An application of differential subordinations

J. PATEL – S. ROUT

Riassunto: Usando la tecnica di subordinazione differenziale si migliorano alcuni risultati classici della teoria delle funzioni univalenti. Si ottengono anche alcuni criteri di univalenza per le funzioni definite nel disco unitario.

Abstract: By using the method of Briot-Bouquet differential subordinations, we prove and sharpen some classical results in univalent function theory. These also lead to some criteria for univalency in the unit disc.

1 – Introduction

Let \( A(p) \) denote the class of functions of form

\[
(1.1) \quad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \text{ is a fixed integer } \geq 1)
\]

which are analytic in the unit disc \( E = \{ z : |z| < 1 \} \). Let \( S, S^*(\alpha) \) and \( K(\alpha) \) (\( 0 \leq \alpha < 1 \)) denote subclasses of functions in \( A(1) \) which are respectively univalent, starlike of order \( \alpha \) and convex of order \( \alpha \). We denote \( S^*(0) = S, K(0) = K \). For given arbitrary numbers \( A, B \) satisfying \( -1 \leq B < A \leq 1 \), we denote by \( P(A, B) \), the class of functions

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of the form

\begin{equation}
    p(z) = 1 + p_1 z + p_2 z^2 + \ldots
\end{equation}

which are analytic in \( E \) and satisfy the condition

\[ p(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in E \]

where “\( \prec \)” stands for subordination. Geometrically, this means that the image of \( E \) under \( p(z) \) is inside the open disc centred on the real axis whose diameter has end points \((1 - A)/(1 - B)\) and \((1 + A)/(1 + B)\). From this we conclude that \( p(z) \) has a positive real part and hence univalent in \( E \) [19]. This class \( P(A, B) \) was investigated by Janowski [8]. We say that a function \( f(z) \in A(1) \) is said to be in the class \( P'(A, B, \alpha) \) if and only if \( f'(z) \in P(A, B) \). It is clear that \( P'(1 - 2\alpha, -1) \equiv P'(\alpha) \) is the class of functions \( f(z) \in A(1) \) for which \( \Re(f'(z)) > \alpha, 0 \leq \alpha < 1 \).

In [6], Goel and Sohi have studied the class of functions \( f(z) \in A(1) \) satisfying

\begin{equation}
    \Re\left\{ \frac{D^{n+1}f(z)}{z} \right\} > \alpha, \quad z \in E, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}
\end{equation}

where \( 0 \leq \alpha < 1 \) and

\[ D^n f(z) = \frac{z}{(1 - z)^{n+1}} \ast f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}. \]

(Here “\( \ast \)” means the Hadamard product of two analytic functions).

Aouf [2], further generalized this class of functions by introducing the class \( V_n(A, B, \alpha) \). Thus, a function \( f(z) \in A(1) \) is said to be in the class \( V_n(A, B, \alpha) \) if

\[ \frac{D^{n+1}f(z)}{z} \prec \frac{1 + \{(1 - \alpha)A + \alpha B\}z}{1 + Bz}, \quad z \in E, \quad n \in \mathbb{N}_0 \]

for \(-1 \leq B < A \leq 1\) and \( 0 \leq \alpha < 1 \). He showed that \( V_{n+1}(A, B, \alpha) \subset V_n(A, B, \alpha) \) for \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \).
Recently, many of the classical results in univalent function theory have been improved and sharpened by the powerful technique of Briot-Bouquet differential subordination [5, 10, 12]. We recall that a function $p(z)$ analytic in $E$ with a power series of the form (1.2) is said to satisfy Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z), \quad z \in E$$  (1.4)

for $\beta$ and $\gamma$ complex constants and $h(z)$ a complex function with $h(0) = 1$, $\Re(\beta h(z) + \gamma) > 0$ in $E$. It is known that [5] if

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (q(0) = 1)$$  (1.5)

has the univalent solution $q(z)$ in $E$, then

$$p(z) < q(z) < h(z)$$

and $q(z)$ is the best dominant of the differential subordination (1.5).

