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Finite differences and orthogonal polynomials

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In memory of Professor A. Ghizzetti

Riassunto: *Il lavoro mostra un'utile applicazione della conoscenza di una rappresentazione esplicita per i polinomi ortogonali di Karlin-McGregor, scoperti indipendentemente anche da Carlitz. La rappresentazione di tali polinomi mediante una serie ipergeometrica mostra già che essi potrebbero, al più, essere funzioni razionali. Nel lavoro viene mostrato, partendo direttamente dalla rappresentazione ipergeometrica, che essi sono in effetti polinomi. Una simile argomentazione è anche usata per ottenere i q-analoghi dei polinomi di Hermite, nel caso continuo, come limite di una classe più generale di polinomi ortogonali.*

Abstract: *Explicit representations of specific sets of orthogonal polynomials are often not as useful as one would like them to be. However, being able to work with them can be useful. There is a set of polynomials found by Karlin and McGregor and by Carlitz at the same time. The representation they found as a hypergeometric series shows these are at least rational functions. I can now show directly from the hypergeometric representation why they are polynomials. A similar argument is used to obtain the continuous q-Hermite polynomials as a limit from more general orthogonal polynomials.*

Part of this research was done while I was visiting the Laboratory for Theoretical Physics at Univ. of Paris VII.
Key Words and Phrases: *Finite differences – Orthogonal polynomials*

1 – Introduction

When trying to find explicit formulas for orthogonal polynomials, one seems to have to use hypergeometric or basic hypergeometric series. Usually no such representation can be found, and even when it can the series is often frustrating, with facts one wants seemingly just out of reach. I have had many such problems, and can now solve a couple of them. The first deals with the polynomial $r_n(x)$ which satisfies the recurrence relation

$$x(n+b)r_n(x) = br_{n+1}(x) + nr_{n-1}(x). \tag{1.1}$$

Carlitz [3] and Karlin and McGregor [6] studied these polynomials, found their orthogonality relation, found a generating function, and discovered the explicit formula

$$r_n(x) = x^{-n}{}_2F_0\left(\begin{matrix} -n, b(1-x^{-2}) \\ \text{—} \end{matrix}; -x^2/b\right). \tag{1.2}$$

It is unlikely there is a nicer explicit representation, yet this is an annoying formula, for it looks like a rational function rather than a polynomial. For years this bothered me, but no one else remarked on it to me, so it was not clear whether I was alone in this feeling. In June, when I was visiting in Paris, G. Valent asked me if I could prove directly from (1.2) that $r_n(x)$ is a polynomial. At the time I had to say no, and mentioned that this had bothered me for a couple of decades. However, a couple of days later I was able to change this answer to yes. The reason is easy but interesting, and a variant of it solves another problem which has annoyed me for almost as long. This deals with the limit

$$\lim_{a\to 0} a^{-n}{}_3\varphi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, 0 \end{matrix}; q, q\right)$$

which leads to the continuous q-Hermite polynomial $H_n(\cos\theta|q)$.

For those who are not used to the notation for hypergeometric and basic hypergeometric series, a hypergeometric series is a series Σc_n with

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c_{n+1}/c_n a rational function of n. If the shifted factorial is defined by

$$(1.3)\qquad \begin{aligned}(a)_n &= a(a+1)\cdots(a+n-1), \quad && n=1,2,\ldots\,,\\ &= 1 \quad , && n=0,\end{aligned}$$

then

$$(1.4)\qquad {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{x^n}{n!}.$$

For basic hypergeometric series, the term ratio is a rational function of q^n. The multiplicative shifted factorial is

$$(1.5)\qquad \begin{aligned}(a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad && n=1,2,\ldots\,,\\ &= 1 \quad , && n=0.\end{aligned}$$

A special case of the general basic hypergeometric series which is sufficient for our purposes here is

$$(1.6)\qquad {}_{p+1}\varphi_p\left(\begin{matrix}a_0,\ldots,a_p\\ b_1,\ldots,b_p\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_0;q)_n\cdots(a_p;q)_n}{(b_1;q)_n\cdots(b_p;q)_n}\,\frac{x^n}{(q;q)_n}.$$

We will use two series which can be summed

$$(1.7)\qquad {}_2F_1\left(\begin{matrix}-n,a\\ c\end{matrix};1\right)=\frac{(c-a)_n}{(c)_n}$$

$$(1.8)\qquad {}_1\varphi_0\left(\begin{matrix}a\\ -\end{matrix};q,x\right)=\frac{(ax;q)_\infty}{(x;q)_\infty}$$

when $|x|<1$, $|q|<1$ and $(a;q)_\infty$ is the limit of (1.5) when $n\to\infty$.

