Rendiconti di Matematica, Serie VII
Volume 14, Roma (1994), 367-384

An application of differential subordinations

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Riassunto: *Usando la tecnica di subordinazione differenziale si migliorano alcuni risultati classici della teoria delle funzioni univalenti. Si ottengono anche alcuni criteri di univalenza per le funzioni definite nel disco unitario.*

Abstract: *By using the method of Briot-Bouquet differential subordinations, we prove and sharpen some classical results in univalent function theory. These also lead to some criteria for univalency in the unit disc.*

1 – Introduction

Let $A(p)$ denote the class of functions of form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \text{ is a fixed integer} \geq 1) \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let S, $S^*(\alpha)$ and $K(\alpha)$ $(0 \leq \alpha < 1)$ denote subclasses of functions in $A(1)$ which are respectively univalent, starlike of order α and convex of order α. We denote $S^*(0) = S$, $K(0) = K$. For given arbitrary numbers A, B satisfying $-1 \leq B < A \leq 1$, we denote by $P(A, B)$, the class of functions

Key Words and Phrases: *Starlike functions – Convex functions – Differential subordination*
A.M.S. Classification: 30C45

of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \tag{1.2}$$

which are analytic in E and satisfy the condition

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \qquad z \in E$$

where "$\prec$" stands for subordination. Geometrically, this means that the image of E under $p(z)$ is inside the open disc centred on the real axis whose diameter has end points $(1-A)/(1-B)$ and $(1+A)/(1+B)$. From this we conclude that $p(z)$ has a positive real part and hence univalent in E [19]. This class $P(A, B)$ was investigated by JANOWSKI [8]. We say that a function $f(z) \in A(1)$ is said to be in the class $P'(A, B)$ if and only if $f'(z) \in P(A, B)$. It is clear that $P'(1 - 2\alpha, -1) \equiv P'(\alpha)$ is the class of functions $f(z) \in A(1)$ for which $\Re(f'(z)) > \alpha$, $0 \le \alpha < 1$.

In [6], GOEL and SOHI have studied the class of functions $f(z) \in A(1)$ satisfying

$$\Re\left\{\frac{D^{n+1} f(z)}{z}\right\} > \alpha, \quad z \in E, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \dots\} \tag{1.3}$$

where $0 \le \alpha < 1$ and

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}.$$

(Here "$*$" means the Hadamard product of two analytic functions).

AOUF [2], further generalized this class of functions by introducing the class $V_n(A, B, \alpha)$. Thus, a function $f(z) \in A(1)$ is said to be in the class $V_n(A, B, \alpha)$ if

$$\frac{D^{n+1} f(z)}{z} \prec \frac{1 + \{(1-\alpha)A + \alpha B\}z}{1 + Bz}, \quad z \in E, \quad n \in \mathbb{N}_0$$

for $-1 \le B < A \le 1$ and $0 \le \alpha < 1$. He showed that $V_{n+1}(A, B, \alpha) \subset V_n(A, B, \alpha)$ for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Recently, many of the classical results in univalent function theory have been improved and sharpened by the powerful technique of Briot-Bouquet differential subordination [5, 10, 12]. We recall that a function $p(z)$ analytic in E with a power series of the form (1.2) is said to satisfy Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)\,, \quad z \in E \tag{1.4}$$

for β and γ complex constants and $h(z)$ a complex function with $h(0) = 1$, $\Re(\beta h(z) + \gamma) > 0$ in E. It is known that [5] if

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (q(0) = 1) \tag{1.5}$$

has the univalent solution $q(z)$ in E, then

$$p(z) \prec q(z) \prec h(z)$$

and $q(z)$ is the best dominant of the differential subordination (1.5).

We note that the univalent function $q(z)$ is said to be a dominant of the differential subordination (1.4) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.4). If $\tilde{q}(z)$ is a dominant of (1.4) and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.4), then $\tilde{q}(z)$ is said to be the best dominant of (1.4). We remark that the best dominant is unique up to a rotation of E. More results on differential subordination can be found in [11].