We note that the univalent function $q(z)$ is said to be a dominant of the differential subordination (1.4) if $p(z) < q(z)$ for all $p(z)$ satisfying (1.4). If $\tilde{q}(z)$ is a dominant of (1.4) and $\tilde{q}(z) < q(z)$ for all dominants $q(z)$ of (1.4), then $\tilde{q}(z)$ is said to be the best dominant of (1.4). We remark that the best dominant is unique up to a rotation of $E$. More results on differential subordination can be found in [11].

In this article, we propose to give some more applications of Briot-Bouquet differential subordination which would not only improve and sharpen many of the earlier results contained in [13, 16, 17, 18, 20, 21], but would also give rise to a number of new results for other subclasses as well. This is accomplished by introducing and studying a more general class $T_{\delta,\lambda}(p; A, B)$. Our results also generalize the work of Ponnusamy and Juneja [20], Owa, Obradovic and Nunokawa [18], Obradovic [16] and Nunokawa [14].
2 – Differential subordinations

We now introduce the class $T_{\delta,\lambda}(p; A, B)$ as follows:

Let $A, B, \lambda$ and $\delta$ be fixed real numbers such that $-1 \leq B < A \leq 1$, $\lambda \geq 0$ and $\delta \geq 0$. A function $f(z) \in A(p)$ is said to be in the class $T_{\delta,\lambda}(p; A, B)$ if it satisfies

$$(2.1) \quad J_{\delta,p}(f; \lambda) \prec 1 + Az + Bz, \quad z \in E$$

where

$$J_{\delta,p}(f; \lambda) = (1 - \lambda) \frac{D^{\delta+p-1} f(z)}{z^p} + \lambda \frac{D^{\delta+p} f(z)}{z^p}$$

and

$$D^{\delta+p-1} f(z) = \frac{z^p}{(1 - z)^{\delta+p}} * f(z).$$

It is readily seen that $T_{0,1}(1; 0, -1)$ is the class considered by GOEL and SOHI [6] whereas $T_{0,\lambda}(1; 1 - 2\alpha, -1)(0 \leq \alpha < 1)$ is the class studied by OWA, OBRADOVIC and NUNOKAWA [18]. Further, it is clear that $T_{0,1}(1; A, B) = P'(A, B)$ is the class studied by OBRADOVIC [16]. We denote $T_{\delta,\lambda}(1; A, B)$ by $T_{\delta,\lambda}(A, B)$.

To establish our main results we need the following lemmas.

**Lemma 1.** ([7]). If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in $E$ and $h(z)$ is a convex function in $E$ with $h(0) = 1$ and $\gamma$ is a complex constant such that $\Re(\gamma) > 0$, then

$$(2.1) \quad p(z) + \frac{z p'(z)}{\gamma} \prec h(z)$$

implies

$$p(z) \prec \gamma z^{-\gamma} \int_0^z t^{\gamma-1} h(t) dt = q(z) \prec h(z)$$

and $q(z)$ is the best dominant.
The following lemma is due to Nunokawa [13, Theorem 8].

**Lemma 2.** Let $f(z) = z^p + \sum_{k=p+1}^\infty a_k z^k$ be analytic in $E$. If there exists a $(p - m + 1)$-valent starlike function $g(z) = z^{p - m + 1} + \sum_{k=p-m+2}^\infty a_k z^k$ in $E$ such that
\[ \Re \left\{ \frac{zf^{(m)}(z)}{g(z)} \right\} > 0, \quad z \in E \]
then $f(z)$ is $p$-valent in $E$.

For $a, b, c$ real numbers other than $0, -1, -2, \ldots$, the hypergeometric series
\[ F(a, b; c; z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a + 1)b(b + 1)}{1 \cdot 2 \cdot c(c + 1)} z^2 + \ldots \]
represents an analytic function in $E$ [1, p. 556]. The following identities are well known [1, p. 556-558].

