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Orthogonal polynomials of dimension –1 in the non-definite case

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Dedicated to the memory of Prof. Aldo Ghizzetti

RIASSUNTO: *I polinomi ortogonali di dimensione $d = -1$ sono un caso particolare dei polinomi ortogonali vettoriali i quali, a loro volta, sono un caso particolare dei polinomi biortogonali. In questo lavoro viene data la relazione di ricorrenza a tre termini soddisfatta da tali polinomi nel caso non-definito, ovvero quando alcuni dei polinomi del sistema non esistono. I polinomi ortogonali di dimensione -1 generalizzano i polinomi ortogonali sulla circonferenza unitaria.*

ABSTRACT: *Orthogonal polynomials of dimension $d = -1$ are a particular case of vector orthogonal polynomials which are, themselves, a particular case of biorthogonal polynomials. In this paper, we give the three-term recurrence relationship satisfied by these polynomials in the non-definite case, that is when some of them do not exist. Orthogonal polynomials of dimension -1 generalize orthogonal polynomials on the unit circle.*

Let $L_0, L_1, \ldots$ be linearly independent linear functionals on the space

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of complex polynomials and let us set

$$L_i\left(x^j\right) = c_{i,j}.$$

Formal biorthogonal polynomials with respect to the family L_i were defined in [2]. They satisfy the biorthogonality conditions

$$L_p\left(P_k(x)\right) = 0 \quad \text{for} \quad p = 0, \dots, k-1,$$

and they are given by

$$P_k(x) = D_k \begin{vmatrix} c_{0,0} & \cdots & c_{0,k} \\ \vdots & & \vdots \\ c_{k-1,0} & \cdots & c_{k-1,k} \\ 1 & \cdots & x^k \end{vmatrix}$$

where D_k is an arbitrary nonzero constant.

Assuming that P_k has the exact degree k, which is equivalent to the condition

$$H_k = \begin{vmatrix} c_{0,0} & \cdots & c_{0,k-1} \\ \vdots & & \vdots \\ c_{k-1,0} & \cdots & c_{k-1,k-1} \end{vmatrix}$$

we can consider the monic polynomial P_k that is the polynomial corresponding to the choice $D_k = 1/H_k$. Thus, P_k exists and is unique if and only if $H_k \neq 0$.

Vector orthogonal polynomials of dimension $d \in \mathbf{N}$ were introduced by VAN ISEGHEM [6]. They correspond to the case where the linear functionals L_i are related by

$$L_i\left(x^{j+1}\right) = L_{i+d}\left(x^j\right)$$

that is

$$L_i\left(x^{j+n}\right) = L_{i+nd}\left(x^j\right).$$

When $d = 1$, the usual formal orthogonal polynomials are recovered [1].

Let us now take $d = -1$ in the definition of the vector orthogonal polynomials.

We have

$$L_i\left(x^j\right) = L_{i+1}\left(x^{j+1}\right)$$

that is $c_{i,j} = c_{i+1,j+1}$.
Such polynomials, which generalize the usual orthogonal polynomials on the unit circle, have applications in Laurent-Padé and two-point Padé approximation [4]. They were studied in details in [3]. It was proved that, in the definite case, that is when $\forall k, H_k \neq 0$, these polynomials satisfy a three-term recurrence relationship of the form

$$P_{k+1}(x) = \left(x + B_{k+1}\right) P_k(x) - C_{k+1}\, x\, P_{k-1}(x)$$

with $P_{-1}(x) = 0, P_0(x) = 1$ and

$$C_{k+1} = L_0\left(xP_k\right)/L_0\left(xP_{k-1}\right)$$

$$B_{k+1} = C_{k+1}L_{k-1}\left(P_{k-1}\right)/L_k\left(P_k\right).$$

We shall now study such a relation in the non-definite case, that is when some of the determinants H_k vanish. This case was already treated in [5] by an indirect approach involving the whole table of the usual formal orthogonal polynomials of dimension $d = 1$. Our treatment will be a very direct one.

We shall make use of slightly modified notations, more adapted to our case and which reduce to the preceding ones in the normal case. Since, in the non-definite case, some of our polynomials do not exist, we shall only denote the existing ones, called regular. Thus P_k (instead of P_{n_k}) will now be the monic regular biorthogonal polynomial of dimension $d = -1$ and degree n_k. The next regular polynomial will be denoted by P_{k+1} (instead of $P_{n_{k+1}}$) and its degree will be equal to $n_{k+1} = n_k + m_k$. All the polynomials with the degrees $n_k + 1, \ldots, n_k + m_k - 1$ do not exist and thus they were not given a denomination.