In this article, we propose to give some more applications of Briot-Bouquet differential subordination which would not only improve and sharpen many of the earlier results contained in [13, 16, 17, 18, 20, 21], but would also give rise to a number of new results for other subclasses as well. This is accomplished by introducing and studying a more general class $T_{\delta,\lambda}(p; A, B)$. Our results also generalize the work of Ponnusamy and Juneja [20], Owa, Obradovic and Nunokawa [18], Obradovic [16] and Nunokawa [14].

2 – Differential subordinations

We now introduce the class $T_{\delta,\lambda}(p; A, B)$ as follows:

Let A, B, λ and δ be fixed real numbers such that $-1 \le B < A \le 1$, $\lambda \ge 0$ and $\delta \ge 0$. A function $f(z) \in A(p)$ is said to be in the class $T_{\delta,\lambda}(p; A, B)$ if it satisfies

$$J_{\delta,p}(f;\lambda) \prec \frac{1+Az}{1+Bz}, \quad z \in E \tag{2.1}$$

where

$$J_{\delta,p}(f;\lambda) = (1-\lambda)\frac{D^{\delta+p-1}f(z)}{z^p} + \lambda\frac{D^{\delta+p}f(z)}{z^p}$$

and

$$D^{\delta+p-1}f(z) = \frac{z^p}{(1-z)^{\delta+p}} * f(z)\,.$$

It is readily seen that $T_{n,1}(1; 0, -1)$ is the class considered by Goel and Sohi [6] whereas $T_{0,\lambda}(1; 1-2\alpha, -1)(0 \le \alpha < 1)$ is the class studied by Owa, Obradovic and Nunokawa [18]. Further, it is clear that $T_{0,1}(1; A, B) = P'(A, B)$ is the class studied by Obradovic [16]. We denote $T_{\delta,\lambda}(1; A, B)$ by $T_{\delta,\lambda}(A, B)$.

To establish our main results we need the following lemmas.

Lemma 1. ([7]). *If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in E and $h(z)$ is a convex function in E with $h(0) = 1$ and γ is a complex constant such that $\Re(\gamma) > 0$, then*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \tag{2.1}$$

implies

$$p(z) \prec \gamma z^{-\gamma}\int_0^z t^{\gamma-1}h(t)dt = q(z) \prec h(z)$$

and $q(z)$ is the best dominant.

The following lemma is due to NUNOKAWA [13, Theorem 8].

LEMMA 2. *Let* $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ *be analytic in* E. *If there exists a* $(p-m+1)$*-valent starlike function* $g(z) = z^{p-m+1} + \sum_{k=p-m+2}^{\infty} a_k z^k$ *in* E *such that*

$$\Re\left\{\frac{zf^{(m)}(z)}{g(z)}\right\} > 0\,, \quad z \in E$$

then $f(z)$ *is* p*-valent in* E.

For a, b, c real numbers other than $0, -1, -2, \dots$, the hypergeometric series

$$F(a,b;c;z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \dots$$

represents an analytic function in E [1, p. 556]. The following identities are well known [1, p. 556-558].

LEMMA 3. *For* a, b, c *real numbers other than* $0, -1, -2, \dots$, *and* $c > b > 0$, *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b) \cdot \Gamma(c-b)}{\Gamma(c)} F(a,b;c;z) \tag{2.2}$$

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \tag{2.3}$$

$$F(1,1;2;z) = -z^{-1}\ln(1-z) \tag{2.4}$$

$$c(c-1)(z-1)F(a,b;c-1;z) + c[c-1-(2c-a-b-1)z]F(a,b;c;z) + \\ + (c-a)(c-b)zF(a,b;c+1;z) = 0\,. \tag{2.5}$$

