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L_1-norm convergence of Hermite-Fejér interpolation based on the Laguerre and Hermite abscissas

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Dedicated to the memory of Prof. Aldo Ghizzetti

RIASSUNTO: *In questo lavoro, vengono provati dei risultati di convergenza per l'interpolazione di Hermite-Fejér, basata sugli zeri di polinomi ortogonali, sia di tipo Laguerre generalizzato, che di tipo Hermite. Questi risultati sono poi applicati per dedurre la convergenza di formule di integrazione prodotto.*

ABSTRACT: *Convergence results are proved for Hermite–Fejér interpolation at the zeros of polynomials orthogonal with respect to generalized Laguerre and Hermite weights. These results can be applied to convergence of product integration rules.*

1 – Introduction

The purpose of this paper is the investigation of the convergence, in the weighted L_1–norm, of Hermite–Fejér interpolation, based on the zeros of orthogonal polynomials associated with generalized Laguerre and Hermite weights. While the natural setting for analyzing convergence of Lagrange interpolation at the zeros of orthogonal polynomials is the L_2–

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norm [13], Hermite–Fejér interpolation can conveniently be studied in the L_1–norm, as recent results confirm. Actually, Nevai and Vértesi in [7] posed the problem of the convergence of Hermite–Fejér interpolation in a finite interval $[a, b]$, considering as interpolatory points the zeros of the polynomial $p_n(w)$ orthogonal with respect to a non–negative weight w; they proved the convergence of Hermite–Fejér interpolation for any polynomial; in addition, if $[a, b] = [-1, 1]$ and w is a generalized Jacobi weight, then the convergence is assured also for each $f \in C[-1, 1]$.

The case of Hermite–Fejér interpolation at the zeros of polynomials orthogonal with respect to weights associated to the real line and several features of these weights have been treated in [3–6,10,12]; in particular, in [4] Freud and Erdös weights are dealt with simultaneously, and convergence theorems are given not only for Hermite–Fejér interpolation, but also for Hermite osculatory interpolating polynomials; furthermore, the results are applied to convergence of product integration rules.

In the present paper we shall deal with Hermite–Fejér interpolation based on the abscissas of generalized Laguerre weights, giving some results on the convergence in the weighted L_1–norm. The same is done for the Hermite weight and abscissas. Once these results have been obtained, the convergence of associated product quadrature rules can also be considered.

In fact, the numerical evaluation of integrals of the form

$$\int_{\mathbb{R}} f(x)p(x)\,\mathrm{d}x$$

can be performed by means of product quadratures

$$Q_n(p, f) := \sum_{j=1}^{n} w_{jn}(p) f(x_{jn}) \tag{1.1}$$

where the weights $w_{jn}(p)$, $1 \le j \le n$, are usually determined by integrating a suitable approximation to f; next the convergence of the sequence $\{Q_n(p, f)\}$ can be analyzed.

If, in particular, f is approximated by $H_n(w, f, x) := \sum_{j=1}^{n} h_{jn}(w, x) f(x_{jn})$, then

$$w_{jn}(p) = \int_{\mathbb{R}} h_{jn}(w,x)p(x)\,\mathrm{d}x \ , \tag{1.2}$$

in this case we shall denote the quadrature sum (1.1) by $Q_n(p,f,w)$.

The convergence of the sequences $\{Q_n(p,f,w)\}$, for suitable weights w, can be assured for functions having a convenient growth at infinity.

Results concerning quadrature on infinite intervals are contained in [3,10], while product integration rules based on Hermite–Fejér interpolation is treated in [4,8,9].

After introducing notations and preliminaries in Section 2, we state and discuss the main results for generalized Laguerre weights in Section 3. Analogous result for Hermite weight is developed in Section 4; in Section 5, we prove L_1-norm convergence of Hermite-Lagrange interpolation. Finally, in Section 6, convergence of product quadrature rules is examined.

2 – Notations

$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}^+$ denote the sets of real numbers, real non–negative numbers, integers and non–negative integers, respectively.

