Rendiconti di Matematica, Serie VII
Volume 14, Roma (1994), 415-455

Non linear two-obstacle problems: Pointwise regularity

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Riassunto: *Vengono dimostrate l'esistenza, la regolaritá e il decadimento della energia di soluzioni di problemi a due ostacoli con operatore ellittico degenere e Hamiltoniana quadratica.*

Abstract: *Existence, pointwise regularity and energy decay, are shown for two-obstacle problems involving degenerate elliptic operators and quadratic Hamiltonian.*

1 – Introduction

The investigation of boundary regularity for solutions of the Dirichlet problem in an open region $D \subset \mathbb{R}^N$, $N \geq 3$, mainly carried out by H. Lebesgue around 1920, culminated in the celebrated Wiener Criterion in 1924. Indeed by relying on a fundamental notion of potential theory, namely that of capacity of an arbitrary set of $\mathbb{R}^N$, N.Wiener was able to characterize the boundary regular points as classically defined by H.Lebesgue in terms of an intrinsic condition regarding the neighbourhood of a given point x_0 of the boundary.

Key Words and Phrases: *Two-obstacle problems – Wiener criteria – Energy estimate*
A.M.S. Classification: 35J60 – 35J70 – 35J85

A boundary point x_0 of an open domain D, $D \subset \mathbb{R}^N$, $N \geq 2$ is termed regular if whenever $g(x)$ is a given continuous function on ∂D the corresponding generalized solution u of the Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D \end{cases} \tag{1.1}$$

is such that $u(x)$ has limit $g(x_0)$ as x approaches x_0 from D. In the form given to it by O.D. Kellog and F. Vasilesco (1929) and in terms of relative capacities the Wiener criterion can be written at a given point $x_0 \in \partial D$ as follows:

x_0 is regular if and only if the following Wiener integral diverges

$$\int_0^R \frac{\operatorname{cap}(B_\rho \cap D^c, B_{2\rho})}{\operatorname{cap}(B_\rho, B_{2\rho})} \frac{d\rho}{\rho} = +\infty \tag{1.2}$$

D^c is the complement of D in $\mathbb{R}^N$ and $B_\rho := B_\rho(x_0)$ the ball centered at x_0 of radius ρ.

The capacities involved are the usual harmonic capacities which were defined by Wiener for any arbitrary bounded subset of $\mathbb{R}^N$, as required in the application.

As pointed out by Wiener himself, the characterizations of regularity that had been previously given by Lebesgue and others, all suffered from the defect of involving the geometrical character of the boundary only in a very indirect and devious manner. The novelty of Wiener's approach was the use of the electrostatic notion of capacity to obtain a priori geometrical characterization of the regular points.

The structural nature of Wiener's criterion had been shown by W. Littman, G Stampacchia and H. Weinberger (1963) [20] and G. Stampacchia (1965) [25] who proved that a boundary point of a given domain D is regular with respect to the Laplace operator if and only if it is regular with respect to any arbitrary second order uniformly elliptic operator in divergence form:

$$L_0 = -\sum_{i,j=1}^N \partial_i(a_{ij}\partial_j) \tag{1.3}$$

where $\lambda|\xi|^2 \le \Sigma a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$ a.e. in D, with $\lambda > 0$.

V.G. Maz'ja (1963) brought into light what was implicit in Wiener's proof, namely the relationship between the rate of divergence of the integral (1.2) and the modulus of continuity at x_0.

Around the same period in the theory of variational inequalities H. Brezis, H.Lewy, G. Stampacchia and others initiated the study of the regularity of solutions of a class of free boundary problems, the so called unilateral problem involving a second order elliptic operator L as in (1.3). Of course Wiener' criterion cannot be applied directly to obstacle problems, because the free boundary, i.e., the boundary of the coincidence set E, is not known explicitly. However it is natural to ask whether a pointwise result holds that assure the continuity of u at a given point $x_0 \in \partial E$, under the assumption that Wiener's condition (1.2) with $D^c = E$ is satisfied at x_0 and the obstacle ψ is continuous at x_0 on E.

Many results concerning the continuity of a solution at a point x_0 in the neighborhood of which the obstacle is continuous have been proved (see L.A. Caffarelli, D. Kinderlehrer, J. Frehse, etc.). On the other hand obstacle problems present an interesting feature: discontinuous obstacles may still have continuous solutions. It is the free boundary that self adjusts to keep the solution continuous. This in particular was studied by U. Mosco [24] who developed a theory of pointwise regularity based on the *Wiener criterion* for local solutions of one obstacle problem relative to second order uniformly elliptic operator in divergence form. The contest is that of calculus of variation, combining methods from P.D.E and potential theory. These results were extended to the two obstacle problem in [10] and by [17] and [23] in a more general contest but with different methods.

The theory of two obstacle problems provides a unified framework to study the regular points both for Dirichlet problems and for unilateral one-obstacle problem. The point x_0, at which the regularity is tested, may indeed be a point of a fixed boundary, as in the Dirichlet problems, as well as a point of a free boundary, that is a point where the solution leaves one of the two obstacles. The "geometry" of the obstacles may even be more complicated, the two obstacles may "touch" each other at x_0, while both oscillate very much in an arbitrarily small neighborhood of the point, interpenetrating each other.

In this work we study the two obstacle problem relative to quasi linear degenerate elliptic operator of the form $Lu + H(u)$ and we extend previous results obtained in [4] and in [17] in a quite different context and by different methods.

Here L is a degenerate elliptic second order operator in divergence form with a "weight" $w(x)$ (in the ellipticity conditions) which is assumed to belong to the Muckenhoupts class A_2 see (2.5). The characterization of the regular boundary points for the corresponding linear (i.e. $H \equiv 0$) Dirichlet problem has been given by [12] as generalization of the Wiener criterion established in [20] for uniformly elliptic operators. In their theory an intrinsic notion of w-capacity associated with the weighted Sobolev space $H_0^{1,2}(\Omega, w)$ plays an important role. To emphasize the difference with respect to the usual $H_0^{1,2}$ capacity, let us point out that a single point in $\mathbb{R}^N$ may have for particular weights a positive capacity. Any such point is indeed regular for the Dirichlet problem. The theory of pointwise regularity for one obstacle problems in the linear case was developed by M. Biroli e U. Mosco in [5]. In the same framework we are going to consider the two obstacle problem associated to a quasilinear degenerate elliptic operator $L + H$ (see (2.8) and (3.24)).

In section 2 we introduce notations and some preliminary results, section 3 is devoted to show the existence of bounded weak solutions (th 3.1). We extend to the degenerate case an approximation procedure and some a priori estimates by means of "suitable" test functions used by L. Boccardo, F. Murat, J-P. Puel in the uniformly elliptic case (see [6]). Let us also remark that obstacle problems involving quasilinear uniformly elliptic operators have been extensively studied, let us just mention e.g. [3],[7],[15],[27] and let us refer to the more complete literature there quoted.

In section 4 we evaluate the modulus of continuity of a weak bounded solution in terms of the divergence of the Wiener integral (see (2.19)) of the relevant level sets of the obstacles (th 4.1). Moreover, if the obstacles admit a "regular" separating function (see (4.42)) then we prove that the *energy decay* also can be estimated in terms of the obstacles Wiener integrals (th 4.4).

Finally in section 5 we give the proofs of the intermediary results that are used to prove the main theorems of section 4.

2 – Notations and preliminaries

Let Ω be a bounded connected open set of $\mathbb{R}^N$. We suppose for simplicity that $\operatorname{diam}\Omega \le 1$.

Denote by $L^p(\Omega, w)$ the weighted Lebesgue class with norm

$$\|f\|_{0,p} = \Big(\int_\Omega |f(x)|^p w dx\Big)^{\frac{1}{p}} \quad p \in [1, +\infty) \tag{2.4}$$

where here and in the following the *weight* $w(x)$ is a nonnegative function belonging to $L^1_{loc}(\mathbb{R}^N)$ and satisfying the *Muckenhoupt's condition* A_2:

$$\sup_B\Big(\fint_B w dx \cdot \fint_B w^{-1} dx\Big) \le k_0\,. \tag{2.5}$$

The supremum is taken over all euclidean balls B and

$$\fint_B g dx = \int_B g dx \cdot \Big(\int_B dx\Big)^{-1}.$$

Let us recall that if w satisfies the condition (2.5) then the following duplicating property holds:

$\exists k > 0$ depending only on k_0 and on N such that

$$w(B_{2\rho}(x)) \le k w(B_\rho(x)) \tag{2.6}$$

for each $x \in \mathbb{R}^N$ and for each $\rho > 0$.

