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# Sturm-Liouville boundary value problems and Lagrange interpolation series

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*Dedicated to the memory of Professor Aldo Ghizzetti*

RIASSUNTO: *Il lavoro è dedicato alle connessioni tra il teorema di campionamento di Kramer ed una particolare forma della serie interpolatoria di Lagrange. Questa connessione si rivela particolarmente utile quando il nucleo del tipo di Kramer gode di specifiche proprietà analitiche, poiché ciò determina una corrispondente proprietà di analiticità per i singoli termini della serie di Lagrange.*

*Recenti risultati hanno mostrato che un caso significativo ed importante di tale connessione è da rilevarsi nella costruzione di nuclei del tipo di Kramer a partire da problemi ai limiti per operatori simmetrici di tipo autoaggiunto sull'asse reale.*

*Nel lavoro ci si limita a considerare i classici problemi di Sturm-Liouville per operatori differenziali del secondo ordine, richiedendo tuttavia condizioni minimali sui coefficienti (appartenenza alla classe $L^1_{loc}$). Vengono peraltro considerati diversi casi di punti limite, seguendo la classificazione di Naimark (punti regolari ovvero cicli-limite).*

*Questo modo di procedere segue quello di precedenti lavori di Weiss, Kramer, Campbell e recenti risultati di Butzer, Zayed e Schöttler. I nuovi metodi qui utilizzati potrebbero essere estesi ad altre classificazioni dei punti limite e ad operatori differenziali simmetrici di ordine arbitrario.*

ABSTRACT: *This paper is concerned with the connection between the Kramer sampling theorem and one form of the Lagrange interpolation formula. One particular interest of this connection is when the Kramer-type kernel has certain analytic properties since this leads to corresponding analyticity for the individual terms in the Lagrange interpolation series. Recent results have shown that one important and significant case of this connection is to be found in the generation of these Kramer-type kernels from self-adjoint boundary value problems, determined by symmetric ordinary linear differ-*

*ential expressions defined on intervals of the real line. In these cases the analyticity properties result from the presence of the spectral parameter of the corresponding self-adjoint differential operator. Results in this paper are restricted to consideration of the classic Sturm-Liouville differential expression of the second-order, but under the minimal (locally Lebesgue integrable) conditions on the coefficients; furthermore the expression is taken to be in the regular and/or limit-circle end-point classification.*

*This approach follows earlier work of Weiss, Kramer, Campbell and others, and recent results of Butzer, Zayed and Schöttler. The new methods adopted here should extend to other end-point classifications and to symmetric differential expressions of arbitrary order.*

## 1 – Introduction, preliminary results, statement of main theorem

### 1.1 – The problem

The Lagrange interpolation results of this paper are best seen in the light of the survey article by Butzer, Splettstösser and Stens [5], see Section 6 and the extensive list of references, and especially the work of Kramer [16] and Campbell [6]. The connection with the classical theory of Sturm-Liouville differential equations and special functions can be seen in the work of Higgins [15] and the three recent papers of Zayed, Hinsen and Butzer [21], Zayed [22] and Butzer and Schöttler [4].

Our main concern in this paper is with extending this connection between Lagrange interpolation and Sturm-Liouville theory which follows from the abstract theory of differential operators defined in Lebesgue integrable-square Hilbert functions spaces. The operator theory required is to be found in Akhiezer and Glazman [2, Appendix 2], and Naimark [17, Chapters V and VI]; the connection between this operator theory and the classical theory of Sturm-Liouville differential equations is developed by Titchmarsh [20], Coddington and Levinson [7, Chapter 7] and Everitt [12]. All these results will be deployed in order to obtain a Lagrange interpolation result under minimal conditions on the coefficients of the associated Sturm-Liouville differential equations.

There is an informative introduction to the form of the Lagrange interpolation result which is the concern of this paper, in the first Sections of the two papers [21, (1.10)] and [22, (1.10)]. If $\{\lambda_n : n \in \mathbb{Z}\}$ is a fixed

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sequence of distinct real numbers and if $F : \mathbb{C} \to \mathbb{C}$ is an entire (integral) function on $\mathbb{C}$, of a type to be specified, then we seek an entire function $G$, with simple zeros only at the points $\{\lambda_n\}$, so that $F$ can be represented in the form

$$F(\lambda) = \sum_{n \in \mathbb{Z}} F(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} \qquad (\lambda \in \mathbb{C}) . \tag{1.1}$$

There is a corollary to this result which states that if $\{c_n : n \in \mathbb{Z}\}$ is a given sequence of complex numbers, satisfying a certain growth condition as $n \to \pm\infty$, then we can construct a unique entire function $F$, from the class to be specified, such that

$$F(\lambda_n) = c_n \qquad (n \in \mathbb{Z}) . \tag{1.2}$$

Thus from the given information (1.2) the function $F$ is defined on the whole complex plane $\mathbb{C}$, and uniquely so within the class specified. Since the statement of our results requires a detailed preliminary introduction it is well that at this stage we draw the attention of the reader to the consequences of Theorem 1.1 and Corollary 1.2 at the end of this Section; in particular to the results (1.36) and (1.38).

The analytic methods used in this paper are essentially different from the methods employed in the papers [4], [21], and the methods of Zayed in [22]. We make a comparison between results and methods, as used here and in [22], in Section 7 below.

### 1.2 – Notations

Some notations are: $\mathbb{Z}$ represents the set of integers $\{\ldots -2, -1, 0, 1, 2, \ldots\}$; $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$; $\mathbb{N}_0^- = \{\ldots, -2, -1, 0\}$; $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex number fields; for any open set $V$ of $\mathbb{C}$ the symbol $\mathbf{H}(V)$ denotes the class of functions $F : V \to \mathbb{C}$ such that $F$ is (Cauchy) holomorphic on $V$; $L$ and $AC$ denote Lebesgue integrable and absolutely continuous classes of functions; "loc" denotes a property holding on compact subsets of an open set in $\mathbb{R}$ or $\mathbb{C}$.

If $I$ is an interval of $\mathbb{R}$, $w : I \to \mathbb{R}$ is Lebesgue measurable and $w(x) > 0$ (almost all $x \in I$), then $L^2(I; w)$ denotes the Hilbert function space of equivalence classes taken from

$$\left\{ f : I \to \mathbb{C} : f \quad \text{is Lebesgue measurable and} \quad \int_I |f(x)|^2 w(x)dx < \infty \right\}$$

with norm and inner-product, see e.g. Rudin [18, Chapter 11],

$$\|f\|_w^2 = \int_I |f(x)|^2 w(x)dx\,, \quad (f,g)_w = \int_I f(x)\overline{g(x)}w(x)dx\,.$$

The Sturm-Liouville differential equation is

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \quad (x \in (a,b)) \tag{1.3}$$

where

(1.4)
- (i) the spectral parameter $\lambda \in \mathbb{C}$;
- (ii) $(a,b)$ is an open interval of $\mathbb{R}$ with $-\infty \leq a < b \leq \infty$;
- (iii) the coefficients $p, q, w : (a,b) \to \mathbb{R}$;
- (iv) $p^{-1}(\equiv 1/p), q, w \in L^1_{loc}(a,b)$;
- (v) $w(x) > 0$ for almost all $x \in (a,b)$.

Remarks. 1. The condition (iv) on $p$ implies that $p(x) \neq 0$ for almost all $x \in (a,b)$,

2. Note that there is no sign restriction on the coefficients $p$; it may take positive and negative values in sets of positive (Lebesgue) measure contained in $(a,b)$, or satisfy $p \geq 0$ or $p \leq 0$ on $(a,b)$; we illustrate these possibilities in the examples given in Section 11.

3. The condition $w > 0$ on $(a,b)$ can be reduced to $w \geq 0$ on $(a,b)$, $w(x) > 0$ $(x \in E \subset (a,b))$, $w(x) = 0$ $(x \in (a,b) \setminus E)$ where the (Lebesgue) measures $\lambda(E) > 0$ and $\lambda((a,b) \setminus E) > 0$; however this relaxation introduces additional technicalities which we omit from this paper although the form and results of Theorem 1.1 and Corollary 1.2, given below, remain valid.

4. Solutions of the linear differential equation (1.3) are defined on $(a,b) \times \mathbb{C}$; the required properties of these solutions are discussed in Section 3 below.

5. The condition (iv) above is minimal, within the framework of the Lebesgue integral, for the existence of solutions of (1.3) in the sense of Carathéodory; it is shown in Everitt and Race [13] that condition (iv) is both necessary and sufficient for the existence of such solutions.

### 1.3 – Boundary value problems and structural conditions

We now introduce the boundary value problems and differential operators associated with the differential equation (1.3) which in turn will yield the Lagrange interpolation results. In this paper we require two structural conditions on these boundary value problems; the first condition concerns the classification of the end-points $a$ and $b$ of the differential equation (1.3); the second condition concerns the form of the boundary conditions to be applied at these end-points. Since these conditions are required to define the self-adjoint differential operators in the space $L^2((a,b);w)$ which represent the boundary value problem, we quote the relevant results from the now classic text by Naimark [17].

These two conditions are:

(1.5) 1. *The end-point $a$ of the differential equation is to be regular, or limit-circle in $L^2((a,b);w)$; independently end-point $b$ is to be regular or limit-circle in $L^2((a,b);w)$.*

Remarks. 1. For *regular* see [17, Section 15.1] and for *limit-circle* see [17, Section 17.5] and [20, Chapter II].

2. The condition (1.5) implies that the minimal operator in $L^2((a,b); w)$ associated with equation (1.3), has deficiency index (2,2), see [17, Section 17.4], and so two linearly independent symmetric boundary conditions, applied at end-points $a$ and $b$, are necessary and sufficient to determine all self-adjoint extensions of the minimal operator.

3. Condition (1.5) implies that all solutions $\{y(x,\lambda) : x \in (a,b), \lambda \in \mathbb{C}\}$ of the differential equation (1.3) belong to $L^2((a,b);w)$, i.e.

$$y(\cdot,\lambda) \in L^2((a,b);w) \qquad (\lambda \in \mathbb{C})\,; \tag{1.6}$$

this is clear when both end-points $a$ and $b$ are regular but property (1.6) is a consequence of the limit-circle condition when one or both of the end-points are singular (i.e. nor regular) but limit-circle.

4. Condition (1.5) excludes both $a$ and $b$ from the singular *limit-point* classification, see [17, Section 17.5]; this classification is allowed by ZAYED [22, Section 3].

5. The limit-point case of these problems will be considered by the methods of this paper in a subsequent communication.

The second structural condition is:

(1.7) 2. *The two linearly independent symmetric boundary conditions are separated, with one condition at the end-point a and one condition at end-point b.*

REMARKS. 1. The condition (1.7) excludes the possibility of *coupled* boundary conditions, i.e. those conditions which relate the solution values at end-point $a$ to those at $b$; these conditions are also excluded in [22].

2. Condition (1.7) implies that the self-adjoint operator in $L^2((a,b);w)$ has a simple spectrum, i.e. a spectrum of multiplicity one, see [17, Section 20.3]; the introduction of coupled boundary conditions allows for self-adjoint operators of spectral multiplicity two, the maximum for differential operators generated by second-order differential equations of order two, such as (1.3).

3. The case of coupled boundary conditions will be considered by the methods of this paper in a subsequent communication.

### 1.4 – Glazman-Krein-Naimark boundary conditions

We now introduce the boundary conditions required to determine the boundary value problem and the associated self-adjoint differential operator in $L^2((a,b);w)$, as influenced by the conditions (1.5) and (1.7). The boundary conditions are written in the Glazman-Krein-Naimark form as given in NAIMARK [17, Section 18.1].

The differential expression (quasi-differential expression in the sense of [17, Chapter V]) $M[\cdot]$ is defined by

$$M[f] := -(pf') + qf \quad \text{on} \quad (a,b) \tag{1.8}$$

on functions which satisfy

$$f : (a,b) \to \mathbb{C} \quad \text{and} \quad f, pf' \in AC_{loc}(a,b)\,. \tag{1.9}$$

With the coefficient conditions given in (1.13), (iii) and (iv) of (1.4), $M[\cdot]$ is a Lagrange symmetric (formally self-adjoint) quasi-differential expression in the sense of NAIMARK [17, Section 15].