**Lemma 3.** For $a, b, c$ real numbers other than $0, -1, -2, \ldots$, and $c > b > 0$, we have

\begin{align*}
(2.2) \quad & \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b) \cdot \Gamma(c-b)}{\Gamma(c)} F(a, b; c; z) \\
(2.3) \quad & F(a, b; c; z) = (1 - z)^{-a} F\left(a, c - b; c; \frac{z}{z - 1}\right) \\
(2.4) \quad & F(1, 1; 2; z) = -z^{-1} \ln(1 - z) \\
(2.5) \quad & c(c-1)(z-1)F(a, b; c-1; z) + c[c-1-(2c-a-b-1)z]F(a, b; c; z) + \\
& \quad + (c-a)(c-b)zF(a, b; c+1; z) = 0.
\end{align*}
**Lemma 4.** For any real number $d \neq 0$, we have

\begin{align*}
(2.6) \quad F\left(1, 1; 2; \frac{dz}{dz+1}\right) &= \frac{(1 + dz) \ln(1 + dz)}{dz} \\
(2.7) \quad F\left(1, 1; 3; \frac{dz}{dz+1}\right) &= 2\left(1 + dz\right) \left[1 - \frac{\ln(1 + dz)}{dz}\right] \\
(2.8) \quad F\left(1, 1; 4; \frac{dz}{dz+1}\right) &= \frac{3(1 + dz)}{2(bz)^3} \left[2\ln(1 + dz) - dz(2 - dz)\right] \\
(2.9) \quad F\left(1, 1; 5; \frac{dz}{dz+1}\right) &= \frac{2(1 + dz)}{(dz)^3} \left[\frac{2(bz)^2 - 3bz + 6}{3} - \frac{2\ln(1 + dz)}{dz}\right].
\end{align*}

The proof of Lemma 4 follows from the identities (2.4) and (2.5).

**Theorem 1.** Let the function $f(z)$ defined by (1.1) be in the class $T_{\delta, \lambda}(p; A, B)$. If $\delta + p > \lambda > 0$ then

\begin{equation}
(2.10) \quad \frac{D^{\delta+p-1}f(z)}{z^p} < q(z) < \frac{1 + Az}{1 + Bz}, \quad z \in E
\end{equation}

where

\[
q(z) = (1 + Bz)^{-1} \left[F\left(1, 1; 1 + \frac{\delta + p}{\lambda}; \frac{Bz}{Bz+1}\right) + \frac{(\delta + p)Az}{\delta + p + \lambda} F\left(1, 1; 2 + \frac{\delta + p}{\lambda} \frac{Bz}{Bz+1}\right)\right]
\]

and $q(z)$ is the best dominant. Further more,

\begin{equation}
(2.11) \quad \Re\left\{\frac{D^{\delta+p-1}f(z)}{z^p}\right\} > \rho, \text{ where }
\end{equation}

\[
\rho = (1-B)^{-1} \left[F\left(1, 1; 1 + \frac{\delta + p}{\lambda}; \frac{B}{B-1}\right) - \frac{(\delta + p)A}{\delta + p + \lambda} F\left(1, 1; 2 + \frac{\delta + p}{\lambda}; \frac{B}{B-1}\right)\right].
\]
Proof. Since for $\delta \geq 0$,

$$D^\delta f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(\delta + p + k)a_{p+k}z^{p+k}}{\Gamma(\delta + p)k!},$$

we have

$$z(D^{\delta+p-1}f(z))' = (\delta + p)D^{\delta+p}f(z) - \delta D^{\delta+p-1}f(z).$$

Let $p(z) = D^{\delta+p-1}f(z)/z^p$. Then $p(z)$ is analytic in $E$ with $p(0) = 1$ and as $f(z) \in T_{\delta,\lambda}(p; A, B)$, (2.1) coupled with (2.12) yields

$$p(z) + \left(\frac{\lambda}{\delta + p}\right) z p'(z) = J_{\delta,p}(f; \lambda) \prec \frac{1 + Az}{1 + Bz}, \quad z \in E.$$

Thus, by using Lemma 1 for $\gamma = (\delta + p)/\lambda$, we deduce that

$$\frac{D^{\delta+p-1}f(z)}{z^p} \prec \left(\frac{\delta + p}{\lambda}\right) z^{-\frac{\delta+p}{\lambda}} \frac{\int_0^z t^{\frac{\delta+p}{\lambda}-1}(1 + At)dt}{(1 + Bt)} = q(z), \text{ say.}$$