We shall now prove the following result

Theorem. *Let P_k be the regular monic biorthogonal polynomial of dimension $d = -1$ and degree n_k such that*

$$L_i\left(P_k\right) = 0 \quad \textit{for} \quad i = 0, \ldots, n_k - 1.$$

If $L_0(xP_k) = 0$, then $P_{k+1}(x) = xP_k(x)$ and all the following polynomials of degree $n_k + i$ for $i \geq 2$ do not exist.
If $L_0(xP_k) \neq 0$, then the next regular polynomial P_{k+1} has degree $n_{k+1} = n_k + m_k$ with m_k defined by

$$L_i(P_k) = 0 \quad \textit{for} \quad i = 0, \ldots, n_k + m_k - 2,$$

$$L_{n_k+m_k-1}(P_k) \neq 0,$$

and it holds

$$P_{k+1}(x) = q_k(x)P_k(x) - C_{k+1}x^{m_k}P_{k-1}(x)$$

where $q_k(x) = a_0 + \cdots + a_{m_k-1}x^{m_k-1} + x^{m_k}$ and the a_i's and C_{k+1} are given by

$$\begin{aligned}
a_1L_0(xP_k)+\cdots+a_{m_k-1}L_0(x^{m_k-1}P_k)+L_0(x^{m_k}P_k)-C_{k+1}L_0(x^{m_k}P_{k-1})&=0\\
a_2L_1(x^2P_k)+\cdots+a_{m_k-1}L_1(x^{m_k-1}P_k)+L_1(x^{m_k}P_k)-C_{k+1}L_1(x^{m_k}P_{k-1})&=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
L_{m_k-1}(x^{m_k}P_k)-C_{k+1}L_{m_k-1}(x^{m_k}P_{k-1})&=0\\
a_0L_{n_k+m_k-1}(P_k)-C_{k+1}L_{n_k-1}(P_{k-1})&=0.
\end{aligned}$$

PROOF. Let us assume that

$$\begin{aligned}
L_i\left(P_{k-1}\right) &= 0 \qquad i = 0, \ldots, n_{k-1} + m_{k-1} - 2 = n_k - 2\\
&\neq 0 \qquad i = n_{k-1} + m_{k-1} - 1 = n_k - 1,
\end{aligned}$$

and

$$\begin{aligned}
L_i(P_k) &= 0 \qquad i = 0, \ldots, n_k + m_k - 2\\
&\neq 0 \qquad i = n_k + m_k - 1,
\end{aligned}$$

and that P_{k-1} has the degree n_{k-1} and P_k the degree n_k.
Obviously this is true for $k = 0$.

Let us assume that P_{k+1} has the degree $n_{k+1} = n_k + m_k$ and satisfies

$$P_{k+1}(x) = \left(a_0 + \cdots + a_{m_k-1}x^{m_k-1} + x^{m_k}\right)P_k(x) - C_{k+1}x^{m_k}P_{k-1}(x).$$

We shall prove that the coefficients a_i and C_{k+1} of this relation can be calculated and thus, due to the uniqueness of P_{k+1}, that such a relation holds.

P_{k+1} must satisfy the biorthogonality conditions $L_i(P_{k+1}) = 0$ for $i = 0, \dots, n_k + m_k - 1$, that is

$$L_i(P_{k+1}) = 0 = a_0 L_i(P_k) + \cdots + a_{m_k-1} L_i(x^{m_k-1} P_k) + L_i(x^{m_k} P_k) + \\ - C_{k+1} L_i(x^{m_k} P_{k-1}).$$