The Green's formula for $M[\cdot]$, given $f, g$ satisfying (1.9) and any compact interval $[\alpha, \beta] \subset (a, b)$, is

$$\int_\alpha^\beta \{\bar{g}(x)M[f](x) - f(x)\overline{M[g]}(x)\}dx = [f,g](x)\Big|_\alpha^\beta \tag{1.10}$$

where the skew-symmetric bilinear form $[f, g](\cdot)$ is given by

$$[f,g](x) := (f \cdot p\bar{g}' - p\bar{f}' \cdot \bar{g})(x) \qquad (x \in (a,b)) . \tag{1.11}$$

The maximal domain $\Delta$, a linear manifold of the space $L^2((a,b);w)$, as determined by the coefficients $p, q$ and $w$, is defined by

$$\begin{aligned} \Delta := \{(a,b) \to \mathbb{C} : f, pf' \in AC_{loc}(a,b);\\ f, w^{-1}M[f] \in L^2((a,b);w)\} . \end{aligned} \tag{1.12}$$

In general the form $[f, g](\cdot)$ is defined on the open interval $(a, b)$ for all $f, g$ satisfying (1.9). However if we restrict $f, g$ to belong to $\Delta$ then from the Green's formula (1.10) we can define the form at the end-points $a$ and $b$ including the cases when $a = -\infty$ and $b = \infty$, by

$$[f,g](a) := \lim_{x \to a}[f,g](x), \quad [f,g](b) := \lim_{x\to b}[f,g](x) . \tag{1.13}$$

These limits exist and are finite for all $f, g \in \Delta$ from (1.10) and (1.12).

We can now define and specify our separated, symmetric, independent Naimark type boundary conditions at the end-points $a$ and $b$.

Firstly consider the end-point $b$. Let $\{\kappa_+, \chi_+\}$ be a pair satisfying

$$\begin{aligned} &\text{(i)} \quad \kappa_+, \chi_+ \in \Delta, \qquad \text{(ii)} \quad \kappa_+, \chi_+ : (a,b) \to \mathbb{R}\\ &\text{(iii)} \quad [\kappa_+, \chi_+](b) = 1 . \end{aligned} \tag{1.14}$$

Such pairs exist. From (1.6), following upon (1.5), all solutions of the differential equation (1.3) belong to $\Delta$, and this for all $\lambda \in \mathbb{C}$. If $\mu \in \mathbb{R}$

and we put $\lambda = \mu$ in (1.3) and choose real-valued, linearly independent solutions $y_1$ and $y_2$ then $[y_1, y_2](b) \neq 0$, and if $\kappa_+, \chi_+$ are defined by suitable linear forms on $\{y_1, y_2\}$ the conditions (1.14) can be satisfied. It is not necessary to choose $\{\kappa_+, \chi_+\}$ from real-valued independent solutions solutions of (1.3) and this is one of the significant strengths of the Naimark form of the boundary conditions; we illustrate such possibilities in the examples considered in Section 11 below.

The boundary condition at $b$ then takes the form, respectively for a solution $y$ of (1.3) (recall $y \in \Delta$) or a function $f \in \Delta$,

$$[y, \kappa_+](b) = 0 \quad \text{or} \quad [f, \kappa_+](b) = 0\,. \tag{1.15}$$

We note that such a boundary condition is linear, homogeneous and, in view of (iii) of (1.14), is non-trivial since $\kappa_+$ is not identically zero in neighbourhood of $b$. We note also from (1.11) and (ii) of (1.14) that $[\kappa_+, \kappa_+] = 0$, i.e. $\kappa_+$ satisfies its own boundary condition.

The boundary condition (1.15) is a definite restriction on $y$ or $f$; for example if $f(x) = \chi_+(x)$ $\big(x \in (a,b)\big)$ then $[f, \kappa_+](b) \neq 0$. Thus there are elements of $\Delta$ which do satisfy (1.15), and elements which do not satisfy this condition.

The boundary condition (1.15) is called symmetric since if $f, g \in \Delta$ and both satisfy this condition then, to be proven in Section 4,

$$[f, g](b) = 0\,. \tag{1.16}$$

Secondly consider the end-point $a$. Let the pair $\{\kappa_-, \chi_-\}$ satisfy the conditions

$$\begin{aligned} &\text{(i)} \quad \kappa_-, \chi_- \in \Delta, \qquad \text{(ii)} \quad \kappa_-, \chi_- : (a,b) \to \mathbb{R}\,, \\ &\text{(iii)} \quad [\kappa_-, \chi_-](a) = 1\,. \end{aligned} \tag{1.17}$$

All remarks for the boundary condition at a then hold; in particular the boundary condition takes the form

$$[y, \kappa_-](a) = 0 \quad \text{or} \quad [f, \kappa_-](a) = 0\,, \tag{1.18}$$

and if $f, g \in \Delta$ both satisfy (1.18) then

$$[f, g](a) = 0\,. \tag{1.19}$$

Note that the conditions (1.15) and (1.18) are separated; if $f$ satisfies (1.15) then it may or may not satisfy (1.18).

In the singular limit-case the boundary conditions (1.15) and (1.18) have to remain, in general, in the limit form given by applying the result (1.13).

In the regular case, as in NAIMARK [17, Section 18.2], the limit form (1.13) can be relaxed to write the boundary conditions in the classical, linear, homogeneous form. For a solution $y$ of the equation (1.3) these forms are

$$\begin{aligned} &[y,\kappa_-](a) \equiv (p\kappa_-')(a)\cdot y(a) - \kappa_-(a)\cdot (py')(a) = 0\,,\\ &[y,\kappa_+](b) \equiv (p\kappa_+')(b)\cdot y(b) - \kappa_+(b)\cdot (py')(b) = 0\,. \end{aligned} \tag{1.20}$$

### 1.5 – Self-adjoint differential operators and properties

We can now define the self-adjoint differential operators generated by the symmetric boundary value problem determined by

$$\begin{gathered} M[y] \equiv -(py')' + qy = \lambda w y \quad \text{on} \quad (a,b)\\ [y,\kappa_-](a) = 0, \qquad [y,\kappa_+](b) = 0\,. \end{gathered} \tag{1.21}$$

A solution to this problem is a pair $\{\lambda,\psi\}$ where $\lambda\in\mathbb{C}$ is the eigenvalue and $\psi$ is a solution to (1.21) which also satisfies the non-triviality conditions

$$[\psi,\chi_-](a)\neq 0\,, \qquad [\psi,\chi_+](b)\neq 0\,. \tag{1.22}$$

The solution $\psi$ is called an eigenfunction or, if considered as an element of $L^2\big((a,b);w\big)$, an eigenvector.

To define the operator $T$ select any boundary condition pairs $\{\kappa_+,\chi_+\}$ and $\{\kappa_-,\chi_-\}$ satisfying (1.14) and (1.17) respectively and then

$$T : D(T)\subset\Delta\subset L^2\big((a,b);w\big)\to L^2\big((a,b);w\big)$$

with

(1.23a)

$$\begin{gathered} D(T) := \{f\in\Delta : [f,\kappa_-](a) = 0, \qquad [f,\kappa_+](b) = 0\}\\ Tf := w^{-1}M[f] \qquad (f\in D(T))\,. \end{gathered}$$

The general theory of differential operators as given in NAIMARK [17, Section 18] then gives the following results:

(1.23b)
(i) $T$ is self-adjoint and unbounded in $L^2((a,b);w)$;
(ii) the spectrum $\sigma(T)$ of $T$ is real, and discrete with limit-points at $+\infty$ or $-\infty$ or both;
(iii) the spectrun of $T$ is simple, i.e. each eigenvalue is of multiplicity one;
(iv) the eigenvalues and eigenvectors satisfy the boundary value problem (1.21) and the condition (1.22).

The possibilities for the spectrum $\sigma(T)$ of $T$ are covered by the following three cases: $\sigma(T) = \{\lambda_n\}$ where

$$(1.24)\quad \text{(i)}\quad \lambda_n \in \mathbb{R}\quad (n \in \mathbb{Z});\quad \lambda_n < \lambda_{n+1}\quad (n \in \mathbb{Z});\quad \lim_{n\to\pm\infty} \lambda_n = \pm\infty\,;$$

$$(1.25)\quad \text{(ii)}\quad \lambda_n \in \mathbb{R}\quad (n \in \mathbb{N}_0);\quad \lambda_n < \lambda_{n+1}\quad (n \in \mathbb{N}_0);\quad \lim_{n\to\infty} \lambda_n = \infty\,;$$

$$(1.26)\quad \text{(iii)}\quad \lambda_n \in \mathbb{R}\quad (n \in \mathbb{N}_0^-);\quad \lambda_{n-1} < \lambda_n\quad (n \in \mathbb{N}_0^-);\quad \lim_{n\to-\infty} \lambda_n = -\infty\,.$$

Any one of these cases can occur; this depends on the form of the three coefficients $p, q$ and $w$. The cases (ii) and (iii) can be regarded as equivalent; the one case can be converted to the other on changing the sign of the parameter $\lambda$.

We shall state and prove our results for case (i) where the spectrum $\{\lambda_n : n \in \mathbb{Z}\}$ satisfies (1.24). We leave to the reader the adjustments which are necessary when case (ii) or (iii) holds.

Since the case (1.24) is unusual in the existing literature we consider two examples of such spectral properties in Section 11 below.

If the coefficients $p$ essentially (Lebesgue measure) changes sign on the interval $(a,b)$ then case (i) above holds; however case (i) may hold even if $p$ is of one sign on $(a,b)$. We do not enter into the technicalities of these spectral results but further information is given in the recent paper [3] of BAILEY, EVERITT and ZETTL, see in particular the examples in [3, Section 6].

There are two more properties of the operator $T$ required and which follow from the von Neumann-Stone spectral theorem, as given in [17, Section 20.2 and 3], of self-adjoint operators:

(a) the set of eigenvectors (eigenfunctions) $\{\psi_n : n \in \mathbb{Z}\}$ is orthogonal and complete in $L^2((a,b);w)$; without loss of generality the $\{\psi_n\}$ are normalized to

$$\|\psi_n\|_w^2 = (\psi_n, \psi_n)_w = 1 \qquad (n \in \mathbb{Z})\,; \tag{1.27}$$

(b) the Parseval relationship holds, i.e. if $f \in L^2\big((a,b);w\big)$ and $c_n := (f, \psi_n)_w$ $(n \in \mathbb{Z})$ then

$$\int_a^b \left|f(x)\right|^2 w(x)dx = \sum_{n\in\mathbb{Z}} |c_n|^2\,. \tag{1.28}$$

Let $c \in (a,b)$ be any point of this open interval (a particular choice is often made in the consideration of an example), and let the pair of basis solutions $\{\varphi_1, \varphi_2\}$ of the differential equation (1.3) satisfy the initial conditions, for all $\lambda \in \mathbb{C}$, namely

$$\varphi_1(c,\lambda) = 1, \quad (p\varphi_1')(c,\lambda) = 0, \quad \varphi_2(c,\lambda) = 0, \quad (p\varphi_2')(c,\lambda) = 1\,; \tag{1.29}$$

then $\{\varphi_1, \varphi_2\}$ satisfy the properties (see Section 3 below)

$$\begin{array}{lll} \text{(i)} & \varphi_r(\cdot,\lambda), (p\varphi_r')(\cdot,\lambda) \in AC_{loc}(a,b) & (r = 1,2; \lambda \in \mathbb{C})\,; \\ \text{(ii)} & \varphi_r(x,\cdot), (p\varphi_r')(x,\cdot) \in \mathbf{H}(\mathbb{C}) & (r = 1,2; x \in (a,b))\,; \\ \text{(iii)} & \text{the generalized Wronskian of } \{\varphi_1,\varphi_2\} \text{ satisfies} & \end{array} \tag{1.30}$$

$$W(\varphi_1,\varphi_2)(x,\lambda) := (\varphi_1 \cdot p\varphi_2' - p\varphi_1' \cdot \varphi_2)(x,\lambda) = 1 \quad (x \in (a,b); \lambda \in \mathbb{C})\,.$$

Additionally from the structural condition (1.5) we have

$$\text{(iv)} \quad \varphi_r(\cdot,\lambda) \in \Delta \quad (r = 1,2; \lambda \in \mathbb{C})\,. \tag{1.31}$$

Now define the analytic functions $H_{r,\pm} : \mathbb{C} \to \mathbb{C}$ $(r = 1,2)$ by

$$\begin{aligned} H_{r,+}(\lambda) &:= \big[\varphi_r(\cdot,\lambda), \kappa_+\big](b), \quad (\lambda \in \mathbb{C}; r = 1,2)\,; \\ H_{r,-}(\lambda) &:= \big[\varphi_r(\cdot,\lambda), \kappa_-\big](a), \quad (\lambda \in \mathbb{C}; r = 1,2)\,; \end{aligned} \tag{1.32}$$

then for $r = 1, 2$ the following results hold

$$(1.33)\quad \begin{array}{ll} \text{(i)} & H_{r,+} \in \mathbf{H}(\mathbb{C}), \quad H_{r,-} \in \mathbf{H}(\mathbb{C})\,; \\ \text{(ii)} & H_{r,+}(\lambda) \neq 0, \quad H_{r,-}(\lambda) \neq 0 \quad (\lambda \in \mathbb{C} \setminus \mathbb{R})\,; \\ \text{(iii)} & H_{r,+} \text{ and } H_{r,-} \text{ all have a countable infinity of real zeros;} \\ \text{(iv)} & \text{all the zeros of } H_{r,+} \text{ and } H_{r,-} \text{ are simple;} \\ \text{(v)} & \text{the zeros of } H_{1,+} \text{ (resp. } H_{1,-}\text{) and } H_{2,+} \text{ (resp. } H_{2,-}\text{)} \\ & \text{interlace on } \mathbb{R}. \end{array}$$

For these results see Section 6 below and Everitt [12].