2 – Finite differences and the Carlitz, Karlin-McGregor polynomials

The polynomials $r_n(x)$ mentioned above satisfy the recurrence relation

$$(2.1)\qquad x(n+b)r_n(x)=br_{n+1}(x)+nr_{n-1}(x),$$

$r_{-1}(x) = 0$, $r_0(x) = 1$.

Thus it is clear they are polynomials. However, this is far from clear from the representation

$$r_n(x) = x^{-n} {}_2F_0 \left(\begin{matrix} -n, b(1-x^{-2}) \\ — \end{matrix} ; -\frac{x^2}{b} \right). \tag{2.2}$$

For this to be a polynomial, when the ${}_2F_0$ is written as a series in terms of powers of x, the first n terms must vanish. This must happen, but it was not clear why. To show directly that it does, we take a side excursion involving finite differences.

If $f(x)$ is defined on the integers $x = 0, 1, \ldots$, define

$$\Delta f(x) = f(x+1) - f(x), \tag{2.3}$$

and

$$\Delta^{n+1} f(x) = \Delta[\Delta^n f(x)]. \tag{2.4}$$

A simple induction gives

$$\begin{aligned} \Delta^n f(x) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x+n-k) \\ &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x+k) \\ &= (-1)^n \sum_{k=0}^{n} \frac{(-n)_k}{k!} f(x+k). \end{aligned} \tag{2.5}$$

If $f(x)$ is a polynomial of degree m, then $\Delta f(x)$ is a polynomial of degree $m-1$. This implies that if $f(x)$ is a polynomial of degree m and $m < n$, then

$$\Delta^n f(x) = 0 \tag{2.6}$$

for all x.

In particular, (2.5) then implies that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)_m = m! \sum_{k=0}^{n} \frac{(-n)_k (m+1)_k}{k!(1)_k} = 0 \tag{2.7}$$

for $m = 0, 1, \dots, n-1$. This also follows immediately from (1.7) when $c = 1$ and $a = m + 1$.

To see how this gives the required vanishing of the power series coefficients in the expansion of $r_n(x)$, rewrite this as

$$x^n r_n(x) = \sum_{k=0}^{n} \frac{(-n)_k \left(1 - \frac{x^2}{b} b\right)\left(1 - \frac{x^2}{b}(b+1)\right) \cdots \left(1 - \frac{x^2}{b}(b+k-1)\right)}{k!}. \tag{2.8}$$

Now write this as a series in powers of x. The first two terms are

$$\sum_{k=0}^{n} \frac{(-n)_k}{k!} - \frac{x^2}{b} \sum_{k=0}^{n} \frac{(-n)_k}{k!}[kb + k(k-1)/2].$$

Both of these sums vanish, since they are just instances of (2.6) when $f(x)$ is a polynomial of degree 0, 1 and 2 and the n'th difference is evaluated at $x = 0$. The next term has the factor

$$\frac{x^4}{b^2} \sum_{1 \le j \le \ell \le k} (b + j - 1)(b + \ell - 1)$$

and this is a polynomial of degree 4 in k. The degree goes up two with each successive term, so is the same as the exponent of x, and thus the coefficient of x^{2k} vanishes as long as $2k < n$. This is exactly what we wanted to prove.