Now the function $q(z)$ can be rewritten as

$$q(z) = \left(\frac{\delta + p}{\lambda}\right) \frac{1}{\lambda} \int_0^1 s^{\frac{\delta+p}{\lambda}-1}(1 + Asz)ds$$

$$= \left(\frac{\delta + p}{\lambda}\right) \frac{1}{\lambda} \int_0^1 s^{\frac{\delta+p}{\lambda}-1}(1 + Bsz)^{-1}ds$$

$$+ A \left(\frac{\delta + p}{\lambda}\right) z \int_0^1 s^{\frac{\delta+p}{\lambda}}(1 + Bsz)^{-1}ds =$$

$$= (1 + Bz)^{-1} \left[ F\left(1, 1; 1 + \frac{\delta + p}{\lambda}; \frac{Bz}{Bz + 1}\right) + \frac{(\delta + p)Az}{\delta + p + \lambda} F\left(1, 1; 2 + \frac{\delta + p}{\lambda}; \frac{Bz}{Bz + 1}\right) \right],$$

by using the identities (2.2) and (2.3). This completes the proof of (2.10).
Next to prove (2.11), it suffices to show that

\[(2.13) \quad \inf_{|z|<1} \{q(z)\} = q(-1).\]

Since for \(-1 \leq B < A \leq 1\), \((1 + Az)/(1 + Bz)\) is convex (univalent) in \(E\), we have for \(|z| \leq r < 1\),

\[(2.14) \quad \Re \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}.\]

Setting

\[g(s, z) = \frac{1 + Asz}{1 + Bs z}, \quad 0 \leq s \leq 1, \quad z \in E\]

and

\[d\mu(s) = s^{\frac{\delta + p - 1}{\lambda}} \frac{1}{\lambda} ds\]

which is a positive measure on \([0, 1]\), we get

\[q(z) = \int_{0}^{1} g(s, z) d\mu(s)\]

so that

\[\Re \{q(z)\} = \int_{0}^{1} \Re \left( \frac{1 + Asz}{1 + Bs z} \right) d\mu(s) \geq \int_{0}^{1} \left( \frac{1 - Asr}{1 - Bs r} \right) d\mu(s) = q(-r), \quad |z| \leq r < 1.\]

Now, letting \(r \to 1^-\) in the above inequality, we obtain

\[\Re \{q(z)\} \geq q(-1), \quad z \in E\]

which implies (2.13). Hence the theorem.

Putting \(p = 1\) in the above Theorem, we obtain
**Corollary 1.** Let \( f(z) \in T_{\delta,\lambda}(A, B) \) and \( \delta + 1 > \lambda > 0 \). Then

\[
\frac{D^\delta f(z)}{z} \prec q(z) = (1 + Bz)^{-1} \left[ F\left(1, 1; 1 + \frac{\delta + 1}{\lambda}; \frac{Bz}{Bz + 1}\right) + \frac{(\delta + 1)Az}{\delta + 1 + \lambda} F\left(1, 1; 2 + \frac{\delta + 1}{\lambda}; \frac{Bz}{Bz + 1}\right) \right] \prec \frac{1 + Az}{1 + Bz}, \quad z \in E
\]

and \( q(z) \) is the best dominant. Further more,

\[
\Re\left( \frac{D^\delta f(z)}{z} \right) > \rho, \quad \text{where}
\]

\[
\rho = (1 - B)^{-1} \left[ F\left(1, 1; 1 + \frac{\delta + 1}{\lambda}; \frac{B}{B - 1}\right) - \frac{(\delta + 1)A}{\delta + 1 + \lambda} F\left(1, 1; 2 + \frac{\delta + 1}{\lambda}; \frac{B}{B - 1}\right) \right].
\]

In the case \( \lambda = 1 \) and \( \delta = 0 \), Corollary 1 yields:

**Corollary 2.** Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A(1) \). If

\[
f'(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in E
\]

then

\[
\frac{f(z)}{z} \prec q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right) \frac{\ln(1 + Bz)}{Bz}, & B \neq 0 \\ 1 + \frac{A}{2} z, & B = 0 \end{cases}
\]

and the \( q(z) \) is the best dominant. Further,

\[
\Re\left( \frac{f(z)}{z} \right) \geq \begin{cases} \frac{A}{B} - \left(1 - \frac{A}{B}\right) \frac{\ln(1 - B)}{B}, & B \neq 0 \\ 1 - \frac{A}{2}, & B = 0 \end{cases}
\]
The function $q(z)$ defined above shows that the estimate (2.18) is sharp.