### 1.6 – Statement of main theorem

Formally here is the solution of the problem we described in Section 1.1:
If $\{\lambda : n \in \mathbb{Z}\}$ is a sequence of eigenvalues of a given eigenvalue problem of the type prescribed above, and if $F : \mathbb{C} \to \mathbb{C}$ is determined by

$$F(\lambda) = \int_a^b K(x, \lambda) f(x) w(x) dx\,,$$

where $f \in L^2\big((a, b); w\big)$ and $K(x, \lambda)$ a kernel to be specified, then $F$ has the representation (1.1), i.e.

$$F(\lambda) = \sum_{n \in \mathbb{Z}} F(\lambda_n) \frac{G(\lambda)}{G'(\lambda)(\lambda - \lambda_n)} \qquad (\lambda \in \mathbb{C})\,.$$

The exact conditions for, and properties of the eigenvalue problem, kernel $K(x, \lambda)$ and interpolation function $G$ are specified in

Theorem 1.1. *Let* $(a, b)$ *be an open interval of the real line* $\mathbb{R}$*; let the coefficients* $p, q$ *and* $w$ *satisfy the basic conditions* (1.4)*; let the Sturm-Liouville quasi-differential equation* (1.3) *satisfy the end-point classification* (1.5)*; let the boundary condition functions* $\{\kappa_+, \chi_+\}$ *and* $\{\kappa_-, \chi_-\}$ *satisfy the conditions* (1.14) *and* (1.17) *respectively; let the self-adjoint* *differential operator* $T$ *be determined by the separated, symmetric boundary conditions of* (1.23a)*; let the simple, discrete spectrum* $\{\lambda_n : n \in \mathbb{Z}\}$ *of* $T$ *follow the case* (1.24)*; let the eigenvectors* $\{\psi_n : n \in \mathbb{Z}\}$ *of* $T$ *be normalized as in* (1.27)*; let the basis solutions* $\{\varphi_1, \varphi_2\}$ *of* (1.3) *be determined by the initial conditions* (1.29)*.*

*Define the Kramer-type kernel* $K_- : (a,b) \times \mathbb{C} \to \mathbb{C}$ *by*

$$\begin{aligned} K_-(x,\lambda) := {} & [\varphi_1(\cdot,\lambda), \kappa_-](a)\varphi_2(x,\lambda) \\ & - [\varphi_2(\cdot,\lambda), \kappa_-](a)\varphi_1(x,\lambda) \quad (x \in (a,b), \lambda \in \mathbb{C})\,; \end{aligned} \tag{1.34a}$$

*then* $K_-$ *has the following properties:*

(i) $K_-(\cdot,\lambda)$ *is a solution of* (1.3) *for all* $\lambda \in \mathbb{C}$, *and* $K(\cdot,\lambda) \in \mathbb{R}$ $(\lambda \in \mathbb{R})$*;*

(ii) $K_-(\cdot,\lambda) \in \Delta \subset L^2((a,b);w) \quad (\lambda \in \mathbb{C})\,;$

(iii) $[K_-(\cdot,\lambda), \kappa_-](a) = 0 \quad (\lambda \in \mathbb{C})\,;$

(iv) $[K_-(\cdot,\lambda), \kappa_+](b) = 0$ *if and only if* $\lambda \in \{\lambda_n : n \in \mathbb{Z}\}\,;$

(v) $K_-(x,\cdot) \in \mathbf{H}(\mathbb{C}) \quad (x \in (a,b))\,;$

(vi) $K_-(\cdot,\lambda_n) = k_n\psi_n \quad (n \in \mathbb{Z})$ *where* $k_n \in \mathbb{R} \setminus \{0\}$ $(n \in \mathbb{Z})\,;$

(vii) $K_-$ *as defined by* (1.34a) *is unique up to multiplication by a factor* $e(\cdot)$ *where* $e(\cdot) \in \mathbf{H}(\mathbb{C})$, $e(\lambda) \neq 0 \quad (\lambda \in \mathbb{C})$ *and* $e(\lambda) \in \mathbb{R} \quad (\lambda \in \mathbb{R})$.

*Define the interpolation function* $G : \mathbb{C} \to \mathbb{C}$ *by*

$$G(\lambda) := [K_-(\cdot,\lambda), \kappa_+](b) \qquad (\lambda \in \mathbb{C})\,; \tag{1.34b}$$

*then* $G$ *has the following properties:*

(i) $G \in \mathbf{H}(\mathbb{C}), \quad G(\lambda) \in \mathbb{R} \quad (\lambda \in \mathbb{R})\,;$

(ii) $G(\lambda) = 0$ *if and only if* $\lambda \in \{\lambda_n : n \in \mathbb{Z}\}\,;$

(iii) $G'(\lambda_n) \neq 0 \qquad (n \in \mathbb{Z})\,.$

*Define the analytic family* $\{K_-\}$ *by* $F \in \{K_-\}$ *if for some* $f \in L^2((a,b);w)$

$$F(\lambda) = \int_a^b K_-(x,\lambda) f(x) w(x)\,dx \qquad (\lambda \in \mathbb{C})\,. \tag{1.35}$$

*Then for all* $F \in \{K_-\}$*:* (i) $F \in \mathbf{H}(\mathbb{C})$*;*

$$(1.36) \qquad \text{(ii)} \quad F(\lambda) = \sum_{n\in\mathbb{Z}} F(\lambda_n) \frac{G(\lambda)}{(\lambda - \lambda_n)G'(\lambda_n)} \qquad (\lambda \in \mathbb{C}),$$

*where the infinite series for* $F$ *is absolutely convergent for each* $\lambda \in \mathbb{C}$*, and is uniformly convergent on any compact subset of* $\mathbb{C}$*.*

PROOF. See Section 8, 9, 10 below.

REMARKS. 1. The notation $K_-$ is chosen for technical reasons; there is a kernel $K_+$, with similar properties, but with $a$ and $\kappa_-$ interchanged with $b$ and $\kappa_+$.

2. The interpolation formula (1.36) is the formula (1.1) given at the beginning of this Section.

3. The kernel $K_-$ (also $K_+$) fullfils all the requirements of the original ideas in KRAMER [16].

As indicated earlier there is a Corollary to Theorem 1.1.

COROLLARY 1.2. *Let all the notations and conditions of Theorem 1.1 hold; let* $\{c_n : n \in \mathbb{Z}\}$ *be a sequence of complex numbers which satisfies the condition*

$$(1.37) \qquad \sum_{n\in\mathbb{Z}} \frac{|c_n|^2}{\left\|K_-(\cdot,\lambda_n)\right\|_w^2} < \infty .$$

*Then there exists a unique element* $F \in \{K_-\}$ *such that*

$$(1.38) \qquad F(\lambda_n) = c_n \qquad (n \in \mathbb{Z}) .$$

PROOF. See Section 10 below.

### 1.7 – Contents of the paper

Section 2 reviews previous results and examples in this area of Lagrange interpolation. Sections 3, 4, 5 and 6 concern properties of the differential equation (1.3), and the associated boundary value problems and differential operators which will be needed for the proof of Theorem 1.1. Section 7 returns to the results of [21] and [22] in order to make clear the essential difference between the conditions and theorems in [22], and in this paper. The proofs required for Theorem 1.1 and Corollary 1.2 are given in Sections 8, 9 and 10. Examples are discussed in Section 11; examples considered previously are reworked, but basic new examples are introduced to illustrate the generality of the methods in this paper.

## 2 – Review of previous results

We comment briefly on earlier results and examples. These remarks apply in particular to references [16], [6], [21] and [22] and to the results in the previous Section.

The Kramer result in [16] is concerned with an interval $I \subseteq \mathbb{R}$ and a kernel $K : I \times \mathbb{R} \to \mathbb{C}$ with the properties

(i) $K(\cdot, t) \in L^2(I) \qquad (t \in \mathbb{R})$;

(ii) there exists a sequence $\{t_n : n \in \mathbb{Z}\}$ of real number such that the sequence $\{K(\cdot, t_n) : n \in \mathbb{Z}\}$ is a complete orthogonal set in $L^2(I)$; see [2, Section 10] and [17, Section 10.1].

Then if for some $g \in L^2(I)$ the function $F : \mathbb{R} \to \mathbb{C}$ is defined by

$$F(t) = \int_I K(x, t) g(x) dx \qquad (t \in \mathbb{R}) \tag{2.1}$$

then we obtain the Lagrange interpolation result

$$F(t) = \sum_{n \in \mathbb{Z}} F(t_n) S_n(t) \qquad (t \in \mathbb{R}) \tag{2.2}$$

where, for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$,

$$S_n(t) = \|K(\cdot, t_n)\|^{-2} \int_I K(x,t)\overline{K(x,t_n)}dx$$

and the series (2.2) is absolutely convergent for each $t \in \mathbb{R}$.

Although not given in [16] there is a corollary to this result which states that if $\{c_n : n \in \mathbb{Z}\}$ is a sequence of complex numbers such that the following growth condition holds

$$\sum_{n\in\mathbb{Z}} \|K(\cdot, t_n)\|^{-2} |c_n|^2 < \infty \, ,$$

then there exists a unique $g \in L^2(I)$ such that if $F$ is defined by (2.1) then $F(t_n) = c_n$ $(n \in \mathbb{Z})$.

Thus the Kramer-type interpolation is prescribed by the class of functions determined by (2.1) and the properties (i) and (ii) of the kernel $K$. Surprisingly these conditions make little demand on $K$; condition (i) requires that $K(\cdot, t)$ is Lebesgue measurable for each $t \in \mathbb{R}$, but as a function $K(x, \cdot)$ on $\mathbb{R}$, for each $x \in I$, requires no continuity or analytic properties. If the functions $K(x, \cdot)$ on $\mathbb{R}$ satisfies additional properties then this will in turn reflect on the class of functions defined by (2.1).

If the Kramer conditions are extended to the case when $K : I \times \mathbb{C} \to \mathbb{C}$ with $K(x, \cdot) \in \mathbf{H}(\mathbb{C})$ for each $x \in I$, then it may happen that $F$ defined by (2.1) satisfies $F \in \mathbf{H}(\mathbb{C})$; in this case the interpolation then takes place over the set of integral (entire) functions on $\mathbb{C}$.

The common ground in the references [16], [6], [21] and [22] is to generate Kramer-type kernels $K$ from self-adjoint boundary value problems derived from symmetric (formerly self-adjoint) differential expressions of finite order. If the spectrum of the associated self-adjoint differential operator is both discrete and simple (multiplicity one) then a kernel $\{K(x, \lambda) : x \in I, \lambda \in \mathbb{C}\}$ is generated which, in certain cases, can be proved to satisfy the Kramer conditions and to yield interpolation with integral functions on $\mathbb{C}$. Here the variable $\lambda$ is the spectral parameter for the associated self-adjoint differential operator.