We refer to [11] and [12] for the proof of (2.6).

From now on we use the following notations for any measurable set E:

$$w(E) = \int_E w dx, \quad |E| = \int_E dx$$

Similarly $H^1(\Omega, w)$ is the completion of $\operatorname{Lip}(\Omega)$ for the norm

$$\|f\|_{1,2} = \{\|f\|^2_{0,2} + \|\nabla f\|^2_{0,2}\}^{1/2},$$

and $H^1_0(\Omega, w)$ the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega, w)$. The dual space of $H^1_0(\Omega, w)$ is $H^{-1}(\Omega, w)$.

Finally the following Poincaré inequality, consequence of a stronger inequality, holds for any $f \in C_0^\infty(\Omega)$:

$$\int_\Omega |f|^2 w dx \leq C(\operatorname{diam}\Omega)^2 \int_\Omega |\nabla f|^2 w dx \,. \tag{2.7}$$

See [11] and [12] for details and proofs.

We denote by L a linear bounded operator from $H_0^1(\Omega, w)$ to $H^{-1}(\Omega, w)$:

$$\begin{cases} i) & Lu = -\partial_j(a_{ij}\partial_i u) + a_0 u \equiv L_0 u + a_0 u \;, \qquad a_{ij} = a_{ji} \\ ii) & \lambda w(x)|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda w(x)|\xi|^2 \quad \Lambda, \lambda \in \mathbb{R}, \quad \lambda > 0 \\ iii) & \underline{a}_0 w(x) \leq a_0(x) \leq \overline{a}_0 w(x) \qquad \underline{a}_0, \overline{a}_0 \in \mathbb{R}, \quad \underline{a}_0 > 0 \end{cases} \tag{2.8}$$

In the following we will set $a(u,v) = \int_\Omega a_{ij}\partial_i u \partial_j v dx$, the summation symbol is clearly understood.

For every $y \in \Omega$ there exists the *Green's function* $G_\Omega^y \equiv G^y$ for the Dirichlet problem relative to L_0 in Ω, which is defined as solution of

$$a(v, G_\Omega^y) = v(y) \quad \forall v \in C_0^\infty(\Omega).$$

Such a Green function is symmetric in x and y and satisfies the following growth conditions

$$\gamma_1 \int_r^{R_0} (w(B_s(y)))^{-1} s ds \leq G^y(x) \leq \gamma_2 \int_r^{R_1} w(B_s(y)))^{-1} s ds \tag{2.9}$$

where γ_1, γ_2 are positive constants depending on N, Λ, λ and k_0 in (2.5), $B_{R_0}(y) \subset \Omega \subset B_{R_1}(y)$, and $|x-y| = r < \dfrac{R_0}{2}$.

Let us consider the regularized Green functions $G_{\Omega,r}^y \equiv G_r^y, r > 0$ which is the unique solution in $H_0^1(\Omega, w)$ of the problem:

$$\begin{cases} \int_\Omega a_{ij} v_i (G_r^y)_j dx = \dfrac{1}{w(B_r(y))} \int_{B_r(y)} v(x) w(x) dx \\ \\ \forall v \in C_0^\infty(\Omega)\,. \end{cases} \tag{2.10}$$

Remark 2.1. The following properties hold:

$G_r^y \to G^y$ uniformly in each compact subset of $(\Omega - \{y\})$ and weakly in $L^p(\Omega, w), p \in \Big(1, \dfrac{2N}{2N-1}\Big)$.

For each $r \in (0, R]$ and for each $v \in H_0^1(\Omega, w) \bigcap L^\infty(\Omega)$ such that $v = 0$ a.e. in $B_r(y)$ we have

$$\int_\Omega |\nabla G_r^y|^2 v^2 w dx \leq 4(\frac{\Lambda}{\lambda})^2 \int_\Omega (G_r^y)^2 |\nabla v|^2 w dx \tag{2.11}$$

Let K be a compact subset of Ω. We define

$$\mathrm{cap}_w(K, \Omega) = \inf\{\int_\Omega |\nabla v|^2 w dx; v \in C_0^1(A); v \geq 1 \text{ in } K \ \} \tag{2.12}$$

We can extend naturally to any open set B and any arbitrary set E by the usual passage to the supremum and to the infimum.

We will say that a property $P(x)$ holds in w-capacity almost everywhere (w.q.e) in $E \subset \Omega$ if there exists a set E^0 with $\mathrm{cap}_w(E^0, \Omega) = 0$, such that $P(x)$ holds for every $x \in E - E^0$.

Let v be a function $E \to [-\infty, +\infty]$, defined everywhere except at most on a set of w-capacity zero. Recall that the oscillation of v on a set E with non zero capacity is:

$$\operatorname*{osc}_E v = \sup_E{}^w v - \inf_E{}^w v$$

where here and in the following $\inf_E{}^w$ ($\sup_E{}^w$) denote the essential infimum (supremum) taken in the w-capacity sense, and with the convention that $(+\infty - (-\infty)) = 0$ and $(-\infty - (+\infty)) = 0$. If the w-capacity of E is zero we set the oscillation to be zero.

We say that v is w-quasi continuous if for every $\varepsilon > 0$ there exist $A \subset \Omega$ with $\mathrm{cap}_w(A, \Omega) < \varepsilon$ such that v is continuous in $\Omega \setminus A$. For every $v \in H^1(\Omega, w)$ there exists a w-quasi continuous representative $\tilde{v} = v$ a.e.:

$$\tilde{v}(x) = \lim_{\rho \to 0} \inf (w(B_\rho(x)))^{-1} \int_{B_\rho(x)} v w dx$$

By $v_E = v_{E,\Omega}$ where $\overline{E} \subset \Omega$, we denote the *w-capacitary potential* of E in Ω with respect to the operator L_0 i.e. the unique solution of

$$\begin{cases} v_E \in H_0^1(\Omega, w) & v_E \geq 1 w\text{-q-e in } E \\ a(v_E, v_E - f) \leq 0 & \\ \forall f \in H_0^1(\Omega, w) & f \ \geq 1 w\text{-q-e in} E. \end{cases} \tag{2.13}$$

$L_0 v_E$ belongs to $H^{-1}(\Omega, w)$ and is a positive radon measure μ_E supported on ∂E satisfying:

$$a(v_E, f) = \int_\Omega f d\mu_E \qquad \forall f \in H_0^1(\Omega, w). \tag{2.14}$$

See [5], [11] and [12] for proofs and details.

In section 5 we use the following bounds which are true for every $r \in (0, R/2)$:

$$C_1 \Big(\int_r^R s(w(B_s(x_0)))^{-1} ds \Big)^{-1} \leq \operatorname{cap}_w(B_r(x_0), B_R(x_0)) \leq$$

$$\leq C_2 \Big(\int_r^R s(w(B_s(x_0)))^{-1} ds \Big)^{-1}$$

DEFINITION 2.1. *The set of Radon's measures μ defined on Ω such that:*

$$\lim_{r \to 0^+} \Big(\sup_{x \in \Omega} \int_{B_r(x) \cap \Omega} \Big(\int_{|y-x|}^1 \frac{s^2}{w(B_s(x))} \frac{ds}{s} \Big) d|\mu|(y) \Big) = 0$$

where $|\mu|$ denote the total variation of μ, is called the Kato space $K(\Omega) \equiv K(\Omega, w)$.

This definition actually generalizes the one given by Kato in [16] and is due to DAL MASO and MOSCO in the case $w \equiv 1$, (see [9]).

The natural norm with which $K(\Omega)$ is equipped, is:

$$\|\mu\|_{K(\Omega)} \equiv \sup_{x \in \Omega} \int_\Omega \Big(\int_{|y-x|}^1 \frac{s^2}{w(B_s(x))} \frac{ds}{s} \Big) d|\mu|(y).$$

REMARK 2.2. We can easily check that if $f \in L^p(\Omega, w)$ with $p \geq p_0(N, w)$, then $fw \in K(\Omega)$, in particular if $f \in L^\infty$ then $\|fw\|_{K(\Omega)} \leq C(\operatorname{diam}\Omega)^2$. Moreover it can be seen that $K(\Omega) \subset H^{-1}(\Omega, w)$ (see [1]).

For other properties of Kato spaces, see [1] and [9].