LEMMA 4. *For any real number $d \neq 0$, we have*

$$F\Big(1,1;2;\frac{dz}{dz+1}\Big) = \frac{(1+dz)\ln(1+dz)}{dz} \tag{2.6}$$

$$F\Big(1,1;3;\frac{dz}{dz+1}\Big) = \frac{2(1+dz)}{dz}\Big[1-\frac{\ln(1+dz)}{dz}\Big] \tag{2.7}$$

$$F\Big(1,1;4;\frac{dz}{dz+1}\Big) = \frac{3(1+dz)}{2(bz)^3}[2\ln(1+dz)-dz(2-dz)] \tag{2.8}$$

$$F\Big(1,1;5;\frac{dz}{dz+1}\Big) = \frac{2(1+dz)}{(dz)^3}\Big[\frac{2(bz)^2-3bz+6}{3}-\frac{2\ln(1+dz)}{dz}\Big]. \tag{2.9}$$

The proof of Lemma 4 follows from the identities (2.4) and (2.5).

THEOREM 1. *Let the function $f(z)$ defined by (1.1) be in the class $T_{\delta,\lambda}(p;A,B)$. If $\delta+p>\lambda>0$ then*

$$\frac{D^{\delta+p-1}f(z)}{z^p} \prec q(z) \prec \frac{1+Az}{1+Bz}, \quad z\in E \tag{2.10}$$

where

$$q(z) = (1+Bz)^{-1}\Big[F\Big(1,1;1+\frac{\delta+p}{\lambda};\frac{Bz}{Bz+1}\Big)+ + \frac{(\delta+p)Az}{\delta+p+\lambda}F\Big(1,1;2+\frac{\delta+p}{\lambda}\frac{Bz}{Bz+1}\Big)\Big]$$

and $q(z)$ is the best dominant. Further more,

$$\Re\Big\{\frac{D^{\delta+p-1}f(z)}{z^p}\Big\} > \rho\,,\ \textit{where} \tag{2.11}$$

$$\rho=(1-B)^{-1}\Big[F\Big(1,1;1+\frac{\delta+p}{\lambda};\frac{B}{B-1}\Big)-\frac{(\delta+p)A}{\delta+p+\lambda}F\Big(1,1;2+\frac{\delta+p}{\lambda};\frac{B}{B-1}\Big)\Big].$$

PROOF. Since for $\delta \geq 0$,

$$D^\delta f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(\delta+p+k)a_{p+k}z^{p+k}}{\Gamma(\delta+p)k!}\,,$$

we have

$$z(D^{\delta+p-1}f(z))' = (\delta+p)D^{\delta+p}f(z) - \delta D^{\delta+p-1}f(z)\,. \tag{2.12}$$

Let $p(z) = D^{\delta+p-1}f(z)/z^p$. Then $p(z)$ is analytic in E with $p(0)=1$ and as $f(z) \in T_{\delta,\lambda}(p;A,B)$, (2.1) coupled with (2.12) yields

$$p(z) + \left(\frac{\lambda}{\delta+p}\right) zp'(z) = J_{\delta,p}(f;\lambda) \prec \frac{1+Az}{1+Bz}\,, \quad z \in E\,.$$

Thus, by using Lemma 1 for $\gamma = (\delta+p)/\lambda$, we deduce that

$$\frac{D^{\delta+p-1}f(z)}{z^p} \prec \left(\frac{\delta+p}{\lambda}\right) z^{-(\frac{\delta+p}{\lambda})} \int_0^z \frac{t^{\frac{\delta+p}{\lambda}-1}(1+At)dt}{(1+Bt)} = q(z)\,, \text{say.}$$

Now the function $q(z)$ can be rewritten as

$$\begin{aligned} q(z) &= \left(\frac{\delta+p}{\lambda}\right) \int_0^1 \frac{s^{\frac{\delta+p}{\lambda}-1}(1+Asz)ds}{(1+Bsz)} = \\ &= \left(\frac{\delta+p}{\lambda}\right) \int_0^1 s^{\frac{\delta+p}{\lambda}-1}(1+Bsz)^{-1}ds + \\ &\quad + A\left(\frac{\delta+p}{\lambda}\right) z \cdot \int_0^1 s^{\frac{\delta+p}{\lambda}}(1+Bsz)^{-1}ds = \\ &= (1+Bz)^{-1}\left[F\left(1,1;1+\frac{\delta+p}{\lambda};\frac{Bz}{Bz+1}\right) + \right. \\ &\quad \left. + \frac{(\delta+p)Az}{\delta+p+\lambda} F\left(1,1;2+\frac{\delta+p}{\lambda};\frac{Bz}{Bz+1}\right)\right], \end{aligned}$$

by using the identities (2.2) and (2.3). This completes the proof of (2.10).