The proof of Corollary 2 follows by letting $\delta = 0$ and $\lambda = 1$ in Corollary 1 followed by using the identities (2.6) and (2.7). This result was also obtained by Obradovic [16].

If we put $A = 1 - 2\alpha$, $0 \leq \alpha < 1$ and $B = -1$ in (2.18) of Corollary 2, we obtain

**Corollary 3.** Let $f(z) \in A(1)$ and $\Re(f'(z)) > \alpha$, $0 \leq \alpha < 1$. Then

$$\Re\left(\frac{f(z)}{z}\right) \geq (2\alpha - 1) + 2(1 - \alpha)\ln 2,$$

and this result is sharp.

This improves an earlier result due to Owa and Obradovic [17] where in they proved that if $f(z) \in A(1)$ satisfies $\Re(f'(z)) > \alpha$, for $0 \leq \alpha < 1$ and $z \in E$, then

$$\Re\left(\frac{f(z)}{z}\right) > \frac{2\alpha + 1}{3}.$$

**Corollary 4.** Let $f(z) \in A(1)$ and

$$f'(z) + \frac{1}{2}zf''(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in E$$

then

(2.19) $f'(z) \prec q(z) = \begin{cases} \frac{A}{B} - \frac{2}{B^2}\left(1 - \frac{A}{B}\right)\left[\ln(1 + Bz) - Bz\right], & B \neq 0 \\ 1 + \frac{2A}{3}z, & B = 0 \end{cases}$

and the right hand side of (2.19) is the best dominant. Further more,

(2.20) $\Re(f'(z)) > \begin{cases} \frac{A}{B} - \frac{2}{B^2}\left(1 - \frac{A}{B}\right)[\ln(1 - B) + B], & B \neq 0 \\ 1 - \frac{2}{3}A, & B = 0 \end{cases}$.
The function $q(z)$ in (2.19) shows that the estimate (2.20) is sharp.

The proof of the above Corollary is obtained by setting $\delta = 1$ and $\lambda = 1$ in Corollary 1 and by using the identities (2.7) and (2.8) in the resulting equation.

**COROLLARY 5.** Let $f(z) \in A(1)$ and

$$f'(z) + \frac{1}{3}zf''(z) < \frac{1 + Az}{1 + Bz}, \quad z \in E$$

then

$$f'(z) < q(z) = \begin{cases} 
\frac{A}{B} + \frac{3}{(Bz)^3} \left(1 - \frac{A}{B}\right) \left[\ln(1 + Bz) - (Bz) + \frac{(Bz)^2}{2}\right], & B \neq 0 \\
1 + \frac{3A}{4}z, & B = 0
\end{cases}$$

and the right hand side of (2.21) is the best dominant. Furthermore,

$$\Re(f'(z)) > \begin{cases} 
\frac{A}{B} - \frac{3}{B^3} \left(1 - \frac{A}{B}\right) \left[\ln(1 - B) + B - \frac{B^2}{2}\right], & B \neq 0 \\
1 - \frac{3A}{4}, & B = 0
\end{cases}$$

The result (2.21) is sharp.

The proof of Corollary 5 is obtained by taking $\delta = 1$ and $\lambda = \frac{2}{3}$ in Corollary 1 followed by applying the identities (2.8) and (2.9) in the resulting equation.