We state now a theorem which will be repeatedly used in the proof of the main results.

THEOREM 2.1. *If $\nu \in H^1(B_R(x_0), w)$ and $L_0\nu = \mu$ in the weak sense with $\mu \in K(B_R(x_0))$ then for each $q \in (0,1)$*

$$\operatorname*{osc}_{B_{qR}}{}^2 \nu \leq \frac{C}{w(B_R(x_0))}\|\nu - \nu_R\|^2_{L^2(B_R(x_0))} + C\|\mu\|^2_{K(B_R(x_0))}$$

$$\int_{B_{qR}(x_0)} |\nabla\nu|^2 G^{x_0}_{B_{2R}} w dx \leq \frac{C}{w(B_R(x_0))}\|\nu - \nu_R\|^2_{L^2(B_R(x_0))} + C\|\mu\|^2_{K(B_R(x_0))}$$

where $\nu_R = \dfrac{1}{w(B_R(x_0))}\displaystyle\int_{B_R} \nu w dx$ *and C stands for any positive constant depending only on λ,Λ, N,k_0 and q.*

In particular, ν is continuous in B_R.

The proof of this theorem can be obtained as in [9], just modifying in the most natural way when the presence of w makes it necessary.

We introduce the following level sets:

(2.15)
$$E_1(\varepsilon, \rho) = E_1(\psi_1, x_0; \varepsilon, \rho) = \{x \in B_\rho(x_0) : \psi_1(x) \geq \sup{}^w_{B_\rho(x_0)}\psi_1 - \varepsilon\}$$
$$E_2(\varepsilon, \rho) = E_2(\psi_2, x_0; \varepsilon, \rho) = \{x \in B_\rho(x_0) : \psi_2(x) \leq \inf{}^w_{B_\rho(x_0)}\psi_2 + \varepsilon\}$$

and their relative capacities

$$\delta_1(\varepsilon, \rho) = \frac{\operatorname{cap}_w(E_1(\varepsilon,\rho), B_{2\rho}(x_0))}{\operatorname{cap}_w(B_\rho(x_0), B_{2\rho}(x_0))} \tag{2.16}$$

$$\delta_2(\varepsilon, \rho) = \frac{\operatorname{cap}_w(E_2(\varepsilon,\rho), B_{2\rho}(x_0))}{\operatorname{cap}_w(B_\rho(x_0), B_{2\rho}(x_0))}. \tag{2.17}$$

According to [24] and [10] the Wiener modulus of the obstacle ψ_i for $\sigma > 0$ is defined by:
(2.18)

$$\omega_{i,\sigma_i}(r,R)=\omega_{i,\sigma_i}(\psi_i,x_0;r,R)=\inf\left\{\omega>0:\omega\exp\Big(\int_r^R\delta_i(\sigma\omega,\rho)\frac{d\rho}{\rho}\Big)\geq 1\right\}$$

where $i = 1$ refers to the lower obstacle and $i = 2$ to the upper obstacle. We state now a few properties which are particularly relevant. For their proofs see [24].

LEMMA 2.2. *Let $0 < r \leq R$ be fixed. Then the constant $\varepsilon > 0$, and $\sigma_i > 0$ for $i = 1; 2$ satisfy*

$$\sigma_i = \varepsilon\exp\Big(\int_r^R\delta_i(\varepsilon,\rho)\frac{d\rho}{\rho}\Big)$$

if and only if

$$\omega_{i,\sigma_i}(r,R) = \exp\Big(-\int_r^R\delta_i(\varepsilon,\rho)\frac{d\rho}{\rho}\Big) \text{ and } \sigma_i\omega_{i,\sigma_i}(r,R) = \varepsilon.$$

We shall also refer to the following integral

$$\int_r^R\delta_i^*(\varepsilon,\rho)\frac{d\rho}{\rho} \tag{2.19}$$

where δ_i^* are the relative w-capacity for $i = 1$ and $i = 2$ of, respectively, the level sets

$$E_1^*(\varepsilon,\rho) = \{x \in B_\rho(x_0) : \psi_1(x) \geq \overline{\psi}_1(x_0) - \varepsilon\}$$

$$E_2^*(\varepsilon,\rho) = \{x \in B_\rho(x_0) : \psi_2(x) \leq \underline{\psi}_2(x_0) + \varepsilon\}$$

where the pointwise values $\overline{v}(x_0)$ and $\underline{v}(x_0) \in [-\infty,+\infty]$ of an arbitrary function $v : \Omega \to [-\infty,+\infty]$ are defined by:

$$\overline{v}(x) = \inf_{\rho>0}\big(\sup_{B_\rho(x)}{}^w v\big) \tag{2.20}$$

$$\underline{v}(x) = \sup_{\rho>0}(\inf_{B_\rho(x)} {}^w v) \tag{2.21}$$

The Wiener moduli $\omega^*_{i,\sigma_i}(r,R)$ are defined as $\omega_{i,\sigma_i}(r,R)$ with δ_i replaced by δ^*_i.

Remark 2.3. From the definitions it follows immediately that $\delta_i(\varepsilon,\rho) \le \delta^*_1(\varepsilon,\rho)$ for every $\varepsilon > 0$, $\rho > 0$, hence $\omega_{i,\sigma_i}(r,R) \le \omega^*_{i,\sigma_i}(r,R)$ for every $0 < r \le R$ and $\sigma > 0$. Moreover Lemma 2.2 holds with the δ_i and ω_{i,σ_i} replaced by δ^*_i and ω^*_{i,σ_i}.

The vanishing of $\omega_{i,\sigma_i}(r,R)$ will play an important role in the estimation of the Holder continuity of the solution. The following lemma links the convergence to zero of the Wiener modulus and the behavior of the relative capacity of the level sets.

Lemma 2.3. *Assume that* $-\infty < \overline{\psi}_1(x_0) < +\infty$ *and* $-\infty < \underline{\psi}_2(x_0) < +\infty$. *Then for* $i = 1$ *and* $i = 2$ *the following are equivalent:*

i) *for every* $\varepsilon > 0$ *there exists* $R > 0$ *such that* $\lim_{r\to 0}\omega_{i,\sigma_i}(r,R) = 0$ *for* $\sigma_i = \sigma_i(r)$ *such that* $\sigma_i\omega_{i,\sigma_i}(r,R) = \varepsilon$ *for all* $0 < r \le R$

ii) *for every* $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$\int_0^R \delta_i(\varepsilon,\rho)\frac{d\rho}{\rho} = +\infty \tag{2.22}$$

Definition 2.2. *We say that* x_0 *is a Wiener point of* $\{\psi_1,\psi_2\}$ *if for every* $\varepsilon > 0$ *and for every* $R > 0$

$$\int_0^R \delta^*_i(\varepsilon,\rho)\frac{d\rho}{\rho} = +\infty \tag{2.23}$$

for $i = 1; 2$.

Remark 2.4. If $\overline{\psi}_1(x_0) = -\infty$ and $\underline{\psi}_2(x_0) = +\infty$ then x_0 is trivially a Wiener point. Furthermore x_0 is a Wiener point for the double obstacle problem if and only if it is a Wiener point for both the lower and upper obstacle as defined in [24] and [5].

3 – Existence of solutions

In this section we prove an existence result for two obstacle problems involving quasi linear degenerate elliptic operator.

Let $H(x,\eta,\xi) : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies

$$(3.24)\qquad \begin{cases} H \text{ is a Caratheodory function} \\ H(x,.,.) \text{ is locally Lipschitz continuous for a.e.} x \in \Omega \\ |H(x,\eta,\xi)| \leq (k_1 + k_2(|\eta|)|\xi|^2)w(x) \\ k_2(|\eta|) \text{can be supposed not decreasing in } \eta \end{cases}$$

Consider the problem

$$(3.25)\qquad \begin{cases} u \in \mathcal{K} \cap L^\infty(\Omega) \\ a(u, v-u) + \int_\Omega (a_0 + H(u))(v-u)dx \geq 0 \\ \forall v \in \mathcal{K} \cap L^\infty(\Omega) \end{cases}$$

where $H(u) \equiv H(x, u(x), \nabla u(x))$ and

$$\mathcal{K} = \{v \in H_0^1(\Omega, w) : \psi_1 \leq v \leq \psi_2 \;\; w - \text{q.e.}\}.$$

Suppose

$$(3.26)\qquad \mathcal{K} \cap L^\infty(\Omega) \neq \emptyset$$

THEOREM 3.1. *Assume* (2.8),(3.24) *and* (3.26) *then there exist* u *solution of problem* (3.25).