Next to prove (2.11), it suffices to show that

$$\inf_{|z|<1}\{q(z)\} = q(-1)\,. \tag{2.13}$$

Since for $-1 \le B < A \le 1$, $(1+Az)/(1+Bz)$ is convex (univalent) in E, we have for $|z| \le r < 1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \ge \frac{1-Ar}{1-Br}\,. \tag{2.14}$$

Setting

$$g(s,z) = \frac{1+Asz}{1+Bsz}\,, \quad 0 \le s \le 1\,, \quad z \in E$$

and

$$d\mu(s) = s^{\frac{\delta+p}{\lambda}-1}\frac{(\delta+p)}{\lambda}ds$$

which is a positive measure on $[0,1]$, we get

$$q(z) = \int_0^1 g(s,z)d\mu(s)$$

so that

$$\Re\{q(z)\} = \int_0^1 \Re\left(\frac{1+Asz}{1+Bsz}\right)d\mu(s) \ge \int_0^1 \left(\frac{1-Asr}{1-Bsr}\right)d\mu(s) =$$
$$= q(-r), \quad |z| \le r < 1\,.$$

Now, letting $r \to 1^-$ in the above inequality, we obtain

$$\Re\{q(z)\} \ge q(-1), \quad z \in E$$

which implies (2.13). Hence the theorem.

Putting $p = 1$ in the above Theorem, we obtain

Corollary 1. *Let $f(z) \in T_{\delta,\lambda}(A,B)$ and $\delta+1>\lambda>0$. Then*

$$
\begin{aligned}
\frac{D^{\delta} f(z)}{z} \prec q(z) = (1+Bz)^{-1}\Big[F\Big(1,1;1+\frac{\delta+1}{\lambda};\frac{Bz}{Bz+1}\Big)+ \\
+\frac{(\delta+1)Az}{\delta+1+\lambda}F\Big(1,1;2+\frac{\delta+1}{\lambda};\frac{Bz}{Bz+1}\Big)\Big] \\
\prec \frac{1+Az}{1+Bz}\,, \quad z\in E
\end{aligned} \tag{2.15}
$$

and $q(z)$ is the best dominant. Further more,

$$
\Re\Big(\frac{D^{\delta} f(z)}{z}\Big) > \rho\,, \ \textit{where} \tag{2.16}
$$

$$
\rho=(1-B)^{-1}\Big[F\Big(1,1;1+\frac{\delta+1}{\lambda};\frac{B}{B-1}\Big)-\frac{(\delta+1)A}{\delta+1+\lambda}F\Big(1,1;2+\frac{\delta+1}{\lambda};\frac{B}{B-1}\Big)\Big].
$$

In the case $\lambda=1$ and $\delta=0$, Corollary 1 yields:

Corollary 2. *Let $f(z)=z+\sum\limits_{k=2}^{\infty} a_k z^k \in A(1)$. If*

$$
f'(z) \prec \frac{1+Az}{1+Bz}\,, \quad z\in E
$$

then

$$
\frac{f(z)}{z} \prec q(z) = \begin{cases} \dfrac{A}{B}+\Big(1-\dfrac{A}{B}\Big)\dfrac{\ln(1+Bz)}{Bz}\,, & B\neq 0 \\[2ex] 1+\dfrac{A}{2}z\,, & B=0 \end{cases} \tag{2.17}
$$

and the $q(z)$ is the best dominant. Further,

$$
\Re\Big(\frac{f(z)}{z}\Big) \geq \begin{cases} \dfrac{A}{B}-\Big(1-\dfrac{A}{B}\Big)\dfrac{\ln(1-B)}{B}\,, & B\neq 0 \\[2ex] 1-\dfrac{A}{2}\,, & B=0\,. \end{cases} \tag{2.18}
$$

The function $q(z)$ defined above shows that the estimate (2.18) is sharp.