3 – The continuous q-Hermite polynomials as a limit of more general orthogonal polynomials

The continuous q-Hermite polynomials which L. J. Rogers discovered [10] are the bottom polynomials in one part of a chart of the classical hypergeometric polynomials which are basic hypergeometric series. At the top of this chart is a set of orthogonal polynomials with four degrees of freedom in addition to the q of basic hypergeometric series. The most general orthogonal polynomials are

$$\frac{W_n(\cos\theta; a, b, c, d|q)}{(ab;q)_n (ac;q)_n (ad;q)_n} = a^{-n} {}_4\varphi_3 \left(\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right). \tag{3.1}$$

When $\max(|a|,|b|,|c|,|d|,\pm q)<1$, the orthogonality is

$$\int_{-1}^{1} W_n(x)W_m(x)\frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)} \frac{dx}{(1-x^2)^{1/2}}=0, m\neq n\,. \tag{3.2}$$

See [2] or [4].

The continuous q-Hermite polynomials are defined by

$$H_n(\cos\theta|q)=\sum_{k=0}^{n}\frac{(q;q)_n e^{i(n-2k)\theta}}{(q;q)_k(q;q)_{n-k}} \tag{3.3}$$

and their orthogonality is

$$\int_{-1}^{1} H_n(x|q)H_m(x|q)h(x;1)h(x,q^{1/2})h(x,-1) h(x,-q^{1/2})(1-x^2)^{-1/2}dx=0, \quad m\neq n. \tag{3.4}$$

when $-1<q<1$. See [4]. Thus the continuous q-Hermite polynomials are a multiple of the $W_n(\cos\theta)$ when $a=b=c=d=0$. It is easy to set $b=c=d=0$ in (3.1), but letting $a\to 0$ is not an easy limit to take.

To take this limit, we use an argument similar to the one in section 2, but with the q-binomial theorem (1.8) taking the place of the Chu-Vandermonde sum (1.7). First, use the q-binomial theorem to expand $(ae^{i\theta};q)_m(ae^{-i\theta};q)_m$ as a double series. The result is

$$a^{-n}{}_3\varphi_2\left(\begin{matrix}q^{-n},ae^{i\theta},qe^{-i\theta}\\0,0\end{matrix};q,q\right)= \tag{3.5}$$

$$=a^{-n}\sum_{m=0}^{n}\frac{(q^{-n};q)_m}{(q;q)_m}q^m\sum_{j=0}^{m}\frac{(q^{-m};q)_j}{(q;q)_j}a^je^{ij\theta}q^{mj}\cdot\sum_{k=0}^{m}\frac{(q^{-m};q)_k}{(q;q)_k}a^ke^{-ik\theta}q^{mk}$$

$$=a^{-n}\sum_{j,k}\frac{a^{j+k}e^{i(j-k)\theta}}{(q;q)_j(q;q)_k}\sum_{m}\frac{(q^{-n};q)_m(q^{-m};q)_j(q^{-m};q)_kq^{(j+k+1)m}}{(q;q)_m}\,.$$

When $j+k<n$, the sum on m is

$$\sum_{m=0}^{n}\frac{(q^{-n};q)_mq^m}{(q;q)_m}p_{j+k}(q^m), \tag{3.6}$$

where $p_j(x)$ is a polynomial of degree j in x. Since

$$\sum_{m=0}^{n} \frac{(q^{-n};q)_m}{(q;q)_m} q^{\ell m} = (q^{\ell-n};q)_n = 0. \tag{3.7}$$

when $\ell = 1, 2, \ldots, n$, the sum in (3.6) vanishes. Thus the coefficients of a^{j+k} vanish when $j+k<n$. When $j+k=n$, the coefficient of a^{j+k} is

$$\sum_{k=0}^{n} \frac{e^{i(n-2k)\theta}}{(q;q)_{n-k}(q;q)_k} \sum_{m=0}^{n} \frac{(q^{-n};q)_m}{(q;q)_m} q^m A_m(q)$$

where

$$\begin{aligned} A_m(q) &= (q^m-1)(q^m-q)\cdots(q^m-q^{k-1})(q^m-1)\cdots(q^m-q^{n-k+1}) \\ &= q^{mn} + \text{ lower terms.} \end{aligned}$$

Finally, (3.7) forces all except the term with q^{mn} to vanish, and when $\ell = n+1$, (3.7) gives $(q;q)_n$. The terms with higher powers of a vanish when $a \to 0$, so

$$\lim_{a,b,c,d\to 0} a^{-n} W_n(\cos\theta; q, b, c, d|q) = H_n(\cos\theta|q). \tag{3.8}$$

4 – Comments

There are still results of the above type which I can not do by a similar argument. Two which I have considered off and on for a couple of decades are the following.