To prove Theorem 3.1, we shall use the following approximation procedure (see [6]). We'll find the existence for an approximated problem and then show that the sequence converges to the solution using uniform L^∞ and $H^1(\Omega, w)$ bounds.

Explicitly, consider the sequence of approximating problems

$$(3.27)\qquad \begin{cases} u_m \in \mathcal{K} \\ a(u_m, v - u_m) + \int\limits_\Omega (a_0 u_m + H_m(u_m))(v - u_m)dx \geq 0 \\ \forall v \in \mathcal{K} \end{cases}$$

where $H_m(u) = H_m(x, u(x), \nabla u(x))$ and H_m is defined by

$$(3.28)\qquad H_m(x, \eta, \xi) = \frac{H(x, \eta, \xi)}{1 + m^{-1} w^{-1}(x)|H(x, \eta, \xi)|}$$

so that H_m satisfies

$$(3.29)\qquad \begin{cases} \text{i)} \quad H_m w^{-1} \in L^2(\Omega, w) \text{ and so} H_m \in H^{-1}(\Omega, w) \\ \text{ii)} \quad |H_m| \leq mw \\ \text{iii)} \quad |H_m| \leq |H| \end{cases}$$

PROPOSITION 3.2. *In the previous notations and hypotheses* u_m *solution of problem* (3.27) *exists.*

PROOF. Consider the operators $T_m : H^1_o(\Omega, w) \to \mathcal{K}$ that maps $v \in H^1(\Omega, w)$ into $T_m(v) = g_m$ the unique solution of the following variational inequality which exists by Stampacchia theorem on bilinear, coercive forms:

$$(3.30)\qquad \begin{cases} g_m \in \mathcal{K} \\ a(g_m, f - g_m) + \int\limits_\Omega (a_0 g_m + H_m(v))(f - g_m)dx \geq 0 \\ \forall f \in \mathcal{K} \end{cases}$$

Any fixed point for T_m is a solution of (3.27), thus it will be enough to prove that the assumptions of the Schauder fixed point theorem are satisfied.

First choose as a test function in (3.30) $f = g_m - \frac{1}{2}(g_m - \underline{d}_m)^+ + \frac{1}{2}(g_m - \overline{d}_m)^-$ where $\underline{d}_m = \sup\{\sup^w_\Omega \psi_1, m/\underline{a}_0\}$ and $\overline{d}_m = \inf\{\inf^w_\Omega \psi_2, -m/\overline{a}_0\}$ using the properties of H_m it is easy to obtain

$$(\|(g_m - \underline{d}_m)^+\|)^2_{1,2} + (\|(g_m - \overline{d}_m)^-\|)^2_{1,2} \leq 0$$

hence

$$\|T_m(v)\|_\infty \leq d_m \text{ where } d_m = \underline{d}_m \vee (-\overline{d}_m) \qquad \forall\, v \in H^1_0(\Omega, w).$$

Consider the convex set closed in $H^1_0(\Omega, w)$:

$$\mathcal{B} = \{v \in H^1_0(\Omega, w) \cap L^\infty(\Omega) : \|v\|_\infty \leq d_m\}$$

First claim : T_m is continuous from $\mathcal{B}$ into $\mathcal{B}$ in the $H^1(\Omega, w)$-norm.

Indeed as H_m is a Caratheodory function we can apply the Lebesgue dominated theorem and so if v_n is a sequence of functions in $\mathcal{B}$ converging in the $H^1(\Omega, w)$-norm towards a function $v \in \mathcal{B}$, then:

$$H_m(v_n)w^{-1} \to H_m(v)w^{-1} \text{ in } L^2(\Omega, w).$$

Now consider the problem $(3.30)_n$ where $H_m(v)$ is replaced by $H_m(v_n)$, then choosing as test function $T_m(v_n)$ and $T_m(v)$ respectively in (3.30) and $(3.30)_n$ we obtain

$$(\lambda \wedge \underline{a}_0)\|T_m(v_n) - T_m(v)\|_{1,2} \leq \|(H_m(v_n) - H_m(v))w^{-1}\|_{0,2}$$

and the first claim is proved.

Second claim T_m is relatively compact in $H^1(\Omega, w)$.

The following inequality can be easily derived from (3.30) for a fixed $v_0 \in \mathcal{K}$

$$(\lambda \wedge \underline{a}_0)\|T_m(v)\|^2_{1,2} \leq (\Lambda \vee \overline{a}_0)\|v_0\|_{1,2}\|T_m(v)\|_{1,2} + \\ + m w^{1/2}(\Omega)(\|v_0\|_{0,2} + \|T_m(v)\|_{0,2})$$

hence there is a constant C_m such that

$$\|T_m(v)\|_{1,2} \leq C_m \quad \forall v \in H^1(\Omega, w).$$

Choosing again as a test function $T_m(v_n)$ and $T_m(v_{n'})$ respectively in $(3.30)_{n'}$ and $(3.30)_n$ we obtain

$$(\lambda \wedge \underline{a}_0)\|T_m(v_n) - T_m(v_{n'})\|_{1,2} \leq 2mw^{1/2}(\Omega)\|(H_m(v_n) - H_m(v_{n'}))w^{-1}\|_{0,2}$$

and the second claim is proved.

This concludes the proof.

The following proposition shows that u_m, solution of (3.27), admits uniform bounds in m.

PROPOSITION 3.3. *Assume previous notations and hypotheses then the following estimates hold for any solution u_m of* (3.27)

$$\|u_m\|_\infty \leq k_3 \tag{3.31}$$

$$\|u_m\|_{1,2} \leq k_4 \tag{3.32}$$

where $k_3 = \{\sup^w_\Omega \psi_1 \vee (k_1 \underline{a}_0^{-1}) \vee (-\inf^w_\Omega \psi_2)\}$ and k_4 depends on the structural data of the problem but not on m.

PROOF. We will only introduce convenient test functions, as the proof is analogous to the one of the uniformly elliptic case (see [6]). We choose as a test function in (3.27)

$$v = u_m - \eta(u_m - C_3)^+ \exp t_m((u_m - C_3)^+)^2 +$$
$$+ \eta^*(u_m - C_3^*)^- \exp t_m((u_m - C_3^*)^-)^2$$

where

$$\begin{cases} C_3 = \sup^w_\Omega \psi_1 \vee (k_1 \underline{a}_0^{-1}) \\ C_3^* = \inf^w_\Omega \psi_2 \wedge (-k_1 \overline{a}_0^{-1}) \\ t_m = \lambda^{-2} k_2^2(\|u_m\|_\infty) \\ \eta = \exp(-t_m\{\|u_m\|_\infty + C_3\}^2) \\ \eta^* = \exp(-t_m\{\|u_m\|_\infty + C_3^*\}^2) \end{cases}$$

It is immediate to obtain that

$$C_3^* \leq u_m \leq C_3$$

and thus (3.31). Similarly, for (3.32) we use the test function

$$v = u_m + \eta(v_0 - u_m) \exp t(v_0 - u_m)^2 \tag{3.33}$$

where

$$\begin{cases} t = \lambda^{-2} k_2^2(k_3) \\ \eta = \exp(-t(\|u_m\|_\infty + \|v_0\|_\infty)^2) \\ v_0 \in \mathcal{K} \cap L^\infty(\Omega)\,. \end{cases}$$

It is easy to see that plugging v into (3.27) we obtain (3.32).

PROOF OF THEOREM 3.1. Consider the sequence $\{u_m\}_{m\in N}$ where for each m, u_m is a solution (3.27), by estimate (3.31) and (3.32) we can find a subsequence such that:

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } H_0^1(\Omega, w) \\ u_m \to u & \text{a.e. in } \Omega. \end{cases}$$

in particular u belongs to $\mathcal{K} \cap L^\infty(\Omega)$.

Now we can choose as test function the one introduced in (3.33) with $v_0 := u$ and by the Lebesgue dominated convergence we can check the strong convergence of the u_m towards u in $H^1(\Omega, w)$.

On the other hand the sequence $\{H_m(u_m)\}$ converges a.e. to $H(u)$, hence by (3.24) and (3.29) $H_m(u_m)w^{-1}$ converges L^1-strong to $H(u)w^{-1}$ by Vitali's theorem and then

$$\begin{aligned} a(u,u) &\leq \liminf a(u_m, u_m) \leq \\ &\leq \lim\{a(u_m, v) - \int_\Omega (a_0 u_m + H_m(u_m))(v - u_m)dx\} = \\ &= a(u,v) - \int_\Omega (a_0 u + H(u))(v-u)dx \end{aligned}$$

and the proof is achieved.