The proof of Corollary 2 follows by letting $\delta = 0$ and $\lambda = 1$ in Corollary 1 followed by using the identities (2.6) and (2.7). This result was also obtained by Obradovic [16].

If we put $A = 1 - 2\alpha$, $0 \le \alpha < 1$ and $B = -1$ in (2.18) of Corollary 2, we obtain

Corollary 3. *Let $f(z) \in A(1)$ and $\Re(f'(z)) > \alpha$, $0 \le \alpha < 1$. Then*

$$\Re\left(\frac{f(z)}{z}\right) \ge (2\alpha - 1) + 2(1-\alpha)\ln 2\,,$$

and this result is sharp.

This improves an earlier result due to Owa and Obradovic [17] where in they proved that if $f(z) \in A(1)$ satisfies $\Re(f'(z)) > \alpha$, for $0 \le \alpha < 1$ and $z \in E$, then

$$\Re\left(\frac{f(z)}{z}\right) > \frac{2\alpha+1}{3}\,.$$

Corollary 4. *Let $f(z) \in A(1)$ and*

$$f'(z) + \frac{1}{2} z f''(z) \prec \frac{1+Az}{1+Bz}\,, \quad z \in E$$

then

$$f'(z) \prec q(z) = \begin{cases} \dfrac{A}{B} - \dfrac{2}{B^2}\left(1 - \dfrac{A}{B}\right)\left[\dfrac{\ln(1+Bz) - Bz}{z^2}\right], & B \neq 0 \\[2ex] 1 + \dfrac{2A}{3} z\,, & B = 0 \end{cases} \tag{2.19}$$

and the right hand side of (2.19) *is the best dominant. Further more,*

$$\Re(f'(z)) > \begin{cases} \dfrac{A}{B} - \dfrac{2}{B^2}\left(1 - \dfrac{A}{B}\right)[\ln(1-B) + B]\,, & B \neq 0 \\[2ex] 1 - \dfrac{2}{3} A\,, & B = 0\,. \end{cases} \tag{2.20}$$

The function $q(z)$ in (2.19) shows that the estimate (2.20) is sharp.

The proof of the above Corollary is obtain by setting $\delta = 1$ and $\lambda = 1$ in Corollary 1 and by using the identities (2.7) and (2.8) in the resulting equation.

COROLLARY 5. *Let* $f(z) \in A(1)$ *and*

$$f'(z) + \frac{1}{3} z f''(z) \prec \frac{1+Az}{1+Bz}, \quad z \in E$$

then

(2.21)
$$f'(z) \prec q(z) = \begin{cases} \dfrac{A}{B} + \dfrac{3}{(Bz)^3}\left(1 - \dfrac{A}{B}\right)\left[\ln(1+Bz) - (Bz) + \dfrac{(Bz)^2}{2}\right], & B \neq 0 \\ 1 + \dfrac{3A}{4} z, & B = 0 \end{cases}$$

and the right hand side of (2.21) *is the best dominant. Further more,*

(2.22)
$$\Re(f'(z)) > \begin{cases} \dfrac{A}{B} - \dfrac{3}{B^3}\left(1 - \dfrac{A}{B}\right)\left[\ln(1-B) + B - \dfrac{B^2}{2}\right], & B \neq 0 \\ 1 - \dfrac{3A}{4}, & B = 0\,. \end{cases}$$

The result (2.21) is sharp.

The proof of Corollary 5 is obtained by taking $\delta = 1$ and $\lambda = \dfrac{2}{3}$ in Corollary 1 followed by applying the identities (2.8) and (2.9) in the resulting equation.