These results are best seen in the later papers of Butzer, Hinsen and Zayed [21] and [22]. Here the analytic form of the kernel $K$ allows

for the introduction into (2.2) of the entire function $G$ which in turns yields the form of the Lagrange interpolation formula given in (1.1).
In these papers the construction of $G$ depends on the exponent of convergence of the sequence $\{\lambda_n : n \in \mathbb{Z}\}$ of eigenvalues of the associated self-adjoint differential operator, primary factors and the theorem of Weierstrass; for these ideas and results see Copson [8, Sections 7,43 and 7.2] or Titchmarsh [19, Chapter VIII]. The Hadamard factorization theorem, [8, Section 7.6] shows that $G$ so constructed need not be regarded as unique. (In passing it should be noted that in [22, Theorem 3.1] the author assumes that the exponent of convergence of the spectrum $\{\lambda_n\}$ is finite and this assumption influences the form of the canonical product in the construction of $G$. However, although we omit the details here, it is possible to construct a potential $q : [0,\infty) \to \mathbb{R}$, $q \in C[0,\infty)$, such that the spectrum $\{\lambda_n\}$ of the boundary value problem (here $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$ are part of the construction)

$$-y''(x) + q(x)y(x) = \lambda y(x) \qquad (x \in [0,\infty)), \quad \alpha y(0) + \beta y'(0) = 0$$

has an exponent of convergence which is $+\infty$. This construction requires the use of the inverse spectral theorem of Gelfand and Levitan [14]; details of this theorem are also given in Naimark [17, Chapter, VIII]).

The main examples considered in the references [16], [6], [21] and [22] lead to Lagrange interpolation results derived from Sturm-Liouville differential equations such as:

$$\begin{array}{ll} \text{Fourier} & -y''(x) = \lambda y(x) \quad (x \in (-\infty,\infty)) \\ \text{Bessel} & -y''(x) + \left(\nu^2 - \dfrac{1}{4}\right)x^{-2}y(x) = \lambda y(x) \quad (x \in (0,\infty)) \\ \text{Legendre} & -\left((1-x^2)y'(x)\right)' + \dfrac{1}{4}y(x) = \lambda y(x) \quad (x \in (-1,1)) \,. \end{array}$$

In addition consideration is given to the equations named after Jacobi, Laguerre and Hermite. For certain boundary conditions these examples yield the classical orthogonal polynomials as the associated complete orthogonal sets for the application of the Kramer interpolation theorem.

In Section 7 below we comment further on the comparison between Theorem 1.1 above and the results of [22]; also on some of the examples mentioned above.

In Section 11 we consider the Fourier and Legendre examples in the terms of Theorem 1.1. Further we consider an example in which the leading coefficients $p$ changes sign on the interval $(a,b)$; also an example in which $p$ is of one sign and yet the spectrum (discrete) extends to both $+\infty$ and $-\infty$.

## 3 – The differential equation

In this Section we record the essential properties of the differential equation

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \qquad (x \in (a,b)) \tag{3.1}$$

under the basic conditions (1.4) on the coefficients $p, q$ and $w$.

The basic existence theorem, see [17, Section 16.1 and 2], for quasi-differential equations shows that for any point $c \in (a,b)$ and for any pair $\xi, \eta \in \mathbb{C}$ there exists a unique map $\varphi : (a,b) \times \mathbb{C} \to \mathbb{C}$ with the properties

$$\begin{aligned} &\text{(i)} && \varphi(c,\lambda) = \xi,\ (p\varphi')(c,\lambda) = \eta \qquad (\lambda \in \mathbb{C})\,; \\ &\text{(ii)} && \varphi(\cdot,\lambda), (p\varphi')(\cdot,\lambda) \in AC_{loc}(a,b) \qquad (\lambda \in \mathbb{C})\,; \\ &\text{(iii)} && \varphi(x,\cdot), (p\varphi')(x,\cdot) \in \mathbf{H}(\mathbb{C}) \qquad (x \in (a,b))\,; \\ &\text{(iv)} && \varphi \quad \text{satisfies (3.1) almost everywhere on} \quad (a,b)\,. \end{aligned} \tag{3.2}$$

If in addition $\xi, \eta \in \mathbb{R}$ then for all $x \in (a,b)$

$$\bar{\varphi}(x,\lambda) = \varphi(x,\bar{\lambda}), \quad (\overline{p\varphi'})(x,\lambda) = (p\varphi')(x,\bar{\lambda}) \quad (\lambda \in \mathbb{C})\,. \tag{3.3}$$

We note here that in the quasi-derivative $p\varphi'$, which is $AC_{loc}(a,b)$, the terms $p$ and $\varphi'$ do not in general satisfy this continuity property; whilst $p$ is Lebesgue measurable, it is not in general continuous or differentiable.

The basic solutions $\{\varphi_1, \varphi_2\}$ defined in Section 1.5 with initial conditions (1.29) satisfy all the properties given in (3.2) and (3.3) above. Note in particular that (3.3) implies that, for $r = 1, 2$,

$$\varphi_r(x,\lambda) \in \mathbb{R} \qquad (x \in (a,b), \lambda \in \mathbb{R})\,. \tag{3.4}$$

From the classification condition (1.5) the differential equation (3.1) is in the regular or limit-circle case at both end-points $a$ and $b$. Thus we have, see [10, Chapter 11] or [7, Chapter 7], recall (1.6),

$$\varphi_r(\cdot,\lambda) \in L^2\big((a,b);w\big) \qquad (r=1,2,\lambda\in\mathbb{C})\,. \tag{3.5}$$

Furthermore with this end-point classification we have the properties: given any compact set $C$ of the complex plane $\mathbb{C}$

(3.6)

(i) the integrals $\displaystyle\int_a^b \left|\varphi_r(x,\lambda)\right|^2 w(x)dx \quad (r=1,2)$ are uniformly convergent on $C$ at both end-points $a$ and $b$.

(ii) there exists a positive number $L \equiv L(C)$, which depends on $C$, such that for $r=1,2$

$$\int_a^b \left|\varphi_r(x,\lambda)\right|^2 w(x)dx \le L(C) \qquad (\lambda\in C)\,.$$

For these last results see [10, Lemma 3.1] and [7, Chapter 9, Theorem 2.1].

## 4 – Boundary conditions

We introduced the Naimark type boundary conditions, for the differential equation (1.3), in Section 1.4 above. This requires the choice of the boundary functions pairs $\{\kappa_-,\chi_-\}$ and $\{\kappa_+,\chi_+\}$ satisfying (1.17) and (1.14) respectively. The boundary conditions then take the form (1.15) and (1.18).

To make a strict comparison with the conditions required for an application of the general result, for second-order differential equations, in [17, Section 18.1, Theorem 4], we should re-define $\kappa_-$ and $\kappa_+$ as follows. Choose $\alpha,\beta,\gamma,\delta\in(a,b)$ so that $a<\alpha<\gamma<\delta<\beta<b$; then on using the fundamental result in [17, Section 17.3, Lemma 2] we can change $\kappa_-$ and $\kappa_+$ to $\tilde\kappa_-$ and $\tilde\kappa_+$ so that

$$\tilde\kappa_-(x) = \begin{cases} \kappa_-(x), & x\in(a,\alpha] \\ 0, & x\in[\gamma,b)\,, \end{cases} \qquad \tilde\kappa_+(x) = \begin{cases} 0, & x\in(a,\delta] \\ \kappa_+(x), & x\in[\beta,b) \end{cases}$$

with $\tilde{\kappa}_-$ defined on $[\alpha,\gamma]$ and $\tilde{\kappa}_+$ defined on $[\delta,\beta]$ so that $\tilde{\kappa}_-$ and $\tilde{\kappa}_+$ continue to satisfy (1.17) and (1.14) respectively.

We note that $\tilde{\kappa}_-$ and $\tilde{\kappa}_+$ are unaltered from $\kappa_-$ and $\kappa_+$ in the neighbourhoods of $a$ and $b$ respectively which in turn leaves the boundary conditions (1.15) and (1.18) unchanged. However we note that $\tilde{\kappa}_-$ is identically zero in the neighbourhood of $b$, and *vice versa* for $\tilde{\kappa}_+$.

For the application, and in the notation, of [17, Section 18.1, Theorem 4] we now define $w_1 = \tilde{\kappa}_-$ and $w_2 = \tilde{\kappa}_+$ on $(a,b)$ to give

(i) $w_r \in \Delta \qquad (r = 1,2)$;

(ii) $\{w_1, w_2\}$ are linearly independent in the sense required by [17, Section 18.1, Theorem 4];

(iii) the required symmetry conditions are satisfied, i.e.

$$[w_r, w_s](b) - [w_r, w_s](a) = 0 \qquad (r, s = 1, 2)\,.$$

The definition of the self-adjoint operator $T$ in (1.23) now falls under [17, Section 18.1, Theorem 4] and the properties of $T$ stated after (1.23) now follow from the results in [17, Chapter V and VI].

With these modifications made we now return to the notation $\kappa_-$ and $\kappa_+$ for the boundary condition functions.

To prove the symmetry properties (1.16) and (1.19) of the boundary conditions (1.15) and (1.18) we introduce an identity which may possibly be due to the German mathematician Julius Plücker (1801-1868); for this reason we use the name Plücker identity. For a proof of the identity see Everitt [10]; see also the remarks in Everitt [12, Section 2].

Let $\{f_r, g_r : r = 1,2,3\}$ be any six functions each of which satisfies the condition (1.9); then the Plücker identity states that the $3 \times 3$-matrix $[[f_r, g_s](x)]$ is singular for all $x \in (a,b)$, i.e.

$$\det\,[[f_r, g_s](x)] = 0 \qquad (x \in (a,b))\,. \tag{4.1}$$

To prove (1.16) we use (4.1), taking the limit as $x \to b$, as follows (we give details here which we omit on later applications):

$$f_1, f_2, f_3 \to f, \kappa_+, \chi_+ \qquad g_1, g_2, g_3 \to g, \kappa_+, \chi_+$$

$$[[f_r, g_s](b)] = \begin{bmatrix} [f,g](b) & [f,\kappa_+](b) & [f,\chi_+](b) \\ [\kappa_+,g](b) & [\kappa_+,\kappa_+](b) & [\kappa_+,\chi_+](b) \\ [\chi_+,g](b) & [\chi_+,\kappa_+](b) & [\chi_+,\chi_+](b) \end{bmatrix} =$$

$$= \begin{bmatrix} [f,g](b) & 0 & [f,\chi_+](b) \\ 0 & 0 & 1 \\ [\chi_+,g](b) & -1 & 0 \end{bmatrix}.$$

From (4.1) the determinant of this matrix is zero and this implies that $[f,g](b) = 0$. There is a similar proof for (1.19).

## 5 – Differential operators

The fundamental self-adjoint differential operator $T$ determined by the symmetric boundary value problem (1.21), is defined in (1.23). The properties of $T$ are subsequently considered in Section 1.5.

In this section we define four additional operators $\{T_{r,-} : r=1,2\}$ and $\{T_{r,+} : r = 1,2\}$, respectively in the spaces $L^2((a,c];w)$ and $L^2([c,b);w)$. We recall that $c \in (a,b)$ is the point at which the basis solutions $\{\varphi_1, \varphi_2\}$ take real, initial values for all $\lambda \in \mathbb{C}$; see (1.29) above. Define these operators as follows: let $\mu \in \mathbb{R}$ and recall that this implies, for $r = 1,2$

$$\varphi_r(\cdot,\mu) \in \Delta \quad \text{and} \quad \varphi_r(x,\mu) \in \mathbb{R} \quad \big(x \in (a,b)\big)\,; \tag{5.1}$$

now put, with $\kappa_\pm$ taken from (1.14) and (1.17),

$$\begin{aligned} D(T_{r,-}) &:= \Big\{ f \in \Delta \big| [f,\varphi_r(\cdot,\mu)](c) = 0, \quad [f,\kappa_-](a) = 0 \Big\} \\ T_{r,-}f &:= w^{-1}M[f] \qquad (f \in D(T_{r,-}))\,; \end{aligned} \tag{5.2}$$

$$\begin{aligned} D(T_{r,+}) &:= \Big\{ f \in \Delta \big| [f,\varphi_r(\cdot,\mu)](c) = 0, \quad [f,\kappa_+](b) = 0 \Big\} \\ T_{r,+}f &:= w^{-1}M[f] \qquad (f \in D(T_{r,+}))\,. \end{aligned} \tag{5.3}$$

These operators $\{T_{r,\pm} : r = 1,2\}$ are again determined by separated symmetric boundary conditions on their respective intervals $(a,c]$ and $[c,b)$. Note that these operators are independent of the choice of parameter $\mu$ in (5.1) since the initial values of the basis pair $\{\varphi_1,\varphi_2\}$ at $c$ are independent of $\lambda$.