The set of polynomials of degree at most n is denoted by $\mathbb{P}_n$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be $\in BI(\mathbb{R})$ if f is bounded and Riemann integrable on every finite interval $A \subset \mathbb{R}$.

If $p \in \mathbb{R}^+$, $p \neq 0$, then $f \in L_p(A)$ on some interval A if $\|f\|_p < \infty$, where

$$\|f\|_p := \left[\int_A |f(x)|^p\,\mathrm{d}x\right]^{1/p} , \quad 0 < p < \infty \ ,$$

and

$$\|f\|_\infty := \sup_{x\in A} |f(x)| \ ;$$

if $v \geq 0$ and $0 < p < \infty$, then $f \in L_{p,v}$ if $\|f\|_{p,v} < \infty$, where

$$\|f\|_{p,v} := \left[\int_{\mathbb{R}} |f(x)|^p v(x)\,\mathrm{d}x\right]^{1/p} .$$

$\|\cdot\|_p$ is a norm if $p \geq 1$. If w is an admissible weight function, i.e. $w \geq 0$ on $\mathbb{R}$, such that $we_k \in L_1(\mathbb{R})$ (where $e_k := x^k$, $k \in \mathbb{N}$) and $\|w\|_1 > 0$, then $p_n(x) := p_n(w,x)$, $\{x_{jn}\}_{j=1}^n$, $n \in \mathbb{N}^+$, denote the corresponding orthonormal polynomial $\in \mathbb{P}_n$

$$p_n(x) = \gamma_n x^n + \dots \ , \tag{2.1}$$

and its zeros, respectively.

By $l_{jn}(x) := l_{jn}(w,x) \in \mathbb{P}_{n-1}$ we indicate the j–th fundamental polynomial of Lagrange interpolation for $\{x_{jn}\}_{j=1}^n$; it fulfils the conditions

$$l_{jn}(x_{kn}) = \delta_{jk} \ , \quad 1 \leq j \ , \quad k \leq n \ ,$$

and the Cotes numbers, defined by

$$\lambda_{jn} := \lambda_{jn}(w) = \int_{\mathbb{R}} l_{jn}^2(x) w(x)\,\mathrm{d}x = \int_{\mathbb{R}} l_{jn}(x) w(x)\,\mathrm{d}x \ , \ 1 \leq j \leq n \ , \tag{2.2}$$

are the weights of the Gaussian quadrature rule

$$\int_{\mathbb{R}} P(x) w(x)\,\mathrm{d}x = \sum_{j=1}^{n} \lambda_{jn} P(x_{jn}) \ , \quad \forall\, P \in \mathbb{P}_{2n-1} \ . \tag{2.3}$$

It is also known [1, p. 48] that

$$l_{jn}(x) = \frac{\gamma_{n-1}}{\gamma_n} \lambda_{jn} p_{n-1}(x_{jn}) \frac{p_n(x)}{x - x_{jn}} \ , \quad 1 \leq j \leq n \ , \tag{2.4}$$

which implies

$$1 = \frac{\gamma_{n-1}}{\gamma_n} \lambda_{jn} p_{n-1}(x_{jn}) p_n'(x_{jn}) \ , \quad 1 \leq j \leq n \ ,$$

and

$$\lambda_{jn} p_{n-1}^2(x_{jn}) = \frac{\gamma_n}{\gamma_{n-1}} \frac{p_{n-1}(x_{jn})}{p_n'(x_{jn})} \ , \quad 1 \leq j \leq n \ . \tag{2.5}$$

The polynomials $h_{jn}(x)$, $k_{jn}(x) \in \mathbb{P}_{2n-1}$ below

$$(2.6) \qquad h_{jn}(x) := \left\{1 - \frac{p_n''(x_{jn})}{p_n'(x_{jn})}(x - x_{jn})\right\} l_{jn}^2(x) \; ; \quad 1 \le j \le n \; ,$$

$$k_{jn}(x) := (x - x_{jn}) l_{jn}^2(x) \; , \quad 1 \le j \le n \; ,$$

are the fundamental polynomials of Hermite interpolation. Hence the Hermite–Fejér interpolating polynomial $H_n(w, f, x) \in \mathbb{P}_{2n-1}$, which satisfies the conditions