For $i = 0$, the first term disappears thanks to the biorthogonality conditions of P_k.
For $i = 1$, the first term disappears for the same reason. Moreover, $L_1(xP_k) = L_0(P_k)$ which is zero by the biorthogonality conditions of P_k, and the second term disappears.
etc, ...
For $i = m_k - 2$, the first term disappears, ..., the $(m_k - 1)^{th}$ term disappears by the biorthogonality conditions of P_k since $L_{m_k-2}(x^{m_k-2} P_k) = L_0(P_k)$.
For $i = m_k - 1$, the first term disappears, ..., the ${m_k}^{th}$ term disappears by the biorthogonality conditions of P_k since $L_{m_k-1}(x^{m_k-1} P_k) = L_0(P_k)$.
For $i = m_k$, the first term disappears, ..., the $(m_k + 1)^{th}$ term disappears by the biorthogonality conditions of P_k since $L_{m_k}(x^{m_k} P_k) = L_0(P_k)$, and the last term vanishes by the biorthogonality conditions of P_{k-1} since $L_{m_k}(x^{m_k} P_{k-1}) = L_0(P_{k-1})$. Thus all the terms are zero and this relation is always satisfied.
etc, ...
For $i = n_k + m_k - 2$, all the terms are zero for the same reason, and this relation is always satisfied.
For $i = n_k + m_k - 1$, the first term does not vanish. All the other terms are zero except the last one which is $L_{n_k+m_k-1}(x^{m_k} P_{k-1}) = L_{n_k-1}(P_{k-1})$.

Thus, the equation for $i = m_k - 1$ gives C_{k+1} since $L_{m_k-1}(x^{m_k} P_{k-1}) = L_0(xP_{k-1}) \neq 0$.
The equation for $i = m_k - 2$ gives a_{m_k-1} if $L_{m_k-2}(x^{m_k-1} P_k) = L_0(xP_k) \neq 0$.
In general, the equation i gives a_{i+1} if $L_i(x^{i+1} P_k) = L_0(xP_k) \neq 0$.
The equation for $i = 0$ gives a_1 if $L_0(xP_k) \neq 0$.
Finally, the equation for $i = n_k + m_k - 1$ gives a_0 since $L_{n_k+m_k-1}(P_k) \neq 0$.

Thus, under the conditions stated in the theorem, the polynomial P_{k+1} can be obtained by the relation given and thus, since P_{k+1} is uniquely determined, this recurrence relation holds. It must be noticed that the condition $L_0(xP_k) \neq 0$ plays a role only if $m_k > 1$.

Let us now examine the condition on $L_0(xP_k)$ in more details.
If $L_0(xP_k) = 0$, then we can only have $m_k = 1$ since, otherwise, a division by zero always occurs in the preceding relations giving the coefficients of the recurrence relation. Thus, the next regular polynomial P_{k+1} has the degree $n_{k+1} = n_k + 1$ and we have $B_{k+1} = C_{k+1} = 0$. It follows that

$$P_{k+1}(x) = xP_k(x).$$

It is easy to check that such a polynomial satisfies the biorthogonality conditions it should satisfy. We must have $L_i\left(P_{k+1}\right) = 0$, that is $L_i\left(xP_k\right) = 0$ for $i = 0, \ldots, n_k$. Indeed we have $L_0\left(xP_k\right) = 0$ and $L_i\left(xP_k\right) = L_{i-1}(P_k) = 0$ for $i = 1, \ldots, n_k$.

Let us now look at the next polynomial P_{k+2}. Since, for obtaining it, it is always necessary, for any value of m_k, to divide by $L_0(xP_k)$ which is zero, then P_{k+2} does not exists. $\square$

Remarks:

1. By the determinantal formula for P_k, it is easy to see that $L_0(xP_k) = 0$ if and only if

$$\begin{vmatrix} L_0(1) & \cdots & L_0\left(x^{n_k}\right) \\ \vdots & & \vdots \\ L_{n_k-1}(1) & \cdots & L_{n_k-1}\left(x^{n_k}\right) \\ L_0(x) & \cdots & L_0\left(x^{n_k+1}\right) \end{vmatrix}$$

is zero, that is, if and only if

$$\begin{vmatrix} L_0(x) & \cdots & L_0\left(x^{n_k+1}\right) \\ L_1(x) & \cdots & L_1\left(x^{n_k+1}\right) \\ \vdots & & \vdots \\ L_{n_k}(x) & \cdots & L_{n_k}\left(x^{n_k+1}\right) \end{vmatrix} = 0.$$

This last determinant is, in fact, equal to the constant term of the polynomial P_{k+1} which is of the degree $n_{k+1} = n_k + 1$ as proved in the theorem and whose constant term is zero since $P_{k+1}(x) = xP_k(x)$.

2. By the determinantal formula for the polynomials, we also see that P_{k+1} of degree $n_{k+1} = n_k + 1$ exists if and only if $L_{n_k}(P_k) \neq 0$.

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