The four operators $\{T_{r,\pm} : r = 1, 2\}$ all satisfy the same properties as $T$; see (1.23 to 28), with appropriate changes for the Hilbert spaces involved.

The boundary conditions at $c$ in (5.2) and (5.3) take the form, see the initial values (1.29),

$$\begin{aligned} &[f, \varphi_1(\cdot, \mu)](c) = 0 \quad \text{if and only if} \quad f(c) = 0\,, \\ &[f, \varphi_2(\cdot, \mu)](c) = 0 \quad \text{if and only if} \quad (pf')(c) = 0\,. \end{aligned} \tag{5.4}$$

These are called, respectively, the Dirichlet and Neumann boundary conditions at the point $c$.

## 6 – Analytic properties

The results from the previous section enable us to state certain properties of analytic functions arising from the boundary value problems associated with the Sturm-Liouville differential equation (1.3) under the classification condition (1.5).

LEMMA 6.1. *Let all the conditions of Theorem 1.1 hold, and let the notations of the previous sections stand; let $f \in L^2\big((a, b); w\big)$ and define $F_r : \mathbb{C} \to \mathbb{C}$, for $r = 1, 2$, by*

$$F_r(\lambda) := \int_a^b \varphi_r(x, \lambda) f(x) w(x) dx \qquad (\lambda \in \mathbb{C})\,. \tag{6.1}$$

*Then, for $r = 1, 2$,*

(i) $F_r \in \mathbf{H}(\mathbb{C})$;

(ii) *if* $f : (a, b) \to \mathbb{R}$ *then* $F_r(\lambda) \in \mathbb{R} \quad (\lambda \in \mathbb{R})$.

Proof. The proof of (i) follows by standard function-analytic arguments from the general properties (3.2) for the solutions $\{\varphi_1, \varphi_2\}$ and the special properties (3.6) which hold when the classification condition (1.5) is satisfied. For details see [12, Lemma 3.1] and [7, Chapter 9, Theorem 2.1].

The proof of (ii) follows from (6.1) and (3.3 and 3.4).

Remark. The result of Lemma 6.1 is fundamental to considering the analytic properties of solutions of equation (1.3) when (1.5) is satisfied.

Lemma 6.2. *Let the conditions of Lemma 6.1 hold; let $\kappa \in \Delta$ and define $H_r : \mathbb{C} \to \mathbb{C}$, for $r = 1, 2$, by*

$$H_r(\lambda) := \big[\varphi_r(\cdot, \lambda), \kappa\big](b) \qquad (\lambda \in \mathbb{C}) \,. \tag{6.2}$$

*Then for $r = 1, 2$*

(i) $H_r \in \mathbf{H}(\mathbb{C})$;

(ii) *if* $\kappa : (a, b) \to \mathbb{R}$ *then* $H_r(\lambda) \in \mathbb{R} \quad (\lambda \in \mathbb{R})$.

*There is a similar result in in* (6.2) *the end-point $b$ is replaced by $a$.*

Proof. Apply the Green's formula to the right-hand side of (6.2) and the results follow on application of (i) and (ii) of Lemma 6.1.

Lemma 6.3. *Let the conditions of Lemma 6.1 hold; let $H_{r,+}$ and $H_{r,-}$ for $r = 1, 2$ be defined by* (1.32)*; then $H_{r,+}$ and $H_{r,-}$ satisfy properties* (i) *to* (v) *of* (1.33).

Proof. The proof of (i) and (ii) of (1.33) follows from Lemma 6.2.

The proof of properties (iii) to (v) of (1.33) follows from the properties of the differential operators $\{T_{r,-} : r = 1, 2\}$ and $\{T_{r,+} : r = 1, 2\}$ introduced and defined in Section 5 above. For details see Everitt [12, Section 5 and 6].

Remark. The reason for stating the results of Lemma 6.3 at the early stage of (1.32 and 1.33) is that the properties (1.33) reflect significantly on the stated properties of the kernel $K_-$ and the function $G$ in Theorem 1.1.

## 7 – Return to comparison with previous results

In [22] Zayed gives general results only for the special equation

$$-y''(x) + q(x)y(x) = \lambda y(x) \qquad (x \in I) \tag{7.1}$$

but calls the more general equation (1.3), but with $w \equiv 1$, into use for the examples. In general it is not possible to transform (1.3) into the form (7.1), see Everitt [11].

To construct the Kramer type kernel $K$ and the interpolation function $G$ in [22] Zayed uses the Weierstrass/Hadamard theory of entire functions and, in the singular case, the Titchmarsh-Weyl $m$-coefficient. However no attempt is made to relate the choice of $m$-coefficient to the boundary condition at a singular limit-circle end-point, nor to the associated differential operator.

However the Zayed method is effective in singular limit-point end-points, provided that the spectrum of the boundary value problem is discrete.

In the singular limit-circle case both the methods of [22] and of this present paper, can be applied. In the limit-circle non-oscillatory cases (for example the Legendre equation considered in Example 3, Section 11 below and in [22, Section 5.3, Example 3]) both theories may well yield the same result. In the limit-circle oscillatory case (see Example 4, Section 11 below) the outcome may be different.

Sections 8, 9, 10 are devoted to the *Proof of Theorem 1.1.*

## 8 – Definition and properties of kernel $K_-(\cdot,\lambda)$

In this section we give the definition of $K_-(\cdot,\lambda)$ and establish the properties (i) to (iv) of the kernel of Theorem 1.1.

Let $\kappa_+, \chi_+, \kappa_-, \chi_- \in \Delta$ be real-valued such that $[\kappa_+, \chi_+](b) = 1$ and $[\kappa_-, \chi_-](a) = 1$ as in (1.14) and (1.17). There exists a fundamental set of solutions $\varphi_1, \varphi_2 \in \Delta$ of $M[y] = \lambda wy$ on $(a,b) \times \mathbb{C}$ having the properties (1.29) for some $c \in (a,b)$ and arbitrary $\lambda \in \mathbb{C}$, i.e. $\big[\varphi_1(\cdot,\lambda), \varphi_2(\cdot,\lambda)\big](x) = 1$ for all $x \in (a,b)$, $\lambda \in \mathbb{C}$.

So there holds for all solutions $y$ of (1.3) and for $\lambda \in \mathbb{C}$ (recall then

$y \in \Delta$)

$$[y, \kappa_+](b) = \alpha[\varphi_1(\cdot, \lambda), \kappa_+](b) + \beta[\varphi_2(\cdot, \lambda), \kappa_+](b)\,,$$
$$[y, \kappa_-](a) = \alpha[\varphi_1(\cdot, \lambda), \kappa_-](a) + \beta[\varphi_2(\cdot, \lambda), \kappa_-](a)\,,$$

since $y(x, \lambda) = \alpha\varphi_1(x, \lambda) + \beta\varphi_2(x, \lambda)$ for some $\alpha, \beta$ depending on $\lambda$. Now define the kernel $K_-(x, \lambda)$ by (1.34a)

$$K_-(x, \lambda) := [\varphi_1(\cdot, \lambda), \kappa_-](a)\varphi_2(x, \lambda) - [\varphi_2(\cdot, \lambda), \kappa_-](a)\varphi_1(x, \lambda) \qquad (x \in (a, b), \lambda \in \mathbb{C})\,.$$

Now $[\varphi_i(\cdot, \lambda), \kappa_-](a)$, $i = 1, 2$, are entire functions on $\mathbb{C}$ by (1.32) and (1.33), each with a countable number of real and simple zeros only. Further $\varphi_1, \varphi_2 \in \Delta$ are non-trivial solutions of the differential equation for all $\lambda \in \mathbb{C}$ and entire functions in $\lambda$ on $\mathbb{C}$ as mentioned in (1.30). Thus $K_-(x, \lambda)$ is also a non-trivial solution of the differential equation for all $\lambda \in \mathbb{C}$ and entire with respect to $\lambda$ for each $x \in (a, b)$.

In view of (3.3) $\varphi_r(x, \lambda)$, $r = 1, 2$, are real whenever $\lambda$ is real and the same holds for the functions $[\varphi_r(\cdot, \lambda), \kappa_+](b)$, $[\varphi_r(\cdot, \lambda), \kappa_-](a)$, $r = 1, 2$, which follows from (ii) of Lemma 6.2. Thus $K_-(x, \lambda)$ is real for $\lambda \in \mathbb{R}$ and fixed $x$ and we achieve properties (i) and (v) of the kernel. $K_-(\cdot, \lambda)$ is in $\Delta$ since $\varphi_r(\cdot, \lambda) \in \Delta$ $(r = 1, 2)$; thus (ii) holds. Concerning property (iii),

$$\begin{aligned}[K_-(\cdot, \lambda), \kappa_-](a) = &[\varphi_1(\cdot, \lambda), \kappa_-](a) \cdot [\varphi_2(\cdot, \lambda), \kappa_-](a) \\ &- [\varphi_2(\cdot, \lambda), \kappa_-](a) \cdot [\varphi_1(\cdot, \lambda), \kappa_-](a) = 0\,,\end{aligned} \tag{8.1}$$

$$\begin{aligned}[K_-(\cdot, \lambda), \kappa_+](b) = &[\varphi_1(\cdot, \lambda), \kappa_-](a) \cdot [\varphi_2(\cdot, \lambda), \kappa_+](b) \\ &- [\varphi_2(\cdot, \lambda), \kappa_-](a) \cdot [\varphi_1(\cdot, \lambda), \kappa_+](b) \quad (\lambda \in \mathbb{C})\,.\end{aligned} \tag{8.2}$$

Since $K_-(\cdot, \lambda) \in \Delta$ fulfills the first boundary condition, and is a non-trivial solution of the differential equation, the second boundary condition $[K_-(\cdot, \lambda), \kappa_+](b) = 0$ is fulfilled if and only if $\lambda$ is an eigenvalue. Thus the eigenvalues are determined as the zeros of (8.2), which are all real and simple. The spectrum is either $\{\lambda_n : n \in \mathbb{N}_0\}$ with $\lim\limits_{n\to\infty} \lambda_n = \pm\infty$ or $\{\lambda_n : n \in \mathbb{Z}\}$ with $\lim\limits_{n\to\pm\infty} \lambda_n = \pm\infty$. We assume the second case. The corresponding real eigenfunctions $\{\psi_n\}$ form a complete normal orthogonal set in $L^2\big((a,b);w\big)$, and in view of (8.2) it follows that $K_-(\cdot,\lambda) = k_n \cdot \psi_n$, $k_n \in \mathbb{R}$, $k_n \neq 0$, $n \in \mathbb{Z}$.

Since $K_-(\cdot,\lambda)$ is real whenever $\lambda$ is real, the coefficients $k_n$ are also real.

Thus properties (i) to (vi) of the kernel $K_-(x,\lambda)$ of Theorem 1.1 hold, and $K_-(x,\lambda)$ is a suitable kernel for Kramer's theorem.

The property (vii) is proved by representing $K_-(x,\lambda)$ in the form $K_-(x,\lambda) = a(\lambda)\varphi_1(x,\lambda) + b(\lambda)\varphi_2(x,\lambda)$ $(x \in (a,b); \lambda \in \mathbb{C})$, where $a(\cdot)$ and $b(\cdot) \in \mathbf{H}(\mathbb{C})$, and using properties (ii) to (vi).