$$\begin{cases} H_n(w, f, x_{jn}) = f(x_{jn}) \\ H_n'(w, f, x_{jn}) = 0 \end{cases}$$

can conveniently be written in the form

$$H_n(w, f, x) := \sum_{j=1}^{n} f(x_{jn}) h_{jn}(x) \; .$$

In addition, we introduce the operator K_n:

$$K_n(w, f, x) := \sum_{j=1}^{n} f(x_{jn}) k_{jn}(x) \; ;$$

thus, the Hermite–Lagrange or osculatory interpolation polynomial is given by

$$F_n(w, f, x) = H_n(w, f, x) + K_n(w, f', x) \; ;$$

the Hermite–Lagrange interpolation formula reads as

$$(2.7) \qquad P(x) = F_n(w, P, x) = \sum_{j=1}^{n} P(x_{jn}) h_{jn}(x) + \sum_{j=1}^{n} P'(x_{jn}) \cdot k_{nj}(x)$$

where $P \in \mathbb{P}_{2n-1}$ [13, p. 331].

Denoting the generalized Laguerre weights by W, i.e.

$$(2.8) \qquad W(x) = \begin{cases} e^{-x} x^{\alpha} \; , & x \ge 0 \, , \alpha > -1 \\ 0 & x < 0 \, , \end{cases}$$

the corresponding generalized Laguerre polynomials $L_n^{(\alpha)}$ fulfil the condition

$$\|L_n^{(\alpha)}\|_{1,W} = 1 \ , \tag{2.9}$$

when γ_n, in (2.1), is given by [2, (3.6.8)]

$$\gamma_n = (-1)^n/[n!\Gamma(\alpha+n+1)]^{1/2} \ . \tag{2.10}$$

In the following, generalized Laguerre weights and polynomials will be referred to as Laguerre weights and polynomials.

For the polynomials $L_n^{(\alpha)}$, the relations below follow from well known formulas [2, (3.6.13), (3.6.10)]

$$L_n^{(\alpha)'}(x_{jn}) = -(n+\alpha)L_{n-1}^{(\alpha)}(x_{jn})/x_{jn} \ , \quad 1 \le j \le n \ , \tag{2.11}$$

$$L_n^{(\alpha)''}(x_{jn})/L_n^{(\alpha)'}(x_{jn}) = (x_{jn}-\alpha-1)/x_{jn} \ , \quad 1 \le j \le n \ . \tag{2.12}$$

We shall also set

$$\rho_n := \gamma_{n-1}/\gamma_n \ , \tag{2.13}$$

thus (2.10) yields, for Laguerre weights,

$$\rho_n = -\sqrt{n(\alpha+n)} \ . \tag{2.14}$$

We shall denote by $V(x)$

$$V(x) := e^{-x^2} \tag{2.15}$$

the Hermite weight on $\mathbb{R}$ and by $\{\mathcal{H}_n(x)\}$ the corresponding system of orthonormal polynomials, for which the following relations hold [2, (3.7.11)]

$$\gamma_n = \sqrt{2^n/(n!\sqrt{\pi})}$$

$$\mathcal{H}_n'(x_{kn}) = \mathcal{H}_{n-1}(x_{kn}) \cdot 2n \ , \quad 1 \le k \le n \tag{2.16}$$

$$\mathcal{H}''(x_{kn})/\mathcal{H}_n'(x_{kn}) = 2x_{kn}\ , \quad 1 \le k \le n \tag{2.17}$$

$$\rho_n = \sqrt{n/2}\ . \tag{2.18}$$

3 – Convergence results

A first convergence result concerns the case of Laguerre weights, and is valid for polynomials

Theorem 1. *Let W be a Laguerre weight, then*

$$\lim_{n\to\infty} \|P - H_n(W,P)\|_{1,W} = 0 \tag{3.1}$$

for every polynomial P.

