Rendiconti di Matematica, Serie VII
Volume 14, Roma (1994), 341-353

Splitting methods for the solution of systems of linear equations with singular matrices

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Dedicated to the memory of Aldo Ghizzetti

Riassunto: *In questo lavoro si sviluppa il metodo della "media aritmetica" per risolvere sistemi di equazioni lineari di grandi dimensioni e sparsi quando la matrice dei coefficienti è una matrice simmetrica semidefinita positiva oppure una M-matrice singolare e irriducibile. Si applica tale metodo al problema del calcolo degli elementi propri di una λ-matrice regolare.*
Il metodo della "media aritmetica" è particolarmente adatto ad essere realizzato su un calcolatore multivettoriale, come, ad es., il CRAY Y-MP.

Abstract: *This paper is concerned with the development of the "arithmetic mean method" for solving large sparse systems of linear equations when the coefficient matrix is a symmetric positive semidefinite matrix or a singular, irreducible M- matrix. This method has been applied for the computation of the minimal eigenpair of the generalized eigenproblem.*
The method of the arithmetic mean is well suited for parallel implementation on a multivector computer, such as the CRAY Y-MP.

1 – Introduction and preliminaries

In this paper, we consider the iterative solution of the large sparse

Key Words and Phrases: *Large sacle linear sistems – Iterative methods – Singular systems*
A.M.S. Classification: 65F10 – 65F35 – 65F15 – 65Y05

linear system

$$(1) \qquad A\mathbf{x} = \mathbf{b}$$

where $A = (a_{ij})$ is an unstructured *singular* matrix of size $n \times n$, $\mathbf{x}$ and $\mathbf{b}$ are n-dimensional vectors.

We assume that (1) is solvable (or consistent), that is, $\mathbf{b} \in \Re(A)$, where $\Re(A)$ denotes the range of A. Also, we assume that the diagonal entries a_{ii} of A are all non zero numbers.

To approximately solve (1), we consider the iterative method of first degree

$$(2) \qquad B\frac{\mathbf{u}_{k+1} - \mathbf{u}_k}{\tau} + A\mathbf{u}_k = \mathbf{b} \qquad k = 0, 1, 2, \dots$$

where $\mathbf{u}_0$ is an arbitrary initial vector approximation to a solution $\mathbf{u}^*$ of (1), B is a non-singular matrix and τ is an iterative parameter ($\tau > 0$). The matrix B is taken to be an easily invertible matrix, i.e., for any vector $\mathbf{r}$ it is relatively easy to solve the system $B\mathbf{w} = \mathbf{r}$.

An alternative form of (2) is given by

$$(3) \qquad \mathbf{u}_{k+1} = \mathbf{u}_k + \tau B^{-1}(\mathbf{b} - A\mathbf{u}_k)$$

or

$$(4) \qquad \mathbf{u}_{k+1} = H\mathbf{u}_k + \mathbf{f}$$

where $H = I - \tau B^{-1}A$ and $\mathbf{f} = \tau B^{-1}\mathbf{b}$.

Since B is invertible, problems (1) and (4) have the same solution set.

If A is non-singular, iteration (4) converges to the unique solution $\mathbf{x}^* = A^{-1}\mathbf{b}$ of (1), for every initial vector $\mathbf{u}_0$, if and only if $\rho(H) < 1$, so that $\lim_{k\to\infty} H^k = 0$. Here, $\rho(H)$ denotes the spectral radius of the iteration matrix H. However, this is not the case when A is singular. We proceed now to analyze this situation.

The matrix H is said to be *convergent* if $\lim_{k\to\infty} H^k$ exists, although it need not to be the zero matrix.

Let λ_i, $i = 1, 2, \dots, n$, denote the eigenvalues of H. Then, we can verify that H is convergent whenever the following property holds: $|\lambda_i| \leq 1$ for $i = 1, 2, \dots, n$, $|\lambda_i| = 1$ implies $\lambda_i = 1$, and all the elementary divisors that correspond to $\lambda_i = 1$ are linear, i.e. there are no principal vectors that correspond to $\lambda_i = 1$.