## 9 – Definition and properties of interpolation function $G$

Let us know define the function, for all $\lambda \in \mathbb{C}$,

$$G(\lambda) := \big[K_-(\cdot,\lambda), \kappa_+\big](b)\,. \tag{9.1}$$

Since $\kappa_+$ is real-valued and $K_-(\cdot,\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$ the same holds for $G(\lambda)$. Thus $G(\lambda)$ is holomorphic on $\mathbb{C}$ in view of formula (8.2), since $\big[\varphi_r(\cdot,\lambda),\kappa_+\big](b)$, $\big[\varphi_r(\cdot,\lambda),\kappa_-\big](a)$ are in $\mathbf{H}(\mathbb{C})$ for $r = 1,2$. Using the definition of $G(\lambda)$ and property (iv) of $K_-(\cdot,\lambda)$ it follows that $G(\lambda) = 0$ if and only if $\lambda \in \{\lambda_n : n \in \mathbb{Z}\}$, and thus we obtain properties (i) and (ii) of the function $G$. Now to property (iii):

Let us apply Green's formula (1.10) to the functions $f := K_-(x,\lambda)$ and $g := K_-(x,\lambda_n)$ for $x \in (a,b)$, $\lambda \in \mathbb{C}$. This yields, as $K_-(x,\lambda_n)$ is real-valued on $(a,b)$,

$$\begin{aligned}\int_a^b \{\bar{g}M[f] - f\overline{M[g]}\}dx &= (\lambda - \lambda_n)\int_a^b K_-(x,\lambda)K_-(x,\lambda_n)w(x)dx = \\ &= \big[K_-(\cdot,\lambda), K_-(\cdot,\lambda_n)\big]_a^b\,.\end{aligned}$$

Using the Plücker identity and setting $f_1(x) = K_-(x,\lambda)$, $f_2(x) = \kappa_+(x)$, $f_3(x) = \chi_+(x)$; $g_1(x) = K_-(x,\lambda_n)$, $g_2(x) = \kappa_+(x)$, $g_3(x) = \chi_+(x)$, and $x = b$ in (4.1), then for all $\lambda \in \mathbb{C}$

$$\begin{vmatrix} [K_-(\cdot,\lambda), K_-(\cdot,\lambda_n)](b) & [K_-(\cdot,\lambda), \kappa_+](b) & [K_-(\cdot,\lambda), \chi_+](b) \\ 0 & 0 & 1 \\ [\chi_+, K_-(\cdot,\lambda_n)](b) & -1 & 0 \end{vmatrix} = 0\,,$$

which yields

$$[K_-(\cdot,\lambda), K_-(\cdot,\lambda_n)](b) + [\chi_+, K_-(\cdot,\lambda_n)](b) \cdot [K_-(\cdot,\lambda), \kappa_+](b) = 0\,.$$

Since $[\chi_+, K_-(\cdot,\lambda_n)](b) = -[K_-(\cdot,\lambda_n), \chi_+](b) = -k_n[\psi_n, \chi_+](b) \neq 0$ (see (vi) of the properties of $K_-(x,\lambda)$ in Theorem 1.1 and (1.22)), we obtain, with $r_n \neq 0$ ($n \in \mathbb{Z}$):

$$[K_-(\cdot,\lambda), K_-(\cdot,\lambda_n)](b) = r_n [K_-(\cdot,\lambda), \kappa_+](b) \qquad (\lambda \in \mathbb{C})\,.$$

Now define the analytic functions $\{G_n; n \in \mathbb{Z}\}$ by

$$\begin{aligned} G_n(\lambda) :&= (\lambda - \lambda_n) \int_a^b K_-(x,\lambda) K_-(x,\lambda_n) w(x)dx = \\ &= [K_-(\cdot,\lambda), K_-(\cdot,\lambda_n)](b) \end{aligned} \tag{9.2}$$

for all $\lambda \in \mathbb{C}$. Thus $G_n(\lambda) = r_n\ G(\lambda)$, and so $G_n(\lambda)$ has the same zeros as $G(\lambda)$, i.e. the eigenvalues $\lambda_n$, and differs from this function only in a real number $r_n \neq 0$, which is independent of $\lambda$.

Thus it is sufficient to examine $G_n'(\lambda_n)$. Now from (9.2), $G_n \in \mathbf{H}(\mathbb{C})$ ($n \in \mathbb{Z}$), and

$$\begin{aligned} G_n'(\lambda) &= \int_a^b K_-(x,\lambda) K_-(x,\lambda_n) w(x)dx + \\ &+ (\lambda - \lambda_n) \frac{d}{d\lambda} \int_a^b K_-(x,\lambda) K_- x, \lambda_n) w(x)dx\,. \end{aligned} \tag{9.3}$$

Since both integrals on the right side of (9.3) are locally uniformly convergent on $\mathbb{C}$ (and also entire functions on $\mathbb{C}$) we let $\lambda \to \lambda$ in (9.3) to deduce $G_n'(\lambda_n) = \int_a^b |K_-(x,\lambda_n)|^2 w(x)dx > 0$. This yields property (iii) of the $G$ of Theorem 1.1.

## 10 – Proof of interpolation results

Finally to the proof of (1.36) and Corollary 1.2. Let

$$F(\lambda) := \int_a^b K_-(x,\lambda) f(x) w(x) dx$$

for some $f \in L^2((a,b);w)$ and all $\lambda \in \mathbb{C}$; thus $F \in \mathbf{H}(\mathbb{C})$ since by Lemma 6.1 $\int_a^b \varphi_r(x,\lambda) f(x) w(x) dx$ are analytic functions for $f \in L^2((a,b);w)$, $r = 1,2$; also noting the definition of $K_-(x,\lambda)$. Then by Kramer's theorem, $F(\lambda) = \sum\limits_n F(\lambda_n) S_n(\lambda)$, where $\lambda_n$, $n \in \mathbb{Z}$, are the eigenvalues, and

$$S_n(\lambda) := \frac{\int_a^b K_-(x,\lambda) K_-(x,\lambda_n) w(x) dx}{\int_a^b \left|K_-(x,\lambda_n)\right|^2 w(x) dx} = \frac{G_n(\lambda)}{(\lambda - \lambda_n) G_n'(\lambda_n)}$$

for all $\lambda \in \mathbb{C}$ by definition (9.2) of $G_n(\lambda)$. Thus for all $\lambda \in \mathbb{C}$ we have

$$F(\lambda) = \sum_{n \in \mathbb{Z}} F(\lambda_n) \frac{G_n(\lambda)}{(\lambda - \lambda_n) G_n'(\lambda_n)} \,.$$

The series is absolutely convergent for each $\lambda \in \mathbb{C}$ by Schwarz's inequality. For

$$\left\{ \sum_{n \in \mathbb{Z}} \left| \frac{F(\lambda_n)}{\left\|K_-(\cdot,\lambda_n)\right\|_w} \cdot \frac{\left\|K_-(\cdot,\lambda_n)\right\|_w G_n(\lambda)}{(\lambda - \lambda_n) G'(\lambda_n)} \right| \right\}^2 \le$$

$$\le \sum_{n \in \mathbb{Z}} \frac{\left|F(\lambda_n)\right|^2}{\left\|K_-(\cdot,\lambda_n)\right\|_w^2} \cdot \sum_{n \in \mathbb{Z}} \frac{\left\|K_-(\cdot,\lambda_n)\right\|_w^2 \left|G_n(\lambda)\right|^2}{|\lambda - \lambda_n|^2 \left|G_n'(\lambda_n)\right|^2} \,.$$

Also the two series on the right side are convergent by Bessel's inequality, since the Fourier coefficients of $f \in L^2((a,b);w)$ with respect to the system $\{K_-(\cdot,\lambda)\}$ are given by $F(\lambda_n) \cdot \left[\int_a^b \left|K_-(x,\lambda_n)\right|^2 w(x) dx\right]^{1/2}$, and the coefficients of $K_-(\cdot,\lambda)$ with respect to the same system are

$$G_n(\lambda) \cdot \left[ \int_a^b \left|K_-(x,\lambda_n)\right|^2 w(x) dx \right]^{-1/2} \cdot \left[(\lambda - \lambda_n) G_n'(\lambda_n)\right]^{-1} .$$

The series is also locally uniformly convergent since (consider $n \in \mathbb{N}_0$; case $n \in \mathbb{Z}$ similarly)

$$\begin{aligned}&\left|F(\lambda) - \sum_{n=0}^{N-1} \frac{F(\lambda_n)G_n(\lambda)}{(\lambda-\lambda_n)G_n'(\lambda_n)}\right| \leq \\ &\leq \left\{\sum_{n=N}^{\infty} \frac{\left|F(\lambda_n)\right|^2}{\left\|K_-(\cdot,\lambda_n)\right\|_w^2}\right\}^{1/2} \left\{\sum_{n=N}^{\infty} \frac{\left\|K_-(\cdot,\lambda_n)\right\|_w^2 \left|G_n(\lambda)\right|^2}{|\lambda-\lambda_n|^2\left|G_n'(\lambda_n)\right|^2}\right\}^{1/2} .\end{aligned} \tag{10.1}$$

The first term on the right side is independent of $\lambda$ and tends to zero if $N$ tends to infinity in view of Bessel's inequality. For the second term there holds - using again Bessel's inequality,

$$\left\{\sum_{n=N}^{\infty} \frac{\left\|K_-(\cdot,\lambda_n)\right\|_w^2 \left|G_n(\lambda)\right|^2}{|\lambda-\lambda_n|^2\left|G_n'(\lambda_n)\right|^2}\right\}^{1/2} \leq \left\{\int_a^b \left|K_-(x,\lambda)\right|^2 w(x)dx\right\}^{1/2} .$$

In view of (3.6) there exists an $L > 0$ such that $\int_a^b \left|K_-(x,\lambda)\right|^2 w(x)dx \leq L$ on any compact subset of $\mathbb{C}$. Hence from (10.1) there follows that

$$\left|F(\lambda) - \sum_{n=0}^{N-1} F(\lambda_n) \frac{G_n(\lambda)}{(\lambda-\lambda_n)G_n'(\lambda_n)}\right| \to 0$$

for $N \to \infty$ uniformly in $\lambda$ on any compact subset of $\mathbb{C}$. Since $G_n(\lambda) = r_n \cdot G(\lambda)$ for all $n \in \mathbb{Z}$, where the $r_n \neq 0$ are independent of $\lambda$, we obtain

$$F(\lambda) = \sum_{n\in\mathbb{Z}} F(\lambda_n) \frac{G(\lambda)}{(\lambda-\lambda_n)G'(\lambda_n)} ,$$

where $G(\lambda)$, having exactly the eigenvalues as zeros, is an analytic function, real-valued when $\lambda$ is real. This completes the proof of the theorem.

Now to the *Proof of Corollary 1.2*:
Let $\{c_n : n \in \mathbb{Z}\}$ be such that (1.37) holds. Define $d_n := \dfrac{c_n}{\left\|K_-(\cdot,\lambda_n)\right\|_w}$, i.e. $\sum\limits_{n=-\infty}^{\infty} |d_n|^2 < \infty$. Thus, from the completeness property (1.28), there exists a unique function $f \in L^2((a,b);w)$ such that

$$d_n = \left\|K_-(\cdot,\lambda_n)\right\|_w^{-1} \int_a^b K_-(x,\lambda_n)f(x)dx .$$

We obtain $c_n = \int_a^b K_-(x,\lambda_n)f(x)dx$, and define for this special $f \in L^2((a,b);w)$ the function $F \in \{K_-\}$ by $F(\lambda) = \int_a^b K_-(x,\lambda)f(x)dx$.

This proves Corollary 1.2.

## 11 – Examples

Let us now give some examples for Theorem 1.1. For all of them it is necessary to define the function $s(z) = \sqrt{z} : \mathbb{C} \to \mathbb{C}$ for $0 \le \arg(z) < 2\pi$ as follows; if $z = r \cdot \mathrm{e}^{i\theta}$, $r \ge 0$, $0 \le \theta < 2\pi$ then $\sqrt{z} = r^{1/2}\mathrm{e}^{i\theta/2}$ where $0 \le \arg(\sqrt{z}) < \pi$, and the definition of the square-root function is thus unique.