4 – Oscillation and energy estimates

In this section we first estimate the oscillation of a solution of the two-obstacle problem then in a slightly more restricted situation we estimate the energy.

Here and throughout this section we shall use the same notations and the same hypotheses on $L, H, \mathcal{K}$, etc. as in the previous sections. We'll assume also that $H(x, \tau, \xi)$ is differentiable in the τ variable for any ξ and a.e. on x and

$$\frac{\partial H}{\partial \tau} + a_0 \geq \alpha_0 w(x) \text{ where } \alpha_0 \in \mathbb{R},\ \alpha_0 > 0. \tag{4.34}$$

For simplicity of notations we denote $\tilde{H}(x, \tau, \xi) = H(x, \tau, \xi) + a_0 \tau$ and $\tilde{H}(u) \equiv \tilde{H}(x, u, \nabla u)$.

We are then concerned with u solution of:

$$\tag{4.35} \begin{cases} u \in H^1(\Omega, w) \cap L^\infty(\Omega) & \psi_1 \leq u \leq \psi_2 \\ a(u, v-u) + \displaystyle\int_\Omega (\tilde{H}(u))(v-u) & dx \geq 0 \\ \forall v \in H^1(\Omega, w) \cap L^\infty(\Omega) & \psi_1 \leq v \leq \psi_2\ , u - v \in H_0^1(\Omega, w). \end{cases}$$

Definition 4.1. *We say that x_0 is a regular point of $\{\psi_1, \psi_2\}$ if any solution u of* (4.35) *is continuous at x_0*

We need some notations to express the modulus of continuity of u at a Wiener point $x_0 \in \Omega$. Here and in the following $B_\rho = B_\rho(x_0)$ unless some ambiguity arise.

$$d_R = \begin{cases} \sup\limits_{B_R} \psi_1 \wedge \inf\limits_{B_R} \psi_2 & \text{if } u_R < \sup\limits_{B_R} \psi_1 \wedge \inf\limits_{B_R} \psi_2 \\ u_R = \dfrac{1}{w(B_R)} \displaystyle\int_{B_R} u(x) w(x) dx & \text{if } \sup\limits_{B_R} \psi_1 \wedge \inf\limits_{B_R} \psi_2 \leq u_R \leq \sup\limits_{B_R} \psi_1 \vee \inf\limits_{B_R} \psi_2 \\ \sup\limits_{B_R} \psi_1 \vee \inf\limits_{B_R} \psi_2 & \text{if } \sup\limits_{B_R} \psi_1 \vee \inf\limits_{B_R} \psi_2 < u_R \end{cases}$$

$$Z(R) = \{[\sup_{B_R} \psi_1 - \inf_{B_R} \psi_2]^+ + w(B_R)^{-\frac{1}{2}} \|u - d_R\|_{L^2(B_R, w)}$$

and finally

$$\Psi(\varepsilon_1,\varepsilon_2,R)=[\underline{\psi}_2(x_0)-\overline{\psi}_1(x_0)]+\{[\sup_{B_R}\psi_1-\underline{\psi}_2(x_0)]\vee\varepsilon_2\}+$$
$$+\{[\overline{\psi}_1(x_0)-\inf_{B_R}\psi_2]\vee\varepsilon_1\}$$

In particular for $0<r\le R$, $\sigma_1>0,\sigma_2>0$ let

$$\Psi_{\sigma_1,\sigma_2}(r,R)=\Psi(\varepsilon_1,\varepsilon_2,R)\quad\text{where}\quad\varepsilon_i=\sigma_i\omega_{i,\sigma_i}(r,R)\text{ for }i=1;i=2.$$

Let us give a first estimate which involves only the oscillation of u.

THEOREM 4.1. *Assume hypothesis* (2.5), (2.8), (3.24) *and* (4.34). *Let* $s\in(0,1)$ *then there exist positive constants* $C=C(\lambda,\Lambda,N,k_0,s)$, $\beta=\beta(\lambda,\Lambda,N,k_0)$ *and* $\alpha=\alpha(\lambda,\Lambda,N,k_0)$ *such that for any solution of* (4.35) *we have:*

$$\operatorname*{osc}_{B_r}u\le\Psi_{\sigma_1,\sigma_2}(r,R)+c\{(Z(R)[\omega^*_{1,\sigma_1}(r,R)+\omega^*_{2,\sigma_2}(r,R)]^\beta+R^\alpha\}\tag{4.36}$$

for every $0<r\le sR$ *and for every* $\sigma_i>0$.

REMARK 4.1. If $-\infty<\overline{\psi}_1(x_0)=\underline{\psi}_2(x_0)<+\infty$ then Theorem 4.1 guarantees that:

x_0 is a Wiener point according to definition 2.2

implies

x_0 is a regular point according to definition 4.1.

More precisely:

$$\Psi_{\sigma_1,\sigma_2}(r,R)=(\sup_{B_R}\psi_1-\overline{\psi}_1(x_0))\vee(\sigma_2\omega^*_{2,\sigma_2}(r,R))+$$
$$+[\underline{\psi}_2(x_0)-\inf_{B_R}\psi_2]\vee(\sigma_1\omega^*_{1,\sigma_1}(r,R))$$

therefore if x_0 is a Wiener point for $\{\psi_1,\psi_2\}$ then the Wiener moduli go to zero as r does and u is continuous at x_0, indeed:

$$\lim_{r\to0^+}\sup\operatorname*{osc}_{B_r}u\le(\sup_{B_R}\psi_1-\overline{\psi}_1(x_0))+(\underline{\psi}_2(x_0)-\inf_{B_R}\psi_2)+CR^\alpha$$

and, of course, the right hand side converges to zero as soon as R does.

To give the proof of Theorem 4.1 we need to state the following propositions that will be proved in the next section :

PROPOSITION 4.2. *In the previous hypotheses and for* $0 < s < \frac{1}{3}$ *there exists* $C = C(\lambda, \Lambda, N, k_0)$ *such that:*

$$\operatorname*{osc}_{B_{sR}} u \leq [\sup_{B_R} \psi_1 - \inf_{B_R} \psi_2]^+ + C\left\{\frac{1}{w(B_R(x_0))^{\frac{1}{2}}} \|u - d_R\|_{L^2(B_R,w)} + R\right\} \tag{4.37}$$

for u *solution of* (4.35).

Next we are giving estimates of the supremum and the infimum of u in term of

$$\Psi_1(\varepsilon, R) = \inf_{B_R} \psi_2 \wedge [\overline{\psi}_1(x_0) - \varepsilon], \qquad \Psi_2(\varepsilon, R) = \sup_{B_R} \psi_1 \vee [\underline{\psi}_2(x_0) + \varepsilon].$$

LEMMA 4.3. *In the previous hypotheses, there exist some positive constants* $C = C(\lambda, \Lambda, N, k_0)$, $\beta = \beta(\lambda, \Lambda, N, k_0)$ *and* $\alpha = \alpha(\lambda, \Lambda, N, k_0)$ *such that*

$$\inf_{B_r} u \geq \Psi_1(\varepsilon, R) - C\left\{[\inf_{B_R} u - \Psi_1(\varepsilon, R)]^- \exp\left(-\beta \int_r^R \delta_1^*(\varepsilon, \rho) \frac{d\rho}{\rho}\right) + R^\alpha\right\} \tag{4.38}$$

$$\sup_{B_r} u \leq \Psi_2(\varepsilon, R) + C\left\{[\sup_{B_R} u - \Psi_2(\varepsilon, R)]^+ \exp\left(-\beta \int_r^R \delta_2^*(\varepsilon, \rho) \frac{d\rho}{\rho}\right) + R^\alpha\right\} \tag{4.39}$$

for any $0 < r \leq R$, *and* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$.