Assume for simplicity that $\lambda_i = 1$ for $i = 1, 2, \dots, t$, and $|\lambda_i| < 1$ for $i = t+1,\ t+2, \dots, n$ (t may be zero). Then, we can say that H is convergent if and only if there exists a non-singular matrix V such that

$$H = V \begin{pmatrix} I_t & 0 \\ 0 & K \end{pmatrix} V^{-1} \tag{5}$$

where I_t is the $t \times t$ unit matrix and K has a Jordan canonical form with eigenvalues λ_i, $i = t+1, t+2, \dots, n$.
We have the following basic result [2].

The following three conditions are equivalent:

a) *The sequence* $\{\mathbf{u}_k\}$ *converges for any choice of* $\mathbf{u}_0$*.*
b) $\lim_{k\to\infty} \|\mathbf{u}_{k+1} - \mathbf{u}_k\| = 0$ *for any choice of* $\mathbf{u}_0$*.*
c) H *is convergent and the linear system* $(I - H)\mathbf{u} = \mathbf{f}$ *is solvable.*

We now consider the asymptotic rate of convergence of the iterative method (4).

If H is convergent, we adopt the notation

$$\delta(H) = \max\{|\lambda| \colon \lambda \in \sigma(H)\,, \lambda \neq 1\} \tag{6}$$

where $\sigma(H)$ denotes the spectrum of H.
Then, if $\rho(H) < 1$, $\delta(H) = \rho(H)$. Otherwise, $\delta(H)$ is the second largest of the moduli of the eigenvalues of H. Therefore, $\delta(H) = \rho(K)$, where K is given by (5). This leads to the observation that if H is convergent then the asymptotic rate of convergence of the iterative method is given by

$$R_\infty(H) = -\ln \delta(H) \tag{7}$$

where $\delta(H)$ is given by (6).

It is useful to review some criteria that help us to decide whether H in (4) is convergent.

PROPOSITION 1[5]. *Assume that* $\mathbf{x}^*$ *solves* (1)*, where* A *is an* $n \times n$ *symmetric positive semidefinite matrix. Then, the error vectors* $\mathbf{e}_k =$

$\mathbf{u}_k - \mathbf{x}^*$, $k = 0, 1, 2, \ldots$, *satisfy*

$$\mathbf{e}_k^T A \mathbf{e}_k - \mathbf{e}_{k+1}^T A \mathbf{e}_{k+1} = (\mathbf{u}_{k+1} - \mathbf{u}_k)^T P (\mathbf{u}_{k+1} - \mathbf{u}_k) \tag{8}$$

where $\mathbf{u}_k$ *is generated by* (4) *and* $P = \dfrac{1}{\tau}(B + B^T) - A$.

Besides, if P *is positive definite, the matrix* H *in (4) is convergent and the sequence* $\{\mathbf{u}_k\}$ *converges for any choice of* $\mathbf{u}_0$.

Definition. A real $n \times n$ matrix A is an M-matrix if there exists a non negative matrix M with maximal eigenvalue $\rho(M)$ such that

$$A = \alpha I - M$$

where $\alpha \geq \rho(M)$.

Note that the main diagonal entries of an M-matrix are non negative and all its other entries are non positive.

The set of $n \times n$ real matrices whose off-diagonal entries are non positive is denoted by $Z^{n\times n}$.

If A is a non-singular M-matrix we must have that $\alpha > \rho(M)$, so that $\dfrac{1}{\alpha}M$ is zero-convergent.

The total class of M-matrices can be thought of as the closure of the class of non singular M-matrices. This is justified by the following result [1].

"Let $A \in Z^{n\times n}$. Then A is an M-matrix if and only if $A + \varepsilon I$ is a non singular M-matrix for all scalars $\varepsilon > 0$".

Some characterizations of M-matrices are given in [1] and [3].

