(x) f''(x)\}.
\end{aligned}
\tag{6.2}
$$

It follows from Green's formula for $M[\cdot]$ that, see [13, pages 41 and 42], for all $f, g \in \Delta$

$$
\lim_{x\to\pm1} [f,g](\pm1) \quad \text{exist and are finite.} \tag{6.3}
$$

By direct computation it follows that

$$
x :\mapsto 1, \quad (1-x^2), \quad (1-x^2)^2 \in \Delta. \tag{6.4}
$$

Recall from part (i) of Theorem 1.1 that for any $f \in \Delta$ we have $f \in AC[-1,1]$ and $f(\pm1) = \lim_{x\to\pm1} f(x)$.

We can now state the following results; for any $f \in \Delta$:

$$
\left.
\begin{aligned}
&\text{(i)} && [f,1](\pm1) = \lim_{x\to\pm1} (\Lambda'(x;f) - \pi(x) f'(x))\\
&\text{(ii)} && [f,1-x^2](\pm1) = 2\Lambda(1;f) - 48(A+2)f(1)\\
& && [f,1-x^2](-1) = -2\Lambda(-1;f) + 48(B+2)f(-1)\\
&\text{(iii)} && [f,(1-x^2)^2](\pm1) = \pm192 f(\pm1)
\end{aligned}
\right] \tag{6.5}
$$

where the positive numbers A and B are those involved in the definition of the measure μ generated from $\hat{\mu}$ in (v) of (1.4), and the differential expression $M[\cdot]$ in (1.8) and (1.13).

The proof of the results in (6.5) follows from direct substitution of the functions in (6.4) into the form (6.2) and proceeding to the limit at ± 1, using the following information:

$$\text{(a)}\quad \Lambda(x;f) = -((1-x^2)^3 f^{(3)}(x))' + \\ + (1-x^2)(12+\alpha(1-x^2))f''(x) \quad (x \in (-1,1))$$

which is the definition (4.1)

$$\text{(b)}\quad [f,1](x) = \Lambda'(x;f) - \pi(x)f'(x) \quad (x \in (-1,1))$$

$$\text{(c)}\quad \lim_{x \to \pm 1}(1-x^2)f'(x) = 0$$

since from part (i) of Theorem 1.1, we have $f'(x) = 0(|\ln(1-x^2)|)$ $(x \to \pm 1)$

$$\text{(d)}\quad \lim_{x \to \pm 1} \int_0^x \Lambda(t;f)dt \quad \text{exist and are finite}$$

using (4.3) or (4.24)

$$\text{(e)}\quad \lim_{x \to \pm 1}(1-x^2)^3 f^{(3)}(x) = 0$$

using (a), (c) and (d) to give that the limits in (e) exist and are finite; if these limits are not zero, repeated integration over $[0,x]$ or $[x,0]$ and then letting $x \to \pm 1$, yields that $f' \notin L^2[0,1)$ or $L^2(-1,0]$ respectively; this contradiction to part (i) of Theorem 1.1 establishes (e).

With the boundary condition functions given by, see (1.19), for all $x \in [-1,1]$,

$$\psi_+(x) = 4(1-x^2)+(A+2)(1-x^2)^2, \quad \psi_-(x) = 4(1-x^2)+(B+2)(1-x^2)^2$$

we find, for $f \in \Delta$ using (6.5)

$$[f,\psi_+](1) = 8\Lambda(1;f) - 192(A+2)f(+1) + 192(A+2)f(1) = \\ = 8\Lambda(1;f)$$
$$[f,\psi_-](-1) = -8\Lambda(-1;f) + 192(B+2)f(-1) - 192(B+2)f(-1) = \\ = -8\Lambda(-1;f)\,.$$

This complete the proof of Lemma 6.1, on recalling the definition of $D(T)$, the condition (6.1), and utilizing the last result above. □

Finally then to prove part (ii) of Theorem 1.1 we have only to note that if $f \in D(T)$, as defined in (1.18) and (1.19), then from Lemma 6.1 it follows that $\Lambda(\pm 1; f) = 0$ and then from Lemma 5.2 that $f^{(3)} \in L^2(-1,1)$. This result implies that all of f, f', f'' may be regarded as in $AC[-1,1]$ provided we define these functions at the endpoints ± 1 by (5.14) to ensure the required continuity.

7 – Best possible result for operator domain $D(T)$

It remains to prove part (ii)* of Theorem 1.1.

The required function g can be obtained by putting

$$g^{(3)}(x) := \left((1-x)^{1/2}\ln(1-x)\right)^{-1} \quad \left(x \in \left[\frac{1}{2}, 1\right)\right) \tag{7.1}$$

and then completing the definition on $\left[-1, \frac{1}{2}\right]$ by polynomial extension so that the resulting function g, say, is in $C^{(3)}[-1,1)$. The function g itself is then defined by

$$g(x) := \int_0^x (x-t)^2 g^{(3)}(t)\,dt \quad (x \in [-1,1]).$$

A computation shows that $g \in \Delta$ and that $\Lambda(\pm 1; g) = 0$; hence $g \in D(T)$. From the definition (7.1), it follows that $g^{(3)} \in L^2(-1,1)$ but $g^{(3)} \notin L^p(-1,1)$ for any $p > 2$.

This completes the proof of Theorem 1.1.

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Norrie Everitt takes this opportunity, with the agreement of his two co-authors, to dedicate this paper to his friend and colleague of many years, Professor Gaetano Fichera.

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INDIRIZZO DEGLI AUTORI:

W.N. Everitt - Department of Mathematics – University of Birmingham – Birmingham – B15 – 2TT – England – United Kingdom

L.L. Littlejohn - Department of Mathematics and Statistic – Utah State University – Logan – Utah – 84322-3900 – U.S.A.

S.M. Loveland - Department of Mathematics – University of Utah – Salt Lake City – Utah – 84112 – U.S.A.