We can now proceed to the

PROOF. Let $t_1 = \Psi_1(\varepsilon_1, sR)$; $t_2 = \Psi_2(\varepsilon_2, sR)$. Subtracting (4.38) to (4.39) we get:

$$\operatorname*{osc}_{B_{sr}} u \le t_2 - t_1 + C\left\{[\inf_{B_{sR}} u - t_1]^- \vee (\sup_{B_{sR}} u - t_2)^+] \sum_{i=1}^{2} \exp\left(-\beta \int_r^R \delta_i^*(\varepsilon_i, \rho)\frac{d\rho}{\rho}\right) + R^\alpha\right\}$$

Considering that $t_1 \le t_2$, $\inf u \le t_2$ and $\sup u \ge t_1$:

$$\operatorname*{osc}_{B_{sR}} u \ge (\inf_{B_{sR}} u - t_1)^- \vee (\sup_{B_{sR}} u - t_2)^+ \tag{4.40}$$

and estimating $\operatorname{osc}_{B_{sR}} u$ in the left hand side of (4.40) by Proposition 4.2, we obtain:

$$\operatorname*{osc}_{B_r} u \le \Psi(\varepsilon_1, \varepsilon_2, R)) + C\left\{Z(R) \sum_{i=1}^{2} \exp\left(-\beta \int_r^R \delta_i^*(\varepsilon_i, \rho)\frac{d\rho}{\rho}\right) + R^\alpha\right\} \tag{4.41}$$

From this and the definition of $\omega_{i,\sigma_i}^*(r, R)$ we have immediately (4.36). □

The next goal is to estimate the potential semi-norm $V(R)$

$$V(R) = \operatorname*{osc}_{B_R} u + \left(\lambda \int_{B_R} |\nabla u|^2 G^{x_0}_{B_{2\frac{R}{q}}(x_0)} w(x)dx\right)^{\frac{1}{2}}$$

of the solution u of problem (4.35), at the point $x_0 \in \Omega$. We recall that here and in the rest of the section $B_r \equiv B_r(x_0)$ when no confusion arise. We obtain an energy decay under the hypothesis that there exists a function ν such that

$$\nu \in H^1(B_{R_1}, w),\ L_0\nu \in K(B_{R_1}) \text{ and } \psi_1 \le \nu \le \psi_2 \text{ q.e. in } B_{R_1} \tag{4.42}$$

for some $R_1 > 0$.

Observe that if $\overline{\psi}_1(x_0) < \underline{\psi}_2(x_0)$ then (4.42) is satisfied by a suitable constant provided that R_1 is small enough.

Theorem 4.4. *Assume* (2.5),(2.8),(3.24),(4.34) *and* (4.42) *then there is a constant* $C = C(\lambda, \Lambda, N, k_0)$ *such that for u solution of* (4.35)

$$\begin{aligned} V(r) \leq & C\{w(B_R(x_0))^{-1/2}\|u-\nu\|_{L^2(B_R,w)}[\omega_{1,\sigma_1}(r,R)+\omega_{2,\sigma_2}(r,R)]^{\beta}+ \\ & + \sigma_1\omega_{1,\sigma_1}(r,R) + \sigma_2\omega_{2,\sigma_2}(r,R)+ \\ & + w(B_R(x_0))^{-1/2}\|\nu - \nu_R\|_{L^2(B_R,w)} + \|L_0\nu\|^{1/2}_{K(B_R)} + R^{\alpha}\} \end{aligned} \tag{4.43}$$

for every $0 < r \leq R < R_1$ *and for every* σ_1 *and* σ_2 *positive,* α *and* β *as in theorem 4.1.*

To prove this theorem we need the following two propositions that will be proved in the next section.

Set $g(R)^2 = R^2 + \|L_0\nu\|_{K(B_R)} + \dfrac{1}{w(B_R)}\|\nu - \nu_R\|^2_{L^2}$.

Proposition 4.5. *In the hypothesis of theorem 4.4 for each* $0 < q < \dfrac{1}{3}$*:*

$$\begin{aligned} & \sup_{B_{qR}}((u-\nu)^{\pm})^2 + \lambda \int_{B_{qR}} |\nabla(u-\nu)^{\pm}|^2 G^{x_0}_{B_{2R}} w(x)dx \leq \\ & \qquad \leq C\{\frac{1}{w(B_R)} \int_{B_R\setminus B_{qR}} |(u-\nu)^{\pm}|^2 w dx + g(R)^2\}. \end{aligned} \tag{4.44}$$

In order to stress the constants that we have to carefully evaluate, we shall denote by C all the harmless ones while those to be evaluated by c_j with $j \in \mathbb{N}$.

Proposition 4.6. *In the hypothesis of theorem 4.4 for every* $\gamma > 0$

$$\lambda \int_{B_{sR}(z)} |\nabla(u-\nu)^{\pm}|^2 G^z_{B_{tR}(z)} w(x)dx + ((u-\nu)^{\pm})^2(z) \leq \tag{4.45}$$

$$(c_1+\gamma)\sup_{B_{tR}(z)}((u-\nu)^{+})^2 + \frac{c_2\lambda}{\gamma}\int_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla(u-\nu)^{\pm}|^2 G^z_{B_{tR}(z)} w(x)dx + Cg(R)^2$$

where $s < \dfrac{2}{3}t$.

Let us introduce now the following quantities

$$m_r = \frac{1}{w(B_r)} \int_{B_r} (u-\nu)w(x)dx$$

$$a_r = \begin{cases} \sup^w_{B_r}(\psi_1 - \nu) & \text{if } m_r < \sup_{B_r}(\psi_1-\nu) \\ m_r & \text{if } \sup_{B_r}(\psi_1-\nu) \le m_r \le \inf_{B_r}(\psi_2-\nu) \\ \inf_{B_r}(\psi_2-\nu) & \text{if } m_r > \inf_{B_r}(\psi_2-\nu) \end{cases}$$

Before starting the proof of theorem 4.4, we state one more lemma which is a refined form of Lemma 4.3 and which will allow us to give an estimate to $|a_r - m_r|$:

LEMMA 4.7. *In the previous hypothesis, there exist two constants* $C = C(\lambda, \Lambda, N, k_0)$ *and* $\beta = \beta(\lambda, \Lambda, N, k_0)$ *such that*

$$\begin{aligned}\inf_{B_r}(u-\nu) \ge & \sup_{B_r}(\psi_1-\nu) - C\Big\{\operatorname*{osc}_{B_R}(u-\nu)\exp\Big(-\beta\int_r^R \delta_1(\varepsilon,\rho)\frac{d\rho}{\rho}\Big)+ \\ & + \varepsilon_1 + w(B_R)^{-\frac{1}{2}}\|\nu-\nu_R\|_{L^2(B_R)} + \|L_0\nu\|_{K(B_R)} + R^\alpha\Big\}\end{aligned} \tag{4.46}$$

$$\begin{aligned}\sup_{B_r}(u-\nu) \le & \inf_{B_r}(\psi_2-\nu) + C\Big\{\operatorname*{osc}_{B_R}(u-\nu)]\exp\Big(-\beta\int_r^R \delta_2(\varepsilon,\rho)\frac{d\rho}{\rho}\Big)+ \\ & + \varepsilon_2 + w(B_R)^{-\frac{1}{2}}\|\nu-\nu_R\|_{L^2(B_R)} + \|L_0\nu\|_{K(B_R)} + R^\alpha\Big\}\end{aligned} \tag{4.47}$$

for every $0 < r \le R$, $\forall\varepsilon > 0$ *and for* α *as in* Theorem 4.1.

REMARK 4.2. From lemma 4.7, we immediately have that

$$\begin{aligned}|a_r - m_r| \le & C\Big[\operatorname*{osc}_{B_R}(u-\nu)\sum_{i=1}^2 \exp\Big(-\beta\int_r^R \delta_i(\varepsilon_i,\rho)\frac{d\rho}{\rho}\Big) + \varepsilon_1 + \\ & + \varepsilon_2 + g(R) + R^\alpha\Big]\end{aligned} \tag{4.48}$$

for $r < R$.

We are giving now the proof of theorem 4.4.

PROOF. In Proposition 4.6 we may choose $z \in B_{qR}(x_0)$ with $q < 1/3$, and thus if we take $s = 2q$ and $t = (1-q)$ in order to have $B_{(1-q)R}(z) - B_{sR}(z) \subset B_R(x_0) - B_{qR}(x_0)$, (4.45) becomes:

$$\sup_{B_{qR}(x_0)} ((u-\nu)^{\pm})^2 \leq (c_1 + \gamma) \sup_{B_R(x_0)} ((u-\nu)^{\pm})^2 +$$

$$+ \frac{c_3 \lambda}{\gamma} \int_{B_R(x_0) \setminus B_{qR}(x_0)} |\nabla (u-\nu)^{\pm}|^2 G^{x_0}_{B_{2R}(x_0)} w dx + C g(R)^2.$$

Observe that we have used the property (2.9) of the Green function. Furthermore Proposition 4.5 gives the following

$$c_4 \lambda \int_{B_{qR}} |\nabla (u-\nu)^{\pm}|^2 G^{x_0}_{B_{2R}} w dx \leq \sup_{B_R} [(u-\nu)^{\pm}]^2 + C g(R)^2$$

where obviously c_4 can be chosen as small as desired.