1) A matrix A in $Z^{n\times n}$ is an M- matrix if and only if all its eigenvalues have a non negative real part.
2) A matrix A in $Z^{n\times n}$ is an M-matrix if and only if all its principal minors are non negative.
3) Let the matrix $A \in Z^{n\times n}$ be symmetric. Then, A is an M-matrix if and only if A is positive semidefinite.
4) A matrix A in $Z^{n\times n}$ is an M-matrix if and only if $A + D$ is non singular for every positive diagonal matrix D.
5) A non singular matrix $A \in Z^{n\times n}$ is an M-matrix if and only if A^{-1} is non negative.

For the last characterization of an M-matrix the following result is very useful.

LEMMA 1. *Let $A = (a_{ij})$ be strictly or irreducibly diagonally dominant and assume that $a_{ij} \leq 0$, $i \neq j$, and $a_{ii} > 0$, $i = 1, 2, \dots, n$. Then, A is a non singular M-matrix, i.e., $A^{-1} \geq 0$.*

The next result is one of the most important and relevant to our study of M-matrices [3].

LEMMA 2. *If A is an irreducible M-matrix, then A is weak semipositive, i.e., there exists a vector $\mathbf{x} > 0$ such that $A\mathbf{x} \geq 0$.*

Thus, if $A = B - C$ is a *weak regular splitting* of A, Theorem 1 in [6] (see, also, [7], pg 273) yields the following condition for the iteration matrix $B^{-1}C$.

LEMMA 3. *Let $A = B - C$ be a weak regular splitting for $A \in R^{n\times n}$. If A is weak semipositive, then $\rho(B^{-1}C) \leq 1$ and $B^{-1}C$ has only linear elementary divisors corresponding to the eigenvalue 1. However, it may well happen that $B^{-1}C$ has some eigenvalues other than 1 on the unit circle.*

(We remember that, for a matrix $A \in R^{n\times n}$, $A = B - C$ is a *splitting* if B is non singular. This splitting is *regular* if $B^{-1} \geq 0$ and $C \geq 0$ and it is *weak regular* if $B^{-1} \geq 0$ and $B^{-1}C \geq 0$).

Since $H = I - \tau B^{-1}A = (1-\tau)I + \tau B^{-1}C$, the eigenvalues λ_i of H, $i = 1, 2, \dots, n$, are of the form $\lambda_i = (1-\tau) + \tau\mu_i$; where μ_i is the i-th eigenvalue of the matrix $B^{-1}C = I - B^{-1}A$. From Lemma 3 we deduce that, for $0 < \tau < 1$, the matrix H has no eigenvalues other than 1 on the unit circle and $\rho(H) \leq 1$.

Combining this result with Lemmas 3 and 2 we have the following important criterion.

PROPOSITION 2. *Let A be an irreducible M-matrix and let $A = B - C$ be a weak regular splitting of A. Then, the matrix H of the iteration method* (4) *is convergent for all $\tau \in (0, 1)$.*

2 – The basic iterative methods

We can express the matrix A as the matrix sum

$$A = L + D + U \tag{9}$$

where $D = \mathrm{diag}\{a_{11}, a_{22}, \ldots, a_{nn}\}$ is the diagonal matrix with the same diagonal elements of A and L and U are strictly lower and upper triangular matrices, respectively.

In literature widely used forms of B in (3) are:

$$B^{-1} = (D + \omega U)^{-1} D (D + \omega L)^{-1} \qquad 0 < \omega < 2 \tag{10}$$

and

$$B^{-1} = \frac{1}{2}\Big(\Big(\frac{1}{\omega}D + L\Big)^{-1}\Big) + \Big(\frac{1}{\omega}D + U\Big)^{-1}\Big) \qquad 0 < \omega < 2\,. \tag{11}$$

If we assume that $A + \dfrac{2-\omega}{\omega} D$ is non-singular, the matrix B of formula (11) has the expression [4]:

$$B = \frac{1}{2}\Big(A + \frac{2-\omega}{\omega}D\Big) - R\Big(\frac{1}{2}\Big(A + \frac{2-\omega}{\omega}D\Big)\Big)^{-1} R \tag{12}$$

where $R = \dfrac{1}{2}(L - U)$.