Proof. Let P be any polynomial $\in \mathbb{P}_{2n-1}$, then from (2.7), with $w = W$, and using also (2.4), (2.6) we get

$$\begin{aligned} P(x) - H_n(W,P,x) &= \sum_{j=1}^{n} P'(x_{jn})(x - x_{jn}) l_{jn}^2(x) = \\ &= \rho_n L_n^{(\alpha)}(x) \sum_{j=1}^{n} P'(x_{jn}) L_{n-1}^{(\alpha)}(x_{jn}) \lambda_{jn} l_{jn}(x) = \\ &= \rho_n L_n^{(\alpha)}(x) \mathcal{L}_n(x)\ , \end{aligned}$$

where $\mathcal{L}_n \in \mathbb{P}_{n-1}$, is the Lagrange interpolation polynomial satisfying

$$\mathcal{L}_n(x_{jn}) = P'(x_{jn}) L_{n-1}^{(\alpha)}(x_{jn}) \lambda_{jn}\ , \quad 1 \le j \le n\ ,$$

and λ_{jn} are the weights (2.2) of the Gaussian integration rule (2.3) relative to W.

Applying the Cauchy-Schwarz inequality, the Gauss-Laguerre quadrature rule, and recalling (2.9), one has

$$\begin{aligned}\|P - H_n(W,P)\|_{1,W} &= |\rho_n| \int_{\mathbb{R}^+} |L_n^{(\alpha)}(x)\mathcal{L}_n(x)| W(x)\,\mathrm{d}x \le \\ &\le |\rho_n| \Big\{ \int_{\mathbb{R}^+} [\mathcal{L}_n(x)]^2 W(x)\,\mathrm{d}x \Big\}^{1/2} = \\ &= |\rho_n| \cdot \Big\{ \sum_{j=1}^{n} \lambda_{jn} [P'(x_{jn}) L_{n-1}^{(\alpha)}(x_{jn}) \lambda_{jn}]^2 \Big\}^{1/2} .\end{aligned}$$

Furthermore, (2.5) gives

$$\lambda_{jn}[L_{n-1}^{(\alpha)}(x_{jn})]^2 = \frac{1}{\rho_n} \frac{L_{n-1}^{(\alpha)}(x_{jn})}{L_n^{(\alpha)'}(x_{jn})} \tag{3.2}$$

and hence (2.11), (2.14), (3.2) yield, for $n \ge$ degree P,

$$\|P - H_n(W,P)\|_{1,W} \le \left(\frac{|\rho_n|}{n+\alpha} \right)^{1/2} \max_{1 \le j \le n} \lambda_{jn}^{1/2} \|[P'(x)]^2 x\|_{1,W} .$$

About the Laguerre weights (2.8), it is known [13, p. 355], that assuming the zeros $\{x_{jn}\}_{j=1}^n$ ordered increasingly, the sequence $\{\lambda_{jn}\}_{j=1}^n$ of the coefficients in (2.2) is increasing for $x_\nu < \alpha + 1/2$ and decreasing for $x_\nu > \alpha + 1/2$; this behaviour, together with the relation [13, (15.3.18)]

$$\lambda_{jn} \sim x_{jn}^{\alpha+1/2} n^{-1/2} , \quad 0 < x_{jn} \le \omega , \quad \text{for some } \omega ,$$

gives

$$\max_{1 \le j \le n} \lambda_{jn} = o\,(1) .$$

Thus, taking into account (2.14), we get (3.1). □

Now, let us denote by $S(\mathbb{R}^+)$ the following class of functions f. A function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is said to be $\in S(\mathbb{R}^+)$ if $f \in BI(\mathbb{R}^+)$, and

$$f(x)x^{-1/2} \in L_{1,W}(\mathbb{R}^+) \tag{3.3}$$

$$(3.4)\qquad |f(x)x^{1/2}W(x)| < \text{ const } x^{-1-\rho}\ , \quad x \text{ large}, \quad \rho > 0\ .$$

In order to prove our main result, the following theorem turns out to be useful.