Adding these two inequalities, multiplying by γ and summing to the resulting inequality the term $c_3 \lambda \int_{B_{qR}(x_0)} |\nabla (u-\nu)^{\pm}|^2 G^{x_0}_{B_{2R}} w dx$ we obtain:

$$\begin{aligned} \gamma \sup_{B_{qR}} ((u-\nu)^{\pm})^2 + (c_4 \lambda \gamma + c_3 \lambda) \int_{B_{qR}} |\nabla (u-\nu)^{\pm}|^2 G^{x_0}_{B_{2R}} w dx \leq \\ \leq \gamma (1 + c_1 + \gamma) \sup_{B_R} ((u-\nu)^{\pm})^2 + \\ + c_3 \lambda \int_{B_R} |\nabla (u-\nu)^{\pm}|^2 G^{x_0}_{B_{2R}} w dx + C g(R)^2. \end{aligned} \tag{4.49}$$

Notice that if ν satisfies (4.42) then so does also $\nu + a_R$, therefore we can rewrite (4.49) with $\nu + a_R$. We make this shift because we want to estimate the *oscillation* of $u - \nu$ and as $\sup_{B_R}(u - \nu - a_R) > 0$ and $\inf_{B_R}(u - \nu - a_R) < 0$ for $R \geq r \geq 0$ these imply that $\mathrm{osc}_{B_R}(u - \nu) =$

$\sup_{B_R}(u-\nu-a_R)^+ + \sup_{B_R}(u-\nu-a_R)^-$. (4.49) becomes:

$$\frac{\gamma}{2}\operatorname*{osc}_{B_{qR}}{}^2(u-\nu) + (c_4\lambda\gamma + c_3\lambda)\int_{B_{qR}} |\nabla(u-\nu)|^2 G^{x_0}_{B_{2R}} w dx \leq$$

$$\leq \gamma(1+c_1+\gamma)\operatorname*{osc}_{B_R}{}^2(u-\nu) + c_3\lambda\int_{B_R} |\nabla(u-\nu)|^2 G^{x_0}_{B_{2R}} w dx + Cg(R)^2.$$

Through the Maximum Principle and (2.9) (see also [5]), we obtain the following inequality that will be useful later.

$$G^{x_0}_{B_{2R(x_0)}} = G^{x_0}_{B_{2Rq^{-1}(x_0)}} + G^{x_0}_{B_{2R(x_0)}} - G^{x_0}_{B_{2Rq^{-1}(x_0)}} \leq$$

$$\leq G^{x_0}_{B_{2Rq^{-1}(x_0)}} - \frac{CR^2}{\lambda w(B_R)}.$$

The following Poincaré inequality (see [11] and [12]) will also be used later:

$$\frac{Cc_3R^2}{w(B_R)}\int_{B_R} |\nabla(u-\nu)|^2 w dx \geq \frac{c_5}{w(B_R)}\int_{B_R} |(u-\nu-m_R)|^2 w dx \geq$$

$$\geq \frac{c_5}{w(B_R)}\int_{B_R} |(u-\nu-a_R)|^2 w dx - C|a_R-m_R|^2 \geq$$

$$\geq c_6\operatorname*{osc}_{B_{qR}}{}^2(u-\nu) - C|a_R-m_R|^2 - Cg(R)^2.$$

The last inequality follows from (4.44). Thus we obtain:

$$\Big(\frac{\gamma}{2}+c_6\Big)\operatorname*{osc}_{B_{qR}}{}^2(u-\nu) + (c_4\lambda\gamma + c_3\lambda)\int_{B_{qR}} |\nabla(u-\nu)|^2 G^{x_0}_{B_{2R}} w dx \leq$$

$$\leq \gamma(1+c_1+\gamma)\operatorname*{osc}_{B_R}{}^2(u-\nu) + \tag{4.50}$$

$$+ c_3\lambda\int_{B_R} |\nabla(u-\nu)|^2 G^{x_0}_{B_{2Rq^{-1}}} w dx + C|a_R-m_R|^2 + Cg(R)^2.$$

We are practically done, we will just use "standard" techniques to get precisely (4.43).

Namely, we will first estimate $|a_R - m_R|$ using (4.48) with $r = R$ and $R = R_{2/2}$, then we will add on both sides of (4.50) $hc_3\lambda\,\mathrm{osc}_{B_{qR}}{}^2(u-\nu)$. Now we can choose γ such that $\frac{\gamma}{2} + c_6 > \gamma(1 + c_1 + \gamma)$ and h such that $c_3\lambda + \frac{1}{h}(\frac{\gamma}{2} + c_6) = c_4\gamma + c_3$.

This guarantees that the constant is larger on the left hand side than on the right hand side, so that (4.50) can be written

$$
\begin{aligned}
&(1 + c_7)V_\nu(qR)^2 \leq \\
(4.51)\qquad &\leq c_7 V_\nu(R)^2 + \tilde{C}\Big(V_\nu\Big(\frac{R_2}{2}\Big)\sum_{i=1}^{2}\exp\Big(-\beta\int_R^{R_2}\delta_i(\varepsilon_i,\rho)\frac{d\rho}{\rho}\Big) + \varepsilon_1 + \\
&+ \varepsilon_2 + g^2(R_2) + R_2^{2\alpha}\Big) \equiv c_7 V_\nu^2(R) + \tilde{C}A^2(R, R_2)
\end{aligned}
$$

where

$$
V_\nu(\rho)^2 \equiv h\,\underset{B_\rho}{\mathrm{osc}}(u-\nu) + \lambda\int_{B_\rho}|\nabla(u-\nu)|^2 G^{x_0}_{B_{2\rho q^{-1}}} w\,dx.
$$

Fix $r \in \Big(0, \frac{qR}{2}\Big)$ and $1 \leq \tau \leq \frac{R}{2r}$, suppose

$$
(4.52)\qquad \tau > q^{-1} \quad \text{and} \quad V_\nu^2(r) \geq 2\tilde{C}A^2(\tau r, R)
$$

then for $q^{-1}r \leq \rho \leq \tau r$:

$$
(1 + C)V_\nu^2(q\rho) \leq CV_\nu^2(\rho)
$$

which implies by a standard analytical lemma (see e.g. [24]):

$$
(4.53)\qquad V_\nu^2(r) \leq C\tau^{-\beta}V_\nu^2(\tau r)\,.
$$

If $1 \leq \tau \leq q^{-1}$ (4.53) holds trivially for $C = q^{-\beta}$.

If V_ν does not satisfy (4.52) then

$$V_\nu^2(r) \le 2\tilde{C}A^2(\tau r, R). \tag{4.54}$$

Therefore in any case we have

$$\begin{aligned} V_\nu^2(r) \le C\Big\{ V_\nu^2\Big(\frac{R}{2}\Big)\Big[\tau^{-\beta} + \tau^\beta \sum_{i=1}^{2} \exp\Big(-\beta \int_r^R \delta_i(\varepsilon_i,\rho)\frac{d\rho}{\rho}\Big)\Big] + \\ + \varepsilon_1^2 + \varepsilon_2^2 + g(R)^2 + R^{2\alpha}\Big\}. \end{aligned} \tag{4.55}$$

We have used the fact that $\delta_i \le 1$ and thus:

$$\tau^\beta \exp\Big(-\beta \int_r^{\tau r} \delta_i(\varepsilon_i,\rho)\frac{d\rho}{\rho}\Big) \ge 1.$$

Consequently if we choose

$$\tau = \Big[\frac{1}{2}\sum_{i=1}^{2} \exp\Big(-\int_r^R \delta_i(\varepsilon_i,\rho)\frac{d\rho}{\rho}\Big)\Big]^{-\frac{1}{2}} < \frac{R}{2r}$$

(4.55) becomes:

$$\begin{aligned} V_\nu^2(r) \le C\Big\{ V_\nu^2(R/2)\Big[\sum_{i=1}^{2} \exp\Big(-\frac{\beta}{2} \int_r^R \delta_i(\varepsilon_i,\rho)\frac{d\rho}{\rho}\Big)\Big] + \\ + \varepsilon_1^2 + \varepsilon_2^2 + g(R)^2 + R^{2\alpha}\Big\} \end{aligned} \tag{4.56}$$