**REMARK 1.** In view of Corollary 2, we note that if $f(z) \in P'(A', B)$, where $A' = (B \ln(1-B))/(B+\ln(1-B))$, $B \neq 0$, then $\Re(f(z)/z) > 0$ in $E$. From this it follows that if $\Re(f'(z)) > (\log 4 - 1)/(\log 4 - 2) = -0.62944$, then $\Re(f(z)/z) > 0$ in $E$.

**REMARK 2.** For $A = (1 - 2\alpha)$, $0 \leq \alpha < 1$ and $B = -1$, Corollary 4 gives the corresponding result obtained by OWA, OBRADOVIC and NUNOKAWA [18].
Remark 3. We observe from Corollary 4 that if for $B \neq 0$
\[ f'(z) + \frac{1}{2}zf''(z) \prec 1 + A''z \frac{1}{1+ Bz} , \quad z \in E \]
where \( A'' = \frac{[2B(B + \ln(1 - B))] / 2[B + \ln(1 - B)] + B^2} {B + \ln(1 - B)} \), then \( \Re(f'(z)) > 0 \)
in \( E \) and hence \( f(z) \) is univalent in \( E \). This gives a new criteria for univalency. Taking \( B = -1 \), we note that if
\[ \Re\left( f'(z) + \frac{1}{2}zf''(z) \right) > \frac{4\ln 2 - 3}{4\ln 2 - 2} = -0.2943 \]
then \( \Re(f'(z)) > 0 \) in \( E \).

Remark 4. It is shown by Saitoh [21] that for \( \lambda > 0 \) and \( 0 \leq \alpha < 1 \),
\[ (2.23) \quad \Re(f'(z) + \lambda zf''(z)) > \alpha \text{ implies } \Re(f'(z)) > \frac{(2\alpha + \lambda)}{(2 + \lambda)} . \]
However, (2.20) and (2.22) shows that \( (A = 1 - 2\alpha, B = -1) \)
\[ \Re\left( f'(z) + \frac{1}{2}zf''(z) \right) > \alpha \implies \Re(f'(z)) > 3 - 2\alpha - 4(1 - \alpha)\ln 2 \]
and
\[ \Re\left( f'(z) + \frac{1}{3}zf''(z) \right) > \alpha \implies \Re(f'(z)) > (2\alpha - 1) + 3(1 - \alpha)(2\ln 2 - 1) . \]
Comparing these results with the result of Saitoh (c.f. (2.23)), we easily conclude that the result (2.23) is not best possible. In that sense, our results contained in Corollary 4 and Corollary 5 is an improvement of the result (2.23).

We next prove

Theorem 2. Let \( f(z) \in A(p) \) for \( p \geq 2 \). If for \( -1 \leq B < 1 \), \( B \neq 0 \)
\[ \frac{f^{(p)}(z)}{p!} < 1 + A'z \frac{1}{1+ Bz} , \quad z \in E \]
where \( A' = \frac{(B\ln(1 - B))}{(B + \ln(1 - B))} \), then \( f(z) \) is \( p \)-valent in \( E \).
Proof. Let $p(z) = f^{(p-1)}(z)/p!z$. Then $p(z)$ is analytic in $E$ with $p(0) = 1$. An easy calculation yields

$$p(z) + zp'(z) = \frac{f^{(p)}(z)}{p!} \prec \frac{1 + A'z}{1 + Bz}.$$  

Thus an application of Lemma 1 for $\gamma = 1$ gives

$$p(z) \prec q(z) = z^{-1} \int_0^z \left( \frac{1 + A't}{1 + Bt} \right) dt =$$

$$= \frac{A'}{B} + \left( 1 - \frac{A'}{B} \right) \frac{\ln(1 + Bz)}{Bz},$$

by (2.6) and (2.7).

Since the right hand side of (2.24) has real coefficients and its image is convex with respect to the real axis, it follows from (2.24) that

$$\Re(p(z)) = \frac{1}{p!} \Re\left( \frac{f^{(p-1)}(z)}{z} \right) > \frac{A'}{B} - \frac{1}{B} \left( 1 - \frac{A'}{B} \right) \ln(1 - B) = 0.$$

This shows that $\Re\left( \frac{f^{(p-1)}(z)}{z} \right) > 0$ in $E$ which is equivalent to

$$\Re\left( z \frac{f^{(p-1)}(z)}{z^2} \right) > 0 \text{ in } E.$$  

Since $g(z) = z^2$ is 2-valently starlike in $E$, in view of Lemma 2, we have that $f(z)$ is $p$-valent in $E$.