THEOREM 2. *Let $g \in S(\mathbb{R}^+)$, then*

$$(3.5)\qquad \lim_{n\to\infty} \sup \|H_n(W,g)\|_{1,W} \leq \text{const}\|g\|_{1,W}\ .$$

PROOF. First, we remark that, by (2.2) one has

$$(3.6)\qquad \int_{\mathbb{R}^+} |\sum_{j=1}^{n} l_{jn}^2(x)g(x_{jn})|W(x)\,\mathrm{d}x \leq \sum_{j=1}^{n} |g(x_{jn})| \int_{\mathbb{R}^+} l_{jn}^2(x)W(x)\,\mathrm{d}x = \\ = \sum_{j=1}^{n} \lambda_{jn}|g(x_{jn})|\ ,$$

moreover, by a theorem on quadrature convergence [14], the condition (3.4) implies that

$$(3.7)\qquad \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn}|g(x_{jn})| = \|g\|_{1,W}\ .$$

Next, from (2.6) and (3.6) one gets

$$\int_{\mathbb{R}^+} |H_n(W,g,x)|W(x)\,\mathrm{d}x \leq \sum_{j=1}^{n} \lambda_{jn}|g(x_{jn})| + I_1\ ,$$

where

$$I_1 := \int_{\mathbb{R}^+} |\sum_{j=1}^{n} g(x_{jn}) \frac{L_n^{(\alpha)''}(x_{jn})}{L_n^{(\alpha)'}(x_{jn})}(x - x_{jn})l_{jn}^2(x)|\,\mathrm{d}x\ .$$

Using (2.4), (2.9), (2.12), (2.13) and the Cauchy–Schwarz inequality, we obtain

$$I_1=\int_{\mathbb{R}^+}|\rho_n L_n^{(\alpha)}(x)\sum_{j=1}^{n} l_{jn}(x)\lambda_{jn}g(x_{jn})L_{n-1}^{(\alpha)}(x_{jn})(x_{jn}-\alpha-1)/x_{jn}|W(x)\,\mathrm{d}x\leq$$

$$\leq|\rho_n|\Big\{\int_{\mathbb{R}^+}\Big[\sum_{j=1}^{n} l_{jn}(x)\lambda_{jn}g(x_{jn})L_{n-1}^{(\alpha)}(x_{jn})(x_{jn}-\alpha-1)/x_{jn}\Big]^2 W(x)\,\mathrm{d}x\Big\}^{1/2}:=$$

$$:=|\rho_n|\Big\{\int_{\mathbb{R}^+}\Big[\sum_{j=1}^{n} l_{jn}(x)F(x_{jn})\Big]^2 W(x)\,\mathrm{d}x\Big\}^{1/2}.$$

If we let $G(x)=\sum_{k=1}^{n} l_{kn}(x)F(x_{kn})\in\mathbb{P}_{n-1}$, then we have $G(x_{jn})=F(x_{jn})$, moreover (2.3) and (2.5), (2.11) allow us to write

$$I_1\leq|\rho_n|\Big\{\sum_{j=1}^{n}\lambda_{jn}[G(x_{jn})]^2\Big\}^{1/2}=$$

$$=\left(\frac{|\rho_n|}{n+\alpha}\right)^{1/2}\Big\{\sum_{j=n}^{n}\lambda_{jn}^2[g(x_{jn})]^2(x_{jn}-\alpha-1)^2/x_{jn}\Big\}^{1/2}\leq$$

$$\leq\left(\frac{|\rho_n|}{n+\alpha}\right)^{1/2}\sum_{j=1}^{n}\lambda_{jn}|g(x_{jn})|\;|x_{jn}-\alpha-1|x_{jn}^{-1/2}.$$

The assumption on g assures that [14]