REMARK 1. In view of Corollary 2, we note that if $f(z) \in P'(A', B)$, where $A' = (B\ln(1-B))/(B+\ln(1-B))$, $B \neq 0$, then $\Re(f(z)/z) > 0$ in E. From this it follows that if $\Re(f'(z)) > (\log 4 - 1)/(\log 4 - 2) = -0.62944$, then $\Re(f(z)/z) > 0$ in E.

REMARK 2. For $A = (1 - 2\alpha)$, $0 \leq \alpha < 1$ and $B = -1$, Corollary 4 gives the corresponding result obtained by OWA, OBRADOVIC and NUNOKAWA [18].

REMARK 3. We observe from Corollary 4 that if for $B \neq 0$

$$f'(z) + \frac{1}{2} z f''(z) \prec \frac{1 + A''z}{1 + Bz}, z \in E$$

where $A'' = [2B(B + \ln(1 - B))]/2[B + \ln(1 - B)] + B^2$, then $\Re(f'(z)) > 0$ in E and hence $f(z)$ is univalent in E. This gives a new criteria for univalency. Taking $B = -1$, we note that if

$$\Re\Big(f'(z) + \frac{1}{2} z f''(z)\Big) > \frac{4\ln 2 - 3}{4\ln 2 - 2} = -0.2943$$

then $\Re(f'(z)) > 0$ in E.

REMARK 4. It is shown by SAITOH [21] that for $\lambda > 0$ and $0 \leq \alpha < 1$,

$$\Re(f'(z) + \lambda z f''(z)) > \alpha \text{ implies } \Re(f'(z)) > (2\alpha + \lambda)/(2 + \lambda). \tag{2.23}$$

However, (2.20) and (2.22) shows that $(A = 1 - 2\alpha, B = -1)$

$$\Re\Big(f'(z) + \frac{1}{2} z f''(z)\Big) > \alpha \implies \Re(f'(z)) > 3 - 2\alpha - 4(1 - \alpha)\ln 2$$

and

$$\Re\Big(f'(z) + \frac{1}{3} z f''(z)\Big) > \alpha \implies \Re(f'(z)) > (2\alpha - 1) + 3(1 - \alpha)(2\ln 2 - 1).$$

Comparing these results with the result of SAITOH (c.f. (2.23)), we easily conclude that the result (2.23) is not best possible. In that sense, our results contained in Corollary 4 and Corollary 5 is an improvement of the result (2.23).

We next prove

THEOREM 2. *Let $f(z) \in A(p)$ for $p \geq 2$. If for $-1 \leq B < 1$, $B \neq 0$*

$$\frac{f^{(p)}(z)}{p!} \prec \frac{1 + A'z}{1 + Bz}, \quad z \in E$$

where $A' = (B\ln(1 - B))/(B + \ln(1 - B))$, then $f(z)$ is p-valent in E.

Proof. Let $p(z) = f^{(p-1)}(z)/p!z$. Then $p(z)$ is analytic in E with $p(0) = 1$. An easy calculation yields

$$p(z) + zp'(z) = \frac{f^{(p)}(z)}{p!} \prec \frac{1 + A'z}{1 + Bz}\,.$$

Thus an application of Lemma 1 for $\gamma = 1$ gives

$$\begin{aligned} p(z) \prec q(z) &= z^{-1}\int_0^z \left(\frac{1 + A't}{1 + Bt}\right) dt = \\ &= \frac{A'}{B} + \left(1 - \frac{A'}{B}\right)\frac{\ln(1 + Bz)}{Bz}\,, \end{aligned} \tag{2.24}$$

by (2.6) and (2.7).

Since the right hand side of (2.24) has real coefficients and its image is convex with respect to the real axis, it follows from (2.24) that

$$\Re(p(z)) = \frac{1}{p!}\Re\left(\frac{f^{(p-1)}(z)}{z}\right) > \frac{A'}{B} - \frac{1}{B}\left(1 - \frac{A'}{B}\right)\ln(1 - B) = 0\,.$$

This shows that $\Re(\frac{f^{(p-1)}(z)}{z}) > 0$ in E which is equivalent to

$$\Re\left(z\frac{f^{(p-1)}(z)}{z^2}\right) > 0 \text{ in } E\,.$$

Since $g(z) = z^2$ is 2-valently starlike in E, in view of Lemma 2, we have that $f(z)$ is p-valent in E.