It is interesting to note that the method (3) with the choice (11) of B^{-1} is characterized by having within its overall mathematical structure certain well-defined substructures that can be executed simultaneously during each iteration k. This feature makes the method (3)-(11) ideally suited for implementation on a multiprocessor system with two or more vector processors, such as the CRAY Y-MP. The lower triangular system $(\dfrac{1}{\omega}D + L\Big)\mathbf{w}_k^{(1)} = \mathbf{r}_k$, and the upper triangular system $\Big(\dfrac{1}{\omega}D + U\Big)\mathbf{w}_k^{(2)} = \mathbf{r}_k$, where $\mathbf{r}_k = \mathbf{b} - A\mathbf{u}_k$, can be solved *simultaneously* on two different vector processors. Hence, the effectiveness of the method (3)-(11) depends on the availability of an efficient parallel algorithm for solving these triangular systems. When we use in (3) the *matrix* B^{-1} of formula (10), during each iteration k, the two triangular systems $(D+\omega L)\mathbf{w}_k^{(1)} = \mathbf{r}_k$ and $(D+\omega U)\mathbf{w}_k^{(2)} = D\mathbf{w}_k^{(1)}$ are solved *sequentially.*

Since the iterative method (2) can be regarded as belonging to a generalized class referred to as the Method of Simultaneous Displacements with some form of "preconditioning" to the original system (1), we call the method (3) with the choice (10) of B^{-1} the *Method of Simultaneous Displacements with the SSOR preconditioner* and the method (3) with the choice (11) of B^{-1} the *Method of Simultaneous Displacements with the additive preconditioner.*

A study of the convergence properties of these two iterative methods is given under the assumption that A is symmetric positive semidefinite or is a singular irreducible M-matrix.

THEOREM 1. *Let A of formula* (9) *be a symmetric positive semidefinite matrix. Then, the iterative method* (3)-(10) *is convergent for $\omega_l \le \omega \le \omega_u$ where $\omega_l = (1-\sqrt{\tau/2})/(2/\tau-1)$ and $\omega_u = (1+\sqrt{\tau/2})/(2/\tau-1)$ and $0<\tau\le 1$, and the iterative method* (3)-(11) *is convergent for $0<\omega<2$ and $0<\tau\le 1$.*

PROOF. When we consider the matrix B of formula (10) we have ($U = L^T$):

$$P = \frac{1}{2}(B+B^T) - A = \frac{2}{\tau}(D+\omega L)D^{-1}(D+\omega L^T) - L - D - L^T =$$

$$= \left(\frac{2}{\tau}-1\right)D + \left(\frac{2\omega}{\tau}-1\right)(L+L^T) + \frac{2\omega^2}{\tau}LD^{-1}L^T\,.$$

Since $LD^{-1}L^T$ is a symmetric positive semidefinite matrix and $(1-\sqrt{\tau/2})/(2/\tau-1) \le \omega \le (1+\sqrt{\tau/2})/(2/\tau-1)$, for any $\mathbf{z}\ne 0$, it is easy to prove that

$$\left(\frac{2\omega}{\tau}-1\right)^2 \mathbf{z}^T LD^{-1}L^T\mathbf{z} \le \frac{2\omega^2}{\tau}\mathbf{z}^T LD^{-1}L^T\mathbf{z}\,.$$

Then, for $0<\tau\le 1$, we have

$$\mathbf{z}^T P\mathbf{z} \ge \left(\frac{2}{\tau}-1\right)\mathbf{z}^T D\mathbf{z} + \left(\frac{2\omega}{\tau}-1\right)\mathbf{z}^T(L+L^T)\mathbf{z} + \left(\frac{2\omega}{\tau}-1\right)^2\mathbf{z}^T LD^{-1}L^T\mathbf{z} =$$

$$= \left(\frac{2}{\tau}-2\right)\mathbf{z}^T D\mathbf{z} + \mathbf{z}^T\left(D+\left(\frac{2\omega}{\tau}-1\right)L\right)D^{-1}\left(D+\left(\frac{2\omega}{\tau}-1\right)L^T\right)\mathbf{z} > 0.$$

Employing Proposition 1, we conclude that the iterative method (3)-(10) is convergent for any choice of $\mathbf{u}_0$.