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn}|g(x_{jn})|\;|x_{jn}-\alpha-1|x_{jn}^{-1/2}=$$

$$=\int_{\mathbb{R}^+}|g(x)|\;|x-\alpha-1|x^{-1/2}W(x)\,\mathrm{d}x:=I_2.$$

Furthermore

$$\int_{\mathbb{R}^+}|g(x)|\;|x-\alpha-1|x^{-1/2}W(x)\,\mathrm{d}x\leq\text{ const }\|g\|_{1,W}\;; \tag{3.8}$$

in fact, since $g\in S(\mathbb{R}^+)$,

$$\forall\,\varepsilon>0\;\exists\delta>0\ni\int_0^{\delta}|g(x)|\;|x-\alpha-1|x^{-1/2}W(x)\,\mathrm{d}x<\varepsilon\,,$$

$$\forall\, \varepsilon > 0\ \exists \tau > 0 \ni \int_{\tau}^{+\infty} |g(x)|\ |x-\alpha-1| x^{-1/2} W(x)\, \mathrm{d}x < \varepsilon\ ,$$

hence

$$I_2 < 2\varepsilon + \tau\delta^{-1/2} \int_{\mathbb{R}^+} |g(x)| W(x)\, \mathrm{d}x\ .$$

Thus (3.6), (3.7), (3.8) give (3.5). □

THEOREM 3. *Let $f \in S(\mathbb{R}^+)$, then*

$$\lim_{n\to\infty} \|f - H_n(W,f)\|_{1,W} = 0\ .$$

PROOF. For any polynomial P one has

$$(3.9)\quad \|f - H_n(W,f)\|_{1,W} \le \|H_n(W, f-P)\|_{1,W} + \|P - H_n(W,P)\|_{1,W} + \\ + \|f-P\|_{1,W} \le \text{ const } \|f-P\|_{1,W} + o(1) + \|f-P\|_{1,W}\ .$$

By Theorem 5.7.2 in [13] the right–hand side of (3.9) can be made arbitrarily small, then the claim follows. □

4 – Hermite–Fejér interpolation based on Hermite abscissas

Following the same line of reasoning as in the previous section ,we may deal with the Hermite–Fejér interpolation associated with the Hermite weight (2.15) and abscissas. Even though this is a special case of the general theorem in [4], nevertheless we give it here since the derivation is so much simpler than there. Furthermore the conditions for convergence are slightly weaker than in [4].

LEMMA 1. *Let V be the Hermite weight, then*

$$(4.1)\qquad \lim_{n\to\infty} \|P - H_n(V,P)\|_{1,V} = 0$$

for every polynomial P.

The proof of this lemma is easily obtained using the same line of demonstration as in Theorem 1; thus one gets, recalling (2.16), (2.18),

$$\|P - H_n(V,P)\|_{1,V} \leq \left(\frac{\rho_n}{2n}\right)^{1/2} \|P'(x)\cdot x\|_{1,V} = \\ = o(1)\|P'(x)\cdot x\|_{1,V}$$

from which the claim follows.

We introduce here the class of functions denoted by $T(\mathbb{R})$. A function $f:\mathbb{R}\to\mathbb{R}$ is said to be $\in T(\mathbb{R})$ if $f\in BI(\mathbb{R})$ and

$$|f(x)xV(x)| < \text{ const } |x|^{-1-\rho}\ , \quad x \text{ large}, \quad \rho > 0\ . \tag{4.2}$$

THEOREM 4. *Let $g\in T(\mathbb{R})$, then*

$$\lim_{n\to\infty} \sup \|H_n(V,g)\|_{1,V} \leq \text{ const } \|g\|_{1,V}\ . \tag{4.3}$$

PROOF. One has

$$\|H_n(g,V)\|_{1,V} \leq \int_{\mathbb{R}} |\sum_{j=1}^{n} l_{jn}^2(x)g(x_{jn})|V(x)\,\mathrm{d}x + \\ + \int_{\mathbb{R}} |\sum_{j=1}^{n} g(x_{jn})\frac{\mathcal{H}_n''(x_{jn})}{\mathcal{H}_n'(x_{jn})}\cdot(x-x_{jn})l_{jn}^2(x)|V(x)\,\mathrm{d}x := I_3 + I_4\ .$$