Remark. In the case $B = -1$, we get $A' = \ln 2/(1 - \ln 2)$ so that Theorem 2 gives the corresponding result obtained by Nunokawa [14].

By a similar method to that used in Theorem 2, we may obtain the following result.

Theorem 3. *Let $f(z) \in A(p)$ for $p \geq 1$. If for $-1 \leq B < A \leq 1$*

$$\frac{f^{(p)}(z)}{p!} \prec \frac{1 + Az}{1 + Bz}\,, \quad z \in E$$

then

$$\frac{f^{(p-1)}(z)}{p!z} \prec q(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)\dfrac{\ln(1+Bz)}{Bz}\,, & B \neq 0 \\[2ex] 1 + \dfrac{A}{2}z\,, & B = 0 \end{cases}$$

and the function $q(z)$ *is the best dominant. Further more,*

$$\Re\left(\frac{f^{(p-1)}(z)}{z}\right) \geq \begin{cases} p!\left[\dfrac{A}{B} - \dfrac{1}{B}\left(1 - \dfrac{A}{B}\right)\ln(1-B)\right], & B \neq 0 \\[2ex] p!\left(1 - \dfrac{A}{2}\right), & B = 0\,. \end{cases}$$

For $A = 1$ and $B = -1$, the above theorem shows that if $f(z) \in A(p)$ and $\Re(f^{(p)}(z)) > 0$ in E, then $\Re[f^{(p-1)}(z)/z] > p!(2\ln 2 - 1)$. This improves a result due to Saiton [21], who proved that if $f(z) \in A(p)$ and $\Re(f^{(p)}(z)) > 0$ in E then $\Re[f^{(p-1)}(z)/z] > p!/3$.

3 – Integral transforms

In [9], Libera defined the integral transforms of $f(z) \in A(1)$ by

$$F_1(z) = \frac{2}{z}\int_0^z f(t)dt$$

and proved that class S^*(resp. K) is preserved under the transformation $F_1(z)$. Bernardi [4], however, showed that the classes S^* and K are also preserved under the more general transformation

$$F_c(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt\,, \quad c > -1\,.$$

This result is then extended for the classes $S^*(\alpha)$ and $K(\alpha)(0 \leq \alpha < 1)$ by Bajpai and Srivastava [3]. In this section we consider the following integral transform

$$F_c(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt\,, \tag{3.1}$$

where $f(z) \in A(p)$ and $c+p>0$. We prove the following theorem.

THEOREM 4. *Let δ be a non-negative real number and c be a real number such that $c+p>0$. If $f(z) \in A(p)$ satisfies*

$$\frac{D^{\delta+p-1}f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \quad z \in E, \quad -1 \le B < A \le 1$$

then

$$\frac{D^{\delta+p-1}F_c(z)}{z^p} \prec q(z) \prec \frac{1+Az}{1+Bz} \tag{3.2}$$

where $F_c(z)$ is defined by (3.1) *and $q(z)$ is given by*

$$q(z) = (1+Bz)^{-1}\Big[F\Big(1,1;c+p+1;\frac{Bz}{Bz+1}\Big)+ \\ + \frac{(c+p)Az}{c+p+1}F\Big(1,1;c+p+2;\frac{Bz}{Bz+1}\Big)\Big].$$

Further more,

$$\Re\Big\{\frac{D^{\delta+p-1}F_c(z)}{z^p}\Big\} \ge (1-B)^{-1}\Big[F\Big(1,1;c+p+1;\frac{B}{B-1}\Big)- \\ - \frac{(c+p)A}{c+p+1}F\Big(1,1;c+p+2;\frac{B}{B-1}\Big)\Big]. \tag{3.3}$$