When we consider the matrix B of formula (11) we have from (12) with $R = \frac{1}{2}(L - L^t)$

$$\begin{aligned} P &= \frac{1}{\tau}(B + B^T) - A = \\ &= \frac{2}{\tau}\Big(\frac{1}{2}\Big(A + \frac{2-\omega}{\omega}D\Big) + R\Big(\frac{1}{2}\Big(A + \frac{2-\omega}{\omega}D\Big)\Big)^{-1} R^T\Big) - A = \\ &= \frac{2-\omega}{\tau\omega}D + \frac{4}{\tau}R\Big(A + \frac{2-\omega}{\omega}D)^{-1}R^T + \Big(\frac{1}{\tau} - 1\Big)A\,. \end{aligned}$$

For $0 < \omega < 2$ the matrices $\frac{2-\omega}{\omega}D$ and $A + \frac{2-\omega}{\omega}D$ are symmetric positive definite. Since the matrix P is the sum of a symmetric positive definite matrix and two symmetric positive semidefinite matrices for $0 < \tau \leq 1$, the matrix P is symmetric positive definite. Employing Proposition 1, we conclude that the iterative method (3)-(11) is convergent for any $\mathbf{u}_0$. □

THEOREM 2. *Let $A = (a_{ij})$ of formula* (9) *be a singular, irreducible M-matrix and assume that the diagonal entries a_{ii} of A are all positive numbers and that $\frac{1}{\omega}D + L$ and $\frac{1}{\omega}D + U$ are strictly diagonally dominant matrices for $0 < \omega \leq 1$. Then, the iterative methods* (3)-(10) *and* (3)-(11) *are convergent for $0 < \omega \leq 1$ and $0 < \tau < 1$.*

PROOF. When we consider the matrix B of formula (10) we have

$$\begin{aligned} C = B - A &= (D + \omega L)D^{-1}(D + \omega U) - L - D - U = \\ &= (\omega - 1)L + (\omega - 1)U + \omega^2 L D^{-1} U\,. \end{aligned}$$

By hypothesis, for $0 < \omega \leq 1$, the matrix C is non-negative. Besides, $D + \omega L$ and $D + \omega U$ are strictly diagonally dominant matrices with positive entries on the diagonal and with non positive off-diagonal elements. Thus, by Lemma 1, B is a non singular M-matrix: $B^{-1} \geq 0$. Therefore, $A = B - C$ is a regular splitting for A.

Now, employing Proposition 2, we conclude that the iterative method (3)-(10) is convergent for any $\mathbf{u}_0$.

By hypothesis, for $0 < \omega \leq 1$, the matrices $\frac{1}{\omega}D + L$ and $\frac{1}{\omega}D + U$ are strictly diagonally dominant and the matrices $\frac{1-\omega}{\omega}D - U$ and $\frac{1-\omega}{\omega}D - L$ are non negative. Thus, by Lemma 1 we deduce that

$$A = \left(\frac{1}{\omega}D + L\right) - \left(\frac{1-\omega}{\omega}D - U\right) = B_1 - C_1$$

$$A = \left(\frac{1}{\omega}D + U\right) - \left(\frac{1-\omega}{\omega}D - L\right) = B_2 - C_2$$

are two regular splittings of A.

When we consider the matrix B of formula (11) we have

$$B^{-1} = \frac{1}{2}B_1^{-1} + \frac{1}{2}B_2^{-1} \geq 0$$

and

$$I - B^{-1}A = I - \frac{1}{2}B_1^{-1}(B_1 - C_1) - \frac{1}{2}B_2^{-1}(B_2 - C_2) =$$

$$= \frac{1}{2}B_1^{-1}C_1 + \frac{1}{2}B_2^{-1}C_2 \geq 0\,.$$

Since $\frac{1}{2}B_1^{-1} + \frac{1}{2}B_2^{-1} = \frac{1}{2}B_1^{-1}(B_1 + B_2)B_2^{-1}$ and the matrix $B_1 + B_2 = \left(\frac{2}{\omega} - 1\right)D + A$ is non singular from the characterization 4) of the M-matrix A for $0 < \omega < 2$, the matrix B is non singular.