Next

$$I_3 \leq \sum_{j=1}^{n} \lambda_{jn}|g(x_{jn})|$$

and, by the assumption on g, one may write

$$\lim_{n\to\infty} I_3 = \|g\|_{1,V}\ .$$

Turning to I_4, one has by (2.16), the Cauchy–Schwarz inequality and the Gauss–Hermite quadrature rule

$$I_4 \leq 2|\rho_n|\left\{\int_R \left(\sum_{j=1}^{n} l_{jn}(x)\lambda_{jn}\mathcal{H}_{n-1}(x_{jn})g(x_{jn})x_{jn}\right)^2 V(x)dx\right\}^{1/2} = \\ = 2|\rho_n|\left\{\sum_{j=1}^{n}\lambda_{jn}\lambda_{jn}^2\mathcal{H}_{n-1}^2(x_{jn})g^2(x_{jn})x_{jn}^2\right\}^{1/2}$$

from which using (2.5) and (2.16) we derive

$$I_4 \leq \left(\frac{2\rho_n}{n}\right)^{1/2} \sum_{j=1}^{n} \lambda_{jn}|x_{jn}g(x_{jn})| = o(1) \sum_{j=1}^{n} \lambda_{jn}|x_{jn}g(x_{jn})| ,$$

the hypothesis on g enables us to write

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn}|x_{jn}g(x_{jn})| = \int_{\mathbb{R}} |xg(x)|V(x)\,\mathrm{d}x \ ,$$

moreover, by argument analogous to those in the proof of Theorem 2,

$$\int_{\mathbb{R}} |xg(x)|V(x)\,\mathrm{d}x \leq \ \text{const}\ \|g\|_{1,V}$$

and the claim follows.

THEOREM 5. *Let $f \in T(\mathbb{R})$, then*

$$\lim_{n\to\infty} \|f - H_n(V, f)\|_{1,V} = 0 \ .$$

The proof is similar to that of Theorem 3, since [13, Theorem 5.7.2] also holds in this case.

5 – Convergence of Hermite–Lagrange polynomials

In Section 2, we introduced the Hermite-Lagrange or Hermite osculatory operator F_n (cf (2.7)); now, we turn to analyze the convergence of Hermite–Lagrange interpolation at the zeros of generalized Laguerre polynomials, or at those of Hermite polynomials.

We first consider the operator K_n given by

$$K_n(w, g) = \sum_{j=1}^{n} g(x_{jn})(x - x_{jn})l_{jn}^2(x) ,$$

where w is an admissible weight, and prove the following Lemma.

Lemma 2. *Let W be a Laguerre weight, let $g\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ satisfy the condition* (3.4), *then*

$$\|K_n(W,g)\|_{1,W} \leq \text{ const } \|g\|_{1,W}\,; \tag{5.1}$$

furthermore, for every polynomial P, there results

$$\|K_n(W,P)\|_{1,W} = o(1)\,. \tag{5.2}$$

Proof. By (2.4) one has

$$\|K_n(W,g)\|_{1,W} = \int_{\mathbb{R}^+} \Big|\sum_{j=1}^n g(x_{jn})(x-x_{jn})l_{jn}^2(x)\Big| W(x)dx =$$

$$= |\rho_n| \cdot \int_{R^+} \Big| L_n^{(\alpha)}(x) \sum_{j=1}^n \lambda_{jn} g(x_{jn}) L_{n-1}^{(\alpha)}(x) l_{jn}(x)\Big| W(x)dx\,;$$

applying the Cauchy-Schwarz inequality, the Gauss–Laguerre quadrature rule yields

$$\|K_n(W,g)\|_{1,W} \leq |\rho_n| \Big\{ \sum_{j=1}^n \lambda_{jn} L_{n-1}^2(x_{jn}) \lambda_{jn}^2 g^2(x_{jn}) \Big\}^{1/2} \tag{5.3}$$

which, by (2.5), (2.11) reduces to

$$\|K_n(W,g)\|_{1,W} \leq \Big(\frac{|\rho_n|}{n+\alpha}\Big)^{1/2} \sum_{j=1}^n \lambda_{jn} |g(x_{jn})| x_{jn}^{1/2}\,. \tag{5.4}$$

Now, the convergence of the Gauss–Laguerre quadrature sums in the right-hand member of (5.4) is assured under the assumption on g, so one has

$$\|K_n(W,g)\|_{1,W} = 0(1) \int_{\mathbb{R}^+} |g(x)| x^{1/2} W(x)dx$$

and the relation (5.1) follows by arguments analogous to those leading to (3.8).