Remark. In the case $B = -1$, we get $A' = \ln 2/(1 - \ln 2)$ so that Theorem 2 gives the corresponding result obtained by Nunokawa [14].

By a similar method to that used in Theorem 2, we may obtain the following result.

Theorem 3. Let $f(z) \in A(p)$ for $p \geq 1$. If for $-1 \leq B < A \leq 1$

$$\frac{f^{(p)}(z)}{p!} \prec \frac{1 + A z}{1 + B z}, \quad z \in E$$
then
\[
\frac{f^{(p-1)}(z)}{p!z} \prec q(z) = \begin{cases} 
\frac{A}{B} + \left(1 - \frac{A}{B}\right) \frac{\ln(1 + Bz)}{Bz}, & B \neq 0 \\
1 + \frac{A}{2}z, & B = 0
\end{cases}
\]

and the function \(q(z)\) is the best dominant. Further more,
\[
\Re\left(\frac{f^{(p-1)}(z)}{z}\right) \geq \begin{cases} 
\frac{p!}{z^2} \left[\frac{A}{B} - \frac{1}{B} \left(1 - \frac{A}{B}\right) \ln(1 - B)\right], & B \neq 0 \\
\frac{p!}{z^2} \left(1 - \frac{A}{2}\right), & B = 0.
\end{cases}
\]

For \(A = 1\) and \(B = -1\), the above theorem shows that if \(f(z) \in A(p)\) and \(\Re(f^{(p)}(z)) > 0\) in \(E\), then \(\Re[f^{(p-1)}(z)/z] > p!(2 \ln 2 - 1)\). This improves a result due to Saito [21], who proved that if \(f(z) \in A(p)\) and \(\Re(f^{(p)}(z)) > 0\) in \(E\) then \(\Re[f^{(p-1)}(z)/z] > p!/3\).

3 – Integral transforms

In [9], Libera defined the integral transforms of \(f(z) \in A(1)\) by
\[
F_1(z) = \frac{2}{z} \int_0^z f(t)dt
\]
and proved that class \(S^*\) (resp. \(K\)) is preserved under the transformation \(F_1(z)\). Bernardi [4], however, showed that the classes \(S^*\) and \(K\) are also preserved under the more general transformation
\[
F_c(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt , \quad c > -1.
\]

This result is then extended for the classes \(S^*(\alpha)\) and \(K(\alpha)(0 \leq \alpha < 1)\) by Bajpai and Srivastava [3]. In this section we consider the following integral transform
\[
(3.1) \quad F_c(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt ,
\]
where $f(z) \in A(p)$ and $c + p > 0$. We prove the following theorem.

**Theorem 4.** Let $\delta$ be a non-negative real number and $c$ be a real number such that $c + p > 0$. If $f(z) \in A(p)$ satisfies

$$\frac{D^{\delta+p-1}f(z)}{z^p} < \frac{1 + Az}{1 + Bz}, \quad z \in E, \quad -1 \leq B < A \leq 1$$

then

$$D^{\delta+p-1}F_c(z) \approx q(z) \approx \frac{1 + Az}{1 + Bz}$$

where $F_c(z)$ is defined by (3.1) and $q(z)$ is given by

$$q(z) = (1 + Bz)^{-1}\left[F\left(1, 1; c + p + 1; \frac{Bz}{Bz + 1}\right) + \frac{(c + p)Az}{c + p + 1}F\left(1, 1; c + p + 2; \frac{Bz}{Bz + 1}\right)\right].$$

Further more,

$$\Re\left\{\frac{D^{\delta+p-1}F_c(z)}{z^p}\right\} \geq (1 - B)^{-1}\left[F\left(1, 1; c + p + 1; \frac{B}{B - 1}\right) - \frac{(c + p)A}{c + p + 1}F\left(1, 1; c + p + 2; \frac{B}{B - 1}\right)\right].$$