Now, we consider the splitting $A = B - C$ for A. We have $B^{-1}C = B^{-1}(B - A) = I - B^{-1}A$. Thus, this splitting $A = B - C$ is a weak regular splitting of A. Employing Proposition 2, we conclude that the iterative method (3)-(11) is convergent for any choice of $\mathbf{u}_0$. $\square$

3 – An application

In many areas of science and technology it is required to compute the smallest eigenvalue and a corresponding eigenvector of the generalized eigenvalue problem

$$(K - \mu N)\mathbf{x} = 0 \tag{13}$$

where K and N are $n \times n$ symmetric positive definite matrices. Furthermore, these matrices are assumed to be large and sparse with irregular structure, so that it is inconvenient to use similarity transformations or to factorize either K or N, or a linear combination of K and N into a product of *simple* matrices.

It is a well-known fact that, if K and N are $n \times n$ symmetric matrices with N positive definite, problem (13) has n real eigenvalues $\mu_1 \leq \mu_2 \leq \dots \mu_n$ and n corresponding linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ that can be chosen to be orthogonal in the inner product $(\mathbf{v}, \mathbf{w})_N = \mathbf{v}^T \cdot N\mathbf{w}$. The matrix $K - \mu_1 N$ is a positive semidefinite matrix and the smallest eigenvalue μ_1 and the largest eigenvalue μ_n are the minimum and the maximum, respectively, of the Rayleigh quotient

$$\rho(\mathbf{x}) = \frac{\mathbf{x}^T K\mathbf{x}}{\mathbf{x}^T N\mathbf{x}} \tag{14}$$

and the minimum and the maximum are taken on at any eigenvector (not necessarily normalized) corresponding to μ_1 and μ_n, respectively. Every eigenvector corresponding to an eigenvalue not equal to μ_1 or μ_n is a stationary point of $\rho(\mathbf{x})$. Moreover, if K is positive definite, then the eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ are all positive. One basic property of the Rayleigh quotient is that if

$$\mathbf{x} = \mathbf{v}_i \cos\theta + \mathbf{w} \sin\theta$$

where $\mathbf{v}_i$ and $\mathbf{w}$ are N-orthonormal (i.e. $\mathbf{v}_i^T \cdot N\mathbf{w} = 0$ and $\mathbf{v}_i^T \cdot N\mathbf{v}_i = 1$, $\mathbf{w}^T \cdot N\mathbf{w} = 1$), makes an angle θ to an eigenvector $\mathbf{v}_i$ for some i ($i = 1, 2, \dots, n$), then

$$\rho(\mathbf{x}) = \mu_i + \Big(\mathbf{w}^T \cdot K\mathbf{w} - \mu_i\Big) \sin^2\theta\,. \tag{15}$$

That is, the approximation error of $\rho(\mathbf{x})$ to μ_i is proportional to the square of the approximation error of $\mathbf{x}$ to the corresponding eigenvector $\mathbf{v}_i$.

One of the most widely used methods for computing the minimal eigenpair in (13) is the Rayleigh Quotient Iteration (RQI) Method [8, pg 319]. This method is described by the following statements.

1. Choose an initial vector $\mathbf{x}^{(0)}$, $\|\mathbf{x}^{(0)}\| = 1$; $1 \to i$.

2. Calculate $\rho^{(i)} = \Big((\mathbf{x}^{(i)})^T \cdot K\mathbf{x}^{(i)}\Big)\Big/\Big((\mathbf{x}^{(i)})^T \cdot N\mathbf{x}^{(i)}\Big)$.

   Solve the system of linear equations for $\mathbf{z}^{(i+1)}$

$$\Big(K - \rho^{(i)}N\Big)\mathbf{z}^{(i+1)} = N\mathbf{x}^{(i)}\,. \tag{16}$$

3. If $\|\mathbf{z}^{(i+1)}\|$ is large enough, goto step 4; otherwise

$$\mathbf{x}^{(i+1)} = \mathbf{z}^{(i+1)}\Big/\|z^{(i+1)}\|$$
$$i + 1 \to i \text{ goto step 2}$$

4. $(\mathbf{x}^{(i)}, \rho^{(i)})$ is the approximate eigenpair of the matrix pencil (K, N).