If g reduces to a polynomial P, we can infer from (5.3)

$$\|K_n(W,P)\|_{1,W} \leq \left(\frac{|\rho_n|}{n+\alpha}\right)^{1/2} \max_{1\leq j\leq n} \lambda_{jn}^{1/2}\left\{\sum_{j=1}^{n}\lambda_{jn}P^2(x_{jn})x_{jn}\right\}^{1/2}.$$

Since the function $h(x)=xP^2(x)$ satisfies the condition

$$|h(x)W(x)| < \text{ const } \cdot x^{-1-\rho}, \quad x \text{ large }, \rho>0$$

one gets, by the results of [14],

$$\|K_n(W,P)\|_{1,W} \leq 0(1)\cdot \max_{1\leq j\leq n} \lambda_{jn}^{1/2}\|xP^2(x)\|_{1,W},$$

and then (5.2) is a consequence of the behaviour of max λ_{jn} already discussed in Section 3. ∎

Theorem 3 and Lemma 2 allow us to state the next theorem.

THEOREM 6. *Let $f \in S(\mathbb{R}^+)$, assume that f' exists in $\mathbb{R}^+$ and satisfies condition* (3.4)*, then*

$$\lim_{n\to\infty}\|f-F_n(W,f)\|_{1,W}=0 \tag{5.5}$$

PROOF. Using (2.7), (5.1), (5.2), we can write

$$\|f-F_n(W,f)\|_{1,W} \leq \|f-H_n(W,f)\|_{1,W} + \|K_n(W,P)\|_{1,W} +$$
$$+\|K_n(W,f'-P)\|_{1,W} = o(1)+o(1)+0(1)\cdot\|f'-P\|_{1,W},$$

since $\|f'-P\|_{1,W}$ can be made arbitrarily small [13, Theorem 5.7.2], (5.5) is proved. ∎

As for the case of Hermite–Lagrange interpolation at the zeros of Hermite polynomials, results on the convergence can be derived similarly as in the previous case, and the theorems below can be easily proved.

LEMMA 3. *Let V be the Hermite weight and let g be $\in T(\mathbb{R})$* (cf Section 4)*, then*

$$\|K_n(V,g)\|_{1,V} \leq \text{ const } \|g\|_{1,V},$$

and

$$\|K_n(V,P)\|_{1,V} = 0(1)$$

for every polynomial P.

The proof is obtained following the same line of reasoning as in Lemma 2. ∎

THEOREM 7. *Let V be the Hermite weight, let $f \in T(\mathbb{R})$, f' exist in $\mathbb{R}$, with $f' \in T(\mathbb{R})$, then*

$$\lim_{n\to\infty} \|f - F_n(V,f)\|_{1,V} = 0\,.$$

The proof is a consequence of Lemma 3 and Theorem 5. ∎

It is to be noted that, although the last result is a particular case of a convergence theorem given in [4] for more general weights, here the convergence is proved under an assumption which is weaker than the condition required in [4].

6 – Convergence of product integration rules

Consider now the product integration rules (1.1), (1.2) based on Laguerre or Hermite abscissas; as a consequence of the results of the previous sections, the following Corollaries are readily proved.

COROLLARY 1. *Let $f \in S(\mathbb{R}^+)$ and p be a measurable function satisfying the relation*

$$\sup_{x\in\mathbb{R}^+} |p(x)/W(x)| < \infty \ ,$$

then

$$\lim_{n\to\infty} Q_n(p,f,W) = \int_{\mathbb{R}^+} f(x)p(x)\,\mathrm{d}x \ .$$

COROLLARY 2. *Let $f \in T(\mathbb{R})$ and p a measurable function satisfying the relation*

$$\sup_{x\in\mathbb{R}} |p(x)/V(x)| < \infty \ ,$$

then

$$\lim_{n\to\infty} Q_n(p,f,V) = \int_{\mathbb{R}} f(x)p(x)\,\mathrm{d}x\ .$$

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