*Example 1.* Given the eigenvalue problem $-y'' = \lambda y$ on $[0,\pi]$, $y'(0) = y'(\pi) = 0$. Both endpoints are regular. The general solution on $[0,\pi]$ is given by $y(x) = A\sin x\sqrt{\lambda} + B\cos x\sqrt{\lambda}$, $A, B \in \mathbb{C}$; a basis at $c = \pi/2$ is $\varphi_1(x,\lambda) = \cos\sqrt{\lambda}\Big(\dfrac{\pi}{2} - x\Big)$, $\varphi_2(x,\lambda) = (1/\sqrt{\lambda})\sin\sqrt{\lambda}\Big(\dfrac{\pi}{2} - x\Big)$. Choose as boundary conditions $\kappa_+(x) = \kappa_-(x) = -1$ (cf. (1.13) to (1.17)) to give $[y,\kappa_+](\pi) = y'(\pi)$ and $[y,\kappa_-](0) = y'(0)$. Then

$$\begin{aligned}
&[\varphi_1(\cdot,\lambda),\kappa](0) = \sqrt{\lambda}\sin\frac{\pi}{2}\sqrt{\lambda}, \quad [\varphi_1(\cdot,\lambda),\kappa_+](\pi) = -\sqrt{\lambda}\sin\frac{\pi}{2}\sqrt{\lambda}\,,\\
&[\varphi_2(\cdot,\lambda),\kappa_-](0) = -\cos\frac{\pi}{2}\sqrt{\lambda} = [\varphi_2(\cdot,\lambda),\kappa_+](\pi)\,,
\end{aligned}$$

and the eigenvalues are determined by $\sqrt{\lambda}\cdot\sin\pi\sqrt{\lambda} = 0$. Thus $\lambda$ is an eigenvalue iff $\lambda = k^2$, $k \in \mathbb{N}_0$. The kernel is given by $K_-(x,\lambda) = \cos x\sqrt{\lambda}$ and the interpolation function is $G(\lambda) = \sqrt{\lambda}\cdot\sin\sqrt{\lambda}$. Altogether we obtain

Lemma 11.1. *If $F \in \{K_-\}$, i.e.*

$$F(\lambda) = \int_0^\pi f(x)\cos x\sqrt{\lambda}dx \qquad (f \in L^2(0,\pi), \lambda \in \mathbb{C})\,,$$

*then $F$ can be represented as*

$$F(\lambda) = \frac{\sin\pi\sqrt{\lambda}}{\pi\sqrt{\lambda}}F(0) + 2\sum_{k=1}^{\infty}F(k^2)\frac{\sqrt{\lambda}\sin(\sqrt{\lambda}-k)\pi}{(\lambda-k^2)\pi} \quad (\lambda \in \mathbb{C})\,.$$

This sampling theorem is the same one that Zayed achieved in [22]. Observe that $G(\lambda)$ differs form the canonical product of the sampling points, which Zayed constructed to obtain in the result above, only by a factor $1/\pi$:

$$\lambda \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{n^2}\right)=\frac{1}{\pi}\sqrt{\lambda}\sin\pi\sqrt{\lambda}\,.$$

Remark. When dealing with the Dirichlet boundary conditions $y(0)=y(\pi)=0$ instead of the Neumann boundary conditions $y'(0)=y'(\pi)=0$ we choose $\kappa_+(x)=x-\pi$ and $\kappa_-(x)=x$ to obtain $[y,\kappa_+](\pi)=y(\pi)$, $[y,\kappa_-](0)=y(0)$.

*Example 2.* Given the eigenvalue problem $-(py')'=\lambda y$ on $[-1,1]$, $y(-1)=y(1)=0$, where

$$p(x)=\begin{cases}-1, & x\in[-1,0)\\ 0, & x=0\\ 1, & x\in(0,1]\,.\end{cases}$$

Both endpoints are regular. The function $p$ changes sing at $x=0$. The general solution on $[-1,1]$ is given by

$$y(x)=\begin{cases}A_1\cosh x\sqrt{\lambda}+A_2\sinh x\sqrt{\lambda}, & x\in[-1,0)\\ B_1\cos x\sqrt{\lambda}+B_2\sin x\sqrt{\lambda}, & x\in(0,1]\,,\end{cases}$$

with $A_i,B_i\in\mathbb{C}$, $i=1,2$. A basis at $c=0$ is

$$\varphi_1(x,\lambda)=\begin{cases}\cosh x\sqrt{\lambda}, & x\in[-1,0]\\ \cos x\sqrt{\lambda}, & x\in[0,1]\,;\end{cases}$$
$$\varphi_2(x,\lambda)=\begin{cases}(-1/\sqrt{\lambda})\sinh x\sqrt{\lambda}, & x\in[-1,0]\\ (1/\sqrt{\lambda})\sin x\sqrt{\lambda}, & x\in[0,1]\,.\end{cases}$$

Choose as boundary condition functions $\kappa_+(x)=x-1$ and $\kappa_-(x)=x+1$ to give $[y,\kappa_+](1)=y(1)$ and $[y,\kappa_-](-1)=y(-1)$. Hence

$$[\varphi_1(\cdot,\lambda),\kappa_-](-1)=-\cosh\sqrt{\lambda},\quad [\varphi_2(\cdot,\lambda),\kappa_-](-1)=(-1/\sqrt{\lambda})\sinh\sqrt{\lambda}\,,$$
$$[\varphi_1(\cdot,\lambda),\kappa_+](1)=\cos\sqrt{\lambda},\qquad [\varphi_2(\cdot,\lambda),\kappa_+](1)=(1/\sqrt{\lambda})\sin\sqrt{\lambda}\,,$$

and the eigenvalues are determined by the zeros of

$$\varrho(\lambda) := (1/\sqrt{\lambda})[\sinh\sqrt{\lambda}\cdot\cos\sqrt{\lambda} - \cosh\sqrt{\lambda}\cdot\sin\sqrt{\lambda}]\,.$$

These zeros are real and simple, and 0 is one of them. Since, on inspection, $\varrho(-\lambda) = -\varrho(\lambda)$, the eigenvalues are symmetric about 0 so that $\lambda_{-n} = -\lambda_n$, $n \in \mathbb{N}$, and $\lambda_0 = 0$.

Since $\cosh t/\sinh t \to 1$ as $t \to \infty$ the eigenvalues for large $n$ are near to the roots of $\cos t = \sin t$, $t \in \mathbb{R}$, i.e. $\lambda_n \sim \left(n + \frac{1}{4}\right)^2 \pi^2$ as $n \to \infty$.

For $\lambda_0 = 0$ it can be shown that the corresponding eigenfunction is given by $\psi_0(x) = 1 - |x|$, $x \in [-1, 1]$. Note that $\psi_0'(0)$ does not exist, but $(p\psi_0')(x) = -1$, $x \in [-1, 1]$.

The kernel is given by

$$K_-(x,\lambda) = (1/\sqrt{\lambda})\sinh\sqrt{\lambda}\cdot\varphi_1(x,\lambda) - \cosh\sqrt{\lambda}\cdot\varphi_2(x,\lambda)\,,$$

and the interpolation function is given by $G(\lambda) = \varrho(\lambda)$. We obtain

Lemma 11.2. *If $F \in \{K_-\}$, i.e.*

$$F(\lambda) = \int_{-1}^{1} f(x)K_-(x,\lambda)dx \quad (f \in L^2(0,\pi), \lambda \in \mathbb{C})\,,$$

*where*

$$K_-(x,\lambda) := \begin{cases} (1/\sqrt{\lambda})(\sinh\sqrt{\lambda}\cdot\cosh x\sqrt{\lambda} + & \\ +\cosh\sqrt{\lambda}\cdot\sinh x\sqrt{\lambda}), & x \in [-1,0] \\ (1/\sqrt{\lambda})(\sinh\sqrt{\lambda}\cdot\cos x\sqrt{\lambda} + & \\ -\cosh\sqrt{\lambda}\cdot\sin x\sqrt{\lambda}), & x \in [0,1]\,, \end{cases}$$

*then $F$ can be represented as*

$$F(\lambda) = \frac{-3\varrho(\lambda)}{2\lambda}F(0) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} F(\lambda_k)\cdot\frac{-\varrho(\lambda)\cdot\lambda_k}{\sinh\sqrt{\lambda_k}\cdot\sin\sqrt{\lambda_k}(\lambda-\lambda_k)} \quad (\lambda \in \mathbb{C})\,,$$

*where $\lambda_{-k} = -\lambda_k$, $k \in \mathbb{Z}$ and $\lambda_k$, $k \in \mathbb{N}$, is the kth positive root of $\varrho(\lambda) = 0$ with*

$$\varrho(\lambda) = (1/\sqrt{\lambda})[\sinh\sqrt{\lambda}\cdot\cos\sqrt{\lambda} - \cosh\sqrt{\lambda}\cdot\sin\sqrt{\lambda}]\,.$$

Observe that this example cannot be handled by Zayed's methods since $p$ changes sign and thus cannot be transformed into the Liouville form. This is an example of a regular eigenvalue problem where the eigenvalues expand over the whole real line.

*Example 3.* a) Given the eigenvalue problem $-\big((1-x^2)y'\big)' = \lambda y$ on $(-1,1)$, $[y,1](1) = [y,1](-1) = 0$. Both endpoints are limit circle and non oscillatory. Boundary condition functions are chosen to be $\kappa_+(x) = \kappa_-(x) = 1$ to give the conditions above. A fundamental system of the differential equation on $(-1,1)$ is given by

$$\left\{ {}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1-x}{2}\Big);\ {}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1+x}{2}\Big)\right\}$$

for all $\lambda \in \mathbb{C}$ except $\lambda = \Big(n+\dfrac{1}{2}\Big)^2$, $n \in \mathbb{N}_0$, which are the eigenvalues of the problem. Working with this system of functions we obtain

$$\begin{aligned}
\varphi_1(x,\lambda) &= \frac{1}{2\varrho(\lambda)}\left\{ {}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1-x}{2}\Big)+\right.\\
&\quad \left. +\,{}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1+x}{2}\Big)\right\},\\
\varphi_2(x,\lambda) &= \frac{1}{2\sigma(\lambda)}\left\{ {}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1-x}{2}\Big)+\right.\\
&\quad \left. +\,{}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1+x}{2}\Big)\right\},
\end{aligned}$$

where

$$\begin{aligned}
\varrho(\lambda) &= {}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1}{2}\Big) =\\
&= \sqrt{\pi}\Big/\left(\Gamma\Big(\frac{3}{4}-\frac{\sqrt{\lambda}}{2}\Big)\Gamma\Big(\frac{3}{4}+\frac{\sqrt{\lambda}}{2}\Big)\right),\\
\sigma(\lambda) &= \frac{1}{2}\Big(\lambda-\frac{1}{4}\Big)\,{}_2F_1\Big(\frac{3}{2}-\sqrt{\lambda},\frac{3}{2}+\sqrt{\lambda};2;\frac{1}{2}\Big) =\\
&= 2\sqrt{\pi}\Big/\left(\Gamma\Big(\frac{1}{4}-\frac{\sqrt{\lambda}}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{\sqrt{\lambda}}{2}\Big)\right);
\end{aligned}$$

see e.g. [1, p. 556]. Although $\varrho(\lambda)$, $\sigma(\lambda)$ have zeros the functions $\varphi_1, \varphi_2$ can be defines at these points by taking limits. In case $\lambda = \Big(2j + 1 + \frac{1}{2}\Big)^2$, $j \in \mathbb{N}_0$, $\varphi_2(x, \lambda)$ tends to $K_1(j) \cdot P_{2j+1}(x)$, $j \in \mathbb{N}_0$, where $K_1(j)$ is a constant depending on $j$, and $P_{2j+1}$ are the Legendre polynomials. $\varphi_1(x, \lambda)$ tends to $K_2(j) \cdot Q_{2j+1}(x)$, where $K_2(j)$ is a constant depending on $j$, and $Q_{2j+1}(x)$ are the associated Legendre functions.