Here, the norm $\|\ \|$ is Euclidean. Note that the matrix of (16) is not positive definite since $\rho^{(i)}$ is always inside the range of eigenvalues of $K - \mu N$. If $\rho^{(i)}$ is really very close to the eigenvalue μ_1, the matrix $(K - \rho^{(i)}N)$ is almost singular and approximately positive semidefinite. When the initial vector $\mathbf{x}^{(0)}$ is good, the method QRI converges very quickly; the convergence rate is cubic. It is therefore usual to recommend this method for solving large size generalized eigenproblems.

At each iteration step i we have to solve the linear system (16) whose coefficient matrix $A = K + \rho^{(i)}N$ is "approximately" positive semidefinite. We solve this system with the splitting method (2), where the preconditioner B has the form (10) or (11). Theorem 1 assures the convergence of this iterative method.

To analyse the *rate of convergence* of methods (2)-(10) and (2)-(11), we remember that, using the Rayleigh quotient in an iterative method for the determination of the eigenvalues, the accuracy of the eigenvalue obtained is the square of that of the corresponding eigenvector (see formula (15)). Hence, it seems reasonable to say that the rate of convergence of the method (2) is determined by the convergence rate of the limiting iteration.

$$B\frac{\mathbf{u}_{k+1} - \mathbf{u}_k}{\tau} + (K - \mu_1 N)\mathbf{u}_k = \mathbf{b} \tag{17}$$

where $\mathbf{b} = N\mathbf{x}^{(i)}$.

The matrix $A = K - \mu_1 N$ is a positive semidefinite matrix.

Since $(I - \tau B^{-1}(K - \mu_1 N))\mathbf{v}_1 = \mathbf{v}_1$, the vector $\mathbf{v}_1$ is an eigenvector of the iteration matrix H of (17) corresponding to the eigenvalue $\lambda_1 = 1$. Besides, the matrix $H = I - \tau B^{-1}(K - \mu_1 N)$ has no principal vectors corresponding the eigenvalue $\lambda_1 = 1$. This follows immediately from the general proof given in [5] by identifying our positive semidefinite matrix $K - \mu_1 N$ with the matrix A there and by setting the matrix N in [5] equal to B. For two successive iterates of the sequence (17) we get the expression

$$\begin{aligned} &\mathbf{u}_k^T(K - \mu_1 N)\mathbf{u}_k - \mathbf{u}_{k+1}^T(K - \mu_1 N)\mathbf{u}_{k+1} = \\ &= (\mathbf{u}_k - \mathbf{u}_{k+1})^T\Big(\frac{1}{\tau}(B + B^T) - (K - \mu_1 N)\Big)(\mathbf{u}_k - \mathbf{u}_{k+1}) \,. \end{aligned}$$

Since $P = \dfrac{1}{\tau}(B + B^T) - (K - \mu_1 N)$ is symmetric positive definite for all $(1 - \sqrt{\tau/2})/(2/\tau - 1) \le \omega \le (1 + \sqrt{\tau/2})/(2/\tau - 1)$ and $0 < \tau \le 1$ when B has the form (10) and for all $0 < \omega < 2$ and $0 < \tau \le 1$ when B has the form (11), the arguments in [5] exclude the existence of principal vectors corresponding to $\lambda_1 = 1$.

Then, for the above values of ω and τ, the spectral radius of H is one and the eigenvalues $\lambda_j \neq 1$ of H have moduli strictly less than one. Therefore, the asymptotic rate of convergence of (2)-(10) and (2)-(11) is determined by the number $R_\infty(H) = -\ln(\max_{\lambda_j \neq 1} |\lambda_j|)$, where λ_j is an eigenvalues of H.

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Lavoro pervenuto alla redazione il 18 aprile 1994

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