Hermite polynomials live at the bottom of the chart of the classical orthogonal polynomials which are hypergeometric series. They are limits of all the polynomials above them. Laguerre polynomials are one level higher. The limit relation from Laguerre to Hermite polynomials was first derived in [9] from differential equations. Laguerre polynomials are orthogonal with respect to $x^\alpha e^{-x}$ on $[0,\infty)$, and Hermite polynomials are orthogonal with respect to e^{-x^2} on $(-\infty,\infty)$. This limit result can also be proven from the orthogonality relation. Shift $[0,\infty)$ to $[-\alpha,\infty)$ to put the maximum at $x = 0$, and then rescale. The measure before rescaling is a constant multiple of

$$e^{-x+\alpha\log(1+\frac{x}{\alpha})} = e^{-x+\alpha\left[\frac{x}{\alpha}-\frac{x^2}{2\alpha^2}\right]+O(x^3\alpha^{-2})} = e^{-x^2/2\alpha+O(x^3\alpha^{-2})}.$$

The rescaling is $x \to (2\alpha)^{1/2}x$. Laguerre polynomials are given by

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k k!}x^k = \frac{(-1)^n x^n}{n!} + \text{ lower terms,} \tag{4.1}$$

and Hermite polynomials satisfy the recurrence relation

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x),$$

so

$$H_n(x) = 2^n x^n + \text{ lower terms.}$$

Thus

$$\lim_{\alpha\to\infty}\left(\frac{2}{\alpha}\right)^n L_n^\alpha((2\alpha)^{1/2}x+\alpha) = \frac{(-1)^n}{n!}H_n(x).$$

I do not know how to do this directly from the representation (4.1), although it is easy to prove from the recurrence relations the polynomials satisfy, from the orthogonality, from the Rodrigues type formulas both polynomials satisfy, or from differential equations.
Pollaczek found a very interesting set of orthogonal polynomials which generalize Legendre polynomials and Szegö extended them to generalize ultraspherical polynomials. With a different normalization, the even case of these polynomials satisfies

$$x[(a+1)n+b]P_n(x) = (an+b)P_{n+1}(x) + nP_{n-1}(x). \tag{4.2}$$

Explicit representations as hypergeometric series can be found. One form is

$$P_n(x) = (a\alpha)^{-n}{}_2F_1\left(\begin{matrix}-n, -B\\ b/a\end{matrix}; -\alpha\right) \tag{4.3}$$

where

$$\begin{aligned} \xi &= [(a+1)^2x^2 - 4a]^{1/2} \\ \alpha &= \frac{x(a+1)+\xi}{2a}, \quad \beta = \frac{x(a+1)-\xi}{2a} \\ B &= b(\beta - x)/\xi. \end{aligned} \tag{4.4}$$

I do not know how to show directly that (4.3) gives a polynomial in x. Such an argument might be related to the argument in section 2, since the polynomials in section 2 are the special case $a = 0$ of these polynomials of Pollaczek and Szegö. See [1] for these polynomials, and a q-extension, for which the same question is still unsolved.

I am afraid that the real message of this paper is that some things which can not be done easily by one method, although they are easy by other methods, can occasionally be done in the hard setting, but so far I have not learned anything new of real interest by doing them this way. That is a disappointment, but life is frequently full of disappointments.

References for other limit relations are given below. See [5], [8]. These show how some limit relations can be proven. The q-chart I mentioned above has not been published, and it is not clear what is the best way to present it. The one for hypergeometric orthogonal polynomials is given in [2] with the continuous Hahn polynomials being missed and replaced by their symmetric special case. An extended printed version of this chart with the continuous Hahn polynomials placed correctly has been made [7], and can be seen in some faculty offices in a number of countries, both in mathematics and physics departments.

Acknowledgements

I thank G. Valent for bringing up the problem solved in section 2, for it got me to think about it seriously, and then later to realize that a similar argument would work to solve the problem treated in section 3. Both of these are problems I wanted to solve, or to have someone else solve.

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