In case $\lambda = \Big(2j+\frac{1}{2}\Big)^2$, $j \in \mathbb{N}_0$, $\varphi_1(x, \lambda)$ tends to $K_3(j) \cdot P_{2j}(x)$, $j \in \mathbb{N}_0$, and $\varphi_2(x, \lambda)$ tends to $K_4(j) \cdot Q_{2j}(x)$. Thus the system $\{\varphi_1(x, \lambda), \varphi_2(x, \lambda)\}$ is a linearly independent set for each $\lambda \in \mathbb{C}$. Now there holds:

$$
\begin{aligned}
[\varphi_1(\cdot, \lambda), \kappa_-](-1) &= \Big(\frac{1}{4} - \lambda\Big) \frac{\Gamma\Big(\frac{3}{4} - \frac{\sqrt{\lambda}}{2}\Big)\Gamma\Big(\frac{3}{4} + \frac{\sqrt{\lambda}}{2}\Big)}{\sqrt{\pi}\Gamma\Big(\frac{3}{2} + \sqrt{\lambda}\Big)\Gamma\Big(\frac{3}{2} - \sqrt{\lambda}\Big)} = \\
&= -[\varphi_1(\cdot, \lambda), \kappa_+](1)\,, \\
[\varphi_2(\cdot, \lambda), \kappa_-](-1) &= -2 \frac{\Gamma\Big(\frac{5}{4} - \frac{\sqrt{\lambda}}{2}\Big)\Gamma\Big(\frac{5}{4} + \frac{\sqrt{\lambda}}{2}\Big)}{\sqrt{\pi}\Gamma\Big(\frac{3}{2} + \sqrt{\lambda}\Big)\Gamma\Big(\frac{3}{2} - \sqrt{\lambda}\Big)} = \\
&= [\varphi_2(\cdot, \lambda), \kappa_+](1)\,.
\end{aligned}
$$

For the calculations which are necessary to achieve these results one needs formulae to be found in [1, Chapters 6 and 15], [9, Chapter 2.8].

The eigenvalues are determined by

$$
G(\lambda) = -2/\Big(\Gamma\Big(\frac{1}{2} + \sqrt{\lambda}\Big)\Gamma\Big(\frac{1}{2} - \sqrt{\lambda}\Big)\Big) = \frac{2}{\pi} \sin \pi \Big(\sqrt{\lambda} - \frac{1}{2}\Big) = 0\,;
$$

these values are $\lambda_n = \Big(n + \frac{1}{2}\Big)^2$, $n \in \mathbb{N}_0$.

The kernel is given as

$$
K_-(x, \lambda) = {}_2F_1\Big( - \sqrt{\lambda} + \frac{1}{2}, \sqrt{\lambda} + \frac{1}{2}; 1; \frac{1-x}{2}\Big) = P_{\sqrt{\lambda}-1/2}(x)
$$

which is the so-called Legendre function. We deduce

Lemma 11.3. *If $F \in \{K_-\}$, i.e.*

$$F(\lambda) = \int_{-1}^{1} f(x) P_{\sqrt{\lambda}-1/2}(x) dx \qquad (f \in L^2(-1,1), \lambda \in \mathbb{C}),$$

*then $F$ can be represented as, for all $\lambda \in \mathbb{C}$,*

$$F(\lambda) = \sum_{k=0}^{\infty} F\left(\left(k+\frac{1}{2}\right)^2\right) \frac{(2k+1) \cdot \sin \pi\left(\sqrt{\lambda} - \frac{1}{2} - k\right)}{\pi\left(\lambda - \left(k+\frac{1}{2}\right)^2\right)} \tag{11.1}$$

Remark. This formula coincides with the one Zayed - Hinsen - Butzer [21] achieved earlier; see also Campbell [6] (where however $F\left(k+\frac{1}{2}\right)$ should read as in (11.1).). The boundary conditions they used are equivalent to the conditions above.

*Example 3.* b) Given the eigenvalue problem $-((1-x^2)^2 y')' = \lambda y$ on $(-1,1)$ with boundary conditions $[y,\kappa_+](1) = [y,\kappa_-](-1) = 0$ where $\kappa_+(x) = \ln(1/(1-x))$, $\kappa_-(x) = \ln(1/(1+x))$.

Working with the fundamental system of solutions $\{\varphi_1, \varphi_2\}$ of example 3a) and using the following asymptotic formulae, both valid for $x \to 1-$,

$$\begin{aligned} {}_2F_1(a,b;a+b;x) = {} & \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left\{\log\frac{1}{1-x} + k_0\right\} + \\ & + 0\left((1-x)\log\frac{1}{1-x}\right); \end{aligned} \tag{11.2}$$

$$\begin{aligned} {}_2F_1'(a,b;a+b;x) = {} & \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{1}{1-x} + \\ & + 0\left(\log\frac{1}{1-x}\right) \quad \text{for} \quad x \to 1- \end{aligned} \tag{11.3}$$

with $k_0 = 2\psi(1) - \psi(a) - \psi(b)$, where $\psi(x)$ is the logarithmic derivative of the $\Gamma$-function (see e.g. [8, p. 267] to derive the formulae and [9, Chapter

1.7] for properties of $\psi$), we obtain

$$\begin{aligned}
\big[\varphi_1(\cdot,\lambda),\kappa_+\big](1) &= \frac{k_0(\lambda)+\ln 2+\Gamma\Big(\dfrac{1}{2}-\sqrt{\lambda}\Big)\Gamma\Big(\dfrac{1}{2}+\sqrt{\lambda}\Big)}{\varrho(\lambda)\Gamma\Big(\dfrac{1}{2}-\sqrt{\lambda}\Big)\Gamma\Big(\dfrac{1}{2}+\sqrt{\lambda}\Big)} = \\
&= -\big[\varphi_1(\cdot,\lambda),\kappa_-\big](-1)\,, \\
\big[\varphi_2(\cdot,\lambda),\kappa_+\big](1) &= \frac{\Gamma\Big(\dfrac{1}{2}\sqrt{\lambda}\Big)\Gamma\Big(\dfrac{1}{2}+\sqrt{\lambda}\Big)-k_0(\lambda)-\ln 2}{\sigma(\lambda)\Gamma\Big(\dfrac{1}{2}-\sqrt{\lambda}\Big)\Gamma\Big(\dfrac{1}{2}+\sqrt{\lambda}\Big)} = \\
&= -\big[\varphi_2(\cdot,\lambda),\kappa_-\big](-1)\,,
\end{aligned}$$

where $k_0(\lambda) = 2\psi(1)-\psi\Big(\dfrac{1}{2}-\sqrt{\lambda}\Big)-\psi\Big(\dfrac{1}{2}+\sqrt{\lambda}\Big)$. Thus after some calculations we have

$$G(\lambda) = -\frac{2}{\pi}\cos\pi\sqrt{\lambda}\left\{\frac{\pi^2}{\cos^2\pi\sqrt{\lambda}}-\big(k_0(\lambda)+\ln 2\big)^2\right\}; \tag{11.4}$$

the zeros of this function determine the eigenvalues. The kernel is given by

$$\begin{aligned}
K_-(x,\lambda) &= \big(k_0(\lambda)+\ln 2\big)\cdot {}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1-x}{2}\Big)+ \\
&+\Gamma\Big(\frac{1}{2}-\sqrt{\lambda}\Big)\Gamma\Big(\frac{1}{2}+\sqrt{\lambda}\Big){}_2F_1\Big(-\sqrt{\lambda}+\frac{1}{2},\sqrt{\lambda}+\frac{1}{2};1;\frac{1-x}{2}\Big) \\
&x\in(-1,1)\,.
\end{aligned}\tag{11.5}$$

Observe that all the functions above are entire in $\lambda$ which is clear by Theorem 1.1 but not obvious. (A more detailed examination of this example will follows in a subsequent communication).

We obtain

LEMMA 11.4. *If* $F\in\{K_-\}$, *i.e.*

$$F(\lambda)=\int_{-1}^{1} f(x)K_-(x,\lambda)dx \quad \big(f\in L^2(-1,1),\lambda\in\mathbb{C}\big)\,,$$

*where* $K_-(x,\lambda)$ *is given by* (11.5), *then* $F$ *can be represented as*

$$F(\lambda)=\sum_{k=0}^{\infty}F(\lambda_k)\frac{G(\lambda)}{(\lambda-\lambda_k)G'(\lambda_k)}$$

*with* $G(\lambda)$ *given as in* (11.4) *and,* $\lambda_k$ *being the kth zero of* $G(\lambda)$,

$$G'(\lambda_k)=-\frac{2}{\pi}\cos\pi\sqrt{\lambda_k}\left\{\frac{\pi^3\sin\pi\sqrt{\lambda_k}}{\sqrt{\lambda_k}\cos^3\pi\sqrt{\lambda_k}}-2\big(k_0(\lambda_k)+\ln 2\big)\cdot\right.$$
$$\left.\cdot\left(\frac{\pi^2}{2\sqrt{\lambda_k}\cos^2\pi\sqrt{\lambda_k}}-\frac{1}{\sqrt{\lambda_k}}\psi'\Big(\frac{1}{2}+\sqrt{\lambda_k}\Big)\right)\right\}.$$

Remark. It could be difficult to handle this example by Zayed's methods, since there is no easy way to construct the $m$-coefficients for this problem.

*Example 4.* Consider the differential equation $-\big(xy'(x)\big)'-xy(x)=\frac{\lambda}{x}y(x)$ on $[1,\infty)$. Choose as boundary conditions $[y,\kappa_-](1)=[y,\kappa_+](\infty)=0$ with $\kappa_-(x)=x-1$, $\kappa_+(x)=x^{-1/2}(\cos x+\sin x)$ to get $[y,\kappa_-](1)=y(1)=0$. The endpoint 1 is regular and the endpoint $\infty$ is limit circle oscillatory in $L^2\big((1,\infty);x^{-1}\big)$.

A fundamental system of the differential equation is given by

$$\varphi_1(x,\lambda)=\frac{\pi}{2\sin(is\pi)}\left\{J'_{is}(1)J_{-is}(x)-J'_{-is}(1)J_{is}(x)\right\},$$
$$\varphi_2(x,\lambda)=\frac{\pi}{2\sin(is\pi)}\left\{J_{-is}(1)J_{is}(x)-J_{is}(1)J_{-is}(x)\right\},$$

where $J_\nu(x)$ is the Bessel function of order $\nu$, and $s=\sqrt{\lambda}$. We deduce

$$\big[\varphi_1(\cdot,\lambda),\kappa_-\big](1)=\varphi_1(1,\lambda)=1,\quad \big[\varphi_2(\cdot,\lambda),\kappa_-\big](1)=\varphi_2(1,\lambda)=0\,,$$

and thus $K_-(x,\lambda)=\varphi_2(x,\lambda)$ and, with $s=\sqrt{\lambda}$,

$$G(\lambda)=\big[K_-(\cdot,\lambda),\kappa_+\big](\infty)=-\sqrt{\lambda}\frac{J_{-is}(1)+J_{is}(1)}{2\cos\left(\frac{1}{2}is\pi\right)}\,. \tag{11.6}$$

Observe that all the functions above are entire in $\lambda$. $G(\lambda)$ has infinitely many real and simple zeros clustering at $-\infty$ and $+\infty$. We achieve

LEMMA 11.5. *If* $F \in \{K_-\}$, *i.e.*

$$F(\lambda) = \int_1^\infty f(x) \frac{\pi}{2\sin(is\pi)} \{J_{-is}(1) J_{is}(x) - J_{is}(1) J_{-is}(x)\} \frac{1}{x} dx \,,$$

*with* $f \in L^2\big((1,\infty); x^{-1}\big)$, $s = \sqrt{\lambda}$, $\lambda \in \mathbb{C}$, *then* $F$ *can be represented as*

$$F(\lambda) = \sum_{k=-\infty}^{\infty} F(\lambda_k) \frac{G(\lambda)}{G'(\lambda_k)(\lambda - \lambda_k)} \qquad (\lambda \in \mathbb{C}) \,,$$

*with* $G$ *given by* (11.6).

REMARK. We have not been able to find a simple closed formula for $G'(\lambda_k)$ in terms of $J_{is}(1)$ and $J_{-is}(1)$.

The denominator $\cos\left(\frac{1}{2} i\sqrt{\lambda}\pi\right)$ in (11.6) prevents the values $\lambda = -(2k+1)^2$, $k \in \mathbb{Z}$, from being eigenvalues (see also Example 3 b)); compare with BAILEY, EVERITT, ZETTL [3, Section 6].

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