**Proof.** Since $F_c(z) = \sum_{k=p}^{\infty} \left(\frac{c + p}{c + k}\right)z^k * f(z)$ and

$$D^{\delta+p-1}f(z) = \frac{z^p}{(1 - z)^{\delta+p}} * f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(p + \delta + k)a_{k+p}z^{p+k}}{\Gamma(\delta + p)k!}$$

a simple calculation gives

$$z(D^{\delta+p-1}F_c(z))' = (c + p)D^{\delta+p-1}f(z) - cD^{\delta+p-1}F_c(z).$$
Let \( p(z) = D^{\delta+p-1}F_c(z)/z^p \). Then \( p(z) \) is analytic in \( E \) with \( p(0) = 1 \). In view of (3.4), we have

\[
p(z) + \frac{zp'(z)}{c+p} = \frac{D^{\delta+p-1}F_c(z)}{z^p} < \frac{1 + Az}{1 + Bz}, \quad z \in E
\]

which with the aid of Lemma 1 for \( \gamma = c + p \) yields

\[
(3.5) \quad \frac{D^{\delta+p-1}f(z)}{z^p} < q(z) = (c+p)z^{-(c+p)} \int_0^z \frac{t^{c+p-1}(1 + At)dt}{(1 + Bt)} < \frac{1 + Az}{1 + Bz}, \quad z \in E.
\]

Applying the identities (2.2) and (2.3) to the right hand side of (3.5), we get

\[
q(z) = (1 + Bz)^{-1} \left[ F\left(1, 1; c + p + 1; \frac{Bz}{Bz + 1}\right) + \frac{(c + p)Az}{c + p + 1} F\left(1, 1; c + p + 2; \frac{Bz}{Bz + 1}\right) \right].
\]

This proves (3.2). The estimate (3.3) can be proved on the same lines as that of (2.11). Hence the theorem.

Taking \( \delta = 0 \), \( p = 1 \), \( A = (1 - 2\alpha) \), \( 0 \leq \alpha < 1 \) and \( B = -1 \) in Theorem 4, we have the following.

**Corollary 6.** Let \( f(z) \in A(1) \). If

\[
\Re\left(\frac{f(z)}{z}\right) > \alpha, \quad 0 \leq \alpha < 1
\]

then

\[
\Re\left(\frac{c + 1}{z^{c+1}} \int_0^z t^{c-1}f(t)dt\right) > \rho,
\]

where \( \rho = \left[ F\left(1, 1; c + 2; \frac{1}{2}\right) + \frac{(c + 1)(2\alpha - 1)}{c + 2} F\left(1, 1; c + 3; \frac{1}{2}\right) \right]/2. \)
In [15], it is proved by Obradovic that if \( f(z) \in A(1) \) and \( \Re(f(z)/z) > \alpha \) for \( 0 \leq \alpha < 1 \) and \( z \in E \), then

\[
(3.6) \quad \Re \left( \frac{c+1}{z^{c+1}} \int_0^z t^{c-1} f(t) dt \right) > \alpha + \frac{1-\alpha}{3+2c}, \quad c > -1
\]

from which we get for \( c = 1 \)

\[
\Re \left( \frac{2}{z^2} \int_0^z f(t) dt \right) = \Re \left( \frac{F_1(z)}{z} \right) > \frac{4\alpha + 1}{5}.
\]

However, our result (Corollary 6) shows that if \( f(z) \in A(1) \) satisfies \( \Re(f(z)/z) > \alpha \), \( 0 \leq \alpha < 1 \), then

\[
\Re \left( \frac{F_1(z)}{z} \right) > (4 \ln 2 - 2)\alpha + (3 - 4 \ln 2).
\]

This shows that the result (3.6) obtained by Obradovic is not best possible. In this sense the result of our Theorem 4 is an improvement of the result (3.6) given in [15].

REFERENCES


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INDIRIZZO DEGLI AUTORI:
J. Patel - S. Rout – Department of Mathematics – Utkal University – Vani Vihar – Bhubaneswar 751004, India