PROOF. Since $F_c(z) = \sum\limits_{k=p}^{\infty}\Big(\frac{c+p}{c+k}\Big)z^k * f(z)$ and

$$D^{\delta+p-1}f(z) = \frac{z^p}{(1-z)^{\delta+p}} * f(z) = \\ = z^p + \sum_{k=1}^{\infty}\frac{\Gamma(p+\delta+k)a_{k+p}z^{p+k}}{\Gamma(\delta+p)k!}$$

a simple calculation gives

$$z(D^{\delta+p-1}F_c(z))' = (c+p)D^{\delta+p-1}f(z) - cD^{\delta+p-1}F_c(z)\,. \tag{3.4}$$

Let $p(z) = D^{\delta+p-1}F_c(z)/z^p$. Then $p(z)$ is analytic in E with $p(0) = 1$. In view of (3.4), we have

$$p(z) + \frac{zp'(z)}{c+p} = \frac{D^{\delta+p-1}F_c(z)}{z^p} \prec \frac{1+Az}{1+Bz}\,, \quad z \in E$$

which with the aid of Lemma 1 for $\gamma = c + p$ yields

$$\frac{D^{\delta+p-1}f(z)}{z^p} \prec q(z) = (c+p)z^{-(c+p)} \int_0^z \frac{t^{c+p-1}(1+At)dt}{(1+Bt)} \prec \frac{1+Az}{1+Bz}\,, \quad z \in E\,. \tag{3.5}$$

Applying the identities (2.2) and (2.3) to the right hand side of (3.5), we get

$$q(z) = (1+Bz)^{-1}\Big[F\Big(1,1;c+p+1;\frac{Bz}{Bz+1}\Big) + \frac{(c+p)Az}{c+p+1}F\Big(1,1;c+p+2;\frac{Bz}{Bz+1}\Big)\Big]\,.$$

This proves (3.2). The estimate (3.3) can be proved on the same lines as that of (2.11). Hence the theorem.

Taking $\delta = 0$, $p = 1$, $A = (1-2\alpha)$, $0 \le \alpha < 1$ and $B = -1$ in Theorem 4, we have the following.

Corollary 6. *Let $f(z) \in A(1)$. If*

$$\Re\Big(\frac{f(z)}{z}\Big) > \alpha, \quad 0 \le \alpha < 1$$

then

$$\Re\Big(\frac{c+1}{z^{c+1}}\int_0^z t^{c-1}f(t)dt\Big) > \rho\,,$$

where $\rho = \Big[F\Big(1,1;c+2;\frac{1}{2}\Big) + \frac{(c+1)(2\alpha-1)}{c+2}F\Big(1,1;c+3;\frac{1}{2}\Big)\Big]\Big/2\,.$

In [15], it is proved by Obradovic that if $f(z) \in A(1)$ and $\Re(f(z)/z) > \alpha$ for $0 \le \alpha < 1$ and $z \in E$, then

$$\Re\left(\frac{c+1}{z^{c+1}}\int_0^z t^{c-1}f(t)dt\right) > \alpha + \frac{1-\alpha}{3+2c}, \quad c > -1 \tag{3.6}$$

from which we get for $c = 1$

$$\Re\left(\frac{2}{z^2}\int_0^z f(t)dt\right) = \Re\left(\frac{F_1(z)}{z}\right) > \frac{4\alpha+1}{5}.$$

However, our result (Corollary 6) shows that if $f(z) \in A(1)$ satisfies $\Re(f(z)/z) > \alpha$, $0 \le \alpha < 1$, then

$$\Re\left(\frac{F_1(z)}{z}\right) > (4\ln 2 - 2)\alpha + (3 - 4\ln 2).$$

This shows that the result (3.6) obtained by Obradovic is not best possible. In this sense the result of our Theorem 4 is an improvement of the result (3.6) given in [15].

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Lavoro pervenuto alla redazione il 7 ottobre 1992
ed accettato per la pubblicazione il 2 marzo 94

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