It is easy to check that (see also Thm 2.1)

$$|V_0(r) - V_\nu(r)| \le \Big[h(\operatorname*{osc}_{B_{R/2}} \nu)^2 + \lambda \int_{B_{R/2}} |\nabla \nu|^2 G^{x_0}_{Rq^{-1}} w dx\Big]^{1/2} \le g(R)$$

We have finally obtained

$$\begin{aligned} V_0(r) \le C\Big\{ V_0(R/2)\Big[\sum_{i=1}^{2} \exp\Big(-\beta \int_r^R \delta_i(\varepsilon_i,\rho)\frac{d\rho}{\rho}\Big)\Big] + \\ + \varepsilon_1 + \varepsilon_2 + g(R)\Big\} \end{aligned} \tag{4.57}$$

Inequality (4.57) obviously holds also with V replacing V_0, thus we obtain the desired inequality (4.43) as soon as we let $\varepsilon_i \to \sigma_i\omega_{i,\sigma_i}(r,R)$ for $i=1; i=2$. $\square$

Remark 4.3. Theorem 4.4 extends the results obtained in [10] for the uniformly elliptic linear case, to the degenerate case with quadratic Hamiltonian (see theorem 2.2 of [10]). But it should be noticed that in the right hand side of (4.43) there is the term $\|L_0\nu\|^{\frac{1}{2}}_{K(B_R)}$ which is somehow less natural then $\|L_0\nu\|_{K(B_R)}$ appearing in theorem 2.2 of [10]. Some comments thus seem necessary.

As previously mentioned , if $\overline{\psi}_1(x_0) < \underline{\psi}_2(x_0)$ then in a small neighborhood of x_0, we can choose a constant as the regular separating function ν of (4.42) and thus, of course, the anomaly disappears. If instead, $\overline{\psi}_1(x_0) = \underline{\psi}_2(x_0)$ then beside (4.43), we can obtain the following inequality,

$$
\begin{aligned}
V(r) \le C\Big\{ & w(B_R(x_0))^{-1/2}\|u-\nu\|_{L^2(B_R,w)}[\omega_{1,\sigma_1}(r,R)+\omega_{2,\sigma_2}(r,R)]^{\beta}+ \\
& + \sigma_1\omega_{1,\sigma_1}(r,R) + \sigma_2\omega_{2,\sigma_2}(r,R) + \\
& + \frac{1}{(w(B_R(x_0)))^{\frac{1}{2}}}\|\nu-\nu_R\|_{L^2(B_R,w)} + \|L_0\nu\|_{K(B_R)} + R^{\alpha} + \\
& + (\sup_{B_R}\psi_1 - \nu_R)^+ + (\nu_R - \inf_{B_R}\psi_2)^+ \Big\}
\end{aligned} \tag{4.58}
$$

Inequality (4.58) doesn't appear to us to be a real improvement with respect to (4.43).

5 – Proofs of the propositions

In this section we give the proof of the propositions and lemmas used to prove the two theorems of the previous section. Most of these are analogous to the proof of proposition 4.5 though usually slightly easier. Thus we will write it down in details, in the other proofs we will just emphasize the differences.

Proof of proposition 4.5. Let $X_1 = \cosh\{M((u-\nu)^+)^2\}$ and $X_2 = \sinh\{M((u-\nu)^+)^2\}$. Similarly to [27] we choose as a test function in (4.35)

$$v = u - \varepsilon[(u-\nu)^+\theta G_\rho X_1]$$

where $G_\rho = G_\rho^z$ refers to the approximate Green's function relative to $G^z_{B_{2R}(z)}$ (see (2.10)) and θ is a suitable function which is zero outside of $B_{tR}(z)$ and that will be determined later.

It is easy to check that for the convexity of $\mathcal{K}$, with ε sufficiently small, v is in $\mathcal{K}$, and (4.35) becomes

$$\begin{aligned}
0 &\geq a(u,(u-\nu)^+\theta G_\rho X_1) + \int\limits_{B_R(z)} \tilde{H}(u)(\theta G_\rho (u-\nu)^+ X_1)dx = \\
&= a((u-\nu)^+,(u-\nu)^+\theta G_\rho X_1) + \int\limits_{B_R(z)} \tilde{H}(u)(\theta G_\rho (u-\nu)^+ X_1)dx + \\
&+ a(\nu,(u-\nu)^+\theta G_\rho X_1) \geq \\
&\geq \lambda \int\limits_{B_R(z)} |\nabla(u-\nu)^+|^2 (X_1 + 2M[(u-\nu)^+]^2 X_2)\theta G_\rho w dx + \\
&+ \int\limits_{B_R(z)} a_{ij}(u-\nu)_i \theta_j G_\rho X_1 (u-\nu)^+ dx + \qquad (5.59)\\
&+ \int\limits_{B_R(z)} a_{ij}(u-\nu)_i (G_\rho)_j \theta X_1 (u-\nu)^+ dx + \\
&- k_1 \int\limits_{B_R(z)} G_\rho \theta X_1 (u-\nu)^+ w dx - k_2 \int\limits_{B_{tR}(z)} |\nabla(u-\nu)|^2 G_\rho \theta X_1 (u-\nu)^+ w dx \\
&- k_2 \int\limits_{B_{tR}(z)} |\nabla \nu|^2 G_\rho \theta X_1 (u-\nu)^+ w dx + a(\nu,(u-\nu)^+\theta G_\rho X_1) \\
&\equiv A_1^\rho + A_2^\rho + A_3^\rho - A_4^\rho - A_5^\rho - A_6^\rho + A_7^\rho.
\end{aligned}$$

Observe that $a(\nu,(u-\nu)^+\theta G_\rho X_1) = \int\limits_{B_R} (G_\rho)\theta(u-\nu)^+ X_1 d(L_0\nu)$ so that letting ρ converge to zero we get from theorem 2.1:

$$A_6^0 - A_7^0 \leq Cg(R)^2.$$

Let $s \in (0,t)$, $t \in (0,1)$ and $\tau \in C^1(\mathbb{R}^N)$ such that $0 \leq \tau \leq 1$, $\tau = 1$ on $B_{sR}(z)$, supp$\tau \subset B_{tR}(z)$ and

$$|\nabla \tau| \leq C(t,s) r^{-1}. \tag{5.60}$$

We now choose $\theta = \tau^2$.

Let us evaluate the terms in (5.59). We first choose M large enough such that

$$A_1^\rho - A_5^\rho \geq \int\limits_{B_{tR}(z)} |\nabla(u-\nu)^+|^2 G_\rho \tau^2 X_1 w dx.$$

Furthermore observe that

$$\begin{aligned}
|A_2^\rho| &= \Big| \int\limits_{B_R(z)} a_{ij}(u-\nu)_i \theta_j G_\rho X_1 (u-\nu)^+ dx \Big| \leq \\
&\leq \Lambda\varepsilon \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla(u-\nu)^+|^2 X_1 \tau^2 G_\rho w dx + \\
&+ \frac{\Lambda}{\varepsilon} \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} |(u-\nu)^+|^2 X_1 |\nabla\tau|^2 G_\rho w dx.
\end{aligned} \tag{5.61}$$

Letting ρ converge to 0 and using (5.60) together with the properties (2.9) of G^y we obtain:

$$\begin{aligned}
|A_2^0| &\leq \Lambda\varepsilon \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla(u-\nu)^+|^2 X_1 \tau^2 G^z_{B_{2R}(z)} w dx + \\
&+ \frac{\Lambda}{\varepsilon} \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} |(u-\nu)^+|^2 X_1 |\nabla\tau|^2 G^z_{B_{2R}(z)} w dx \leq \\
&\leq \frac{\lambda}{6} \int\limits_{B_{tR}(z)} \tau^2 X_1 |\nabla(u-\nu)^+|^2 G^z_{B_{2R}(z)} w dx + \\
&+ \frac{C}{w(B_{sR}(z))} \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} X_1 |(u-\nu)^+|^2 w dx
\end{aligned} \tag{5.62}$$

Similarly, with $\rho < \dfrac{sR}{2}$ and choosing $\tau := \tau_0 \tilde{\tau}$ where $\tau_0 = 0$ on $B_{sR/2}(z)$ and equal to 1 on $B_{tR}(z) - B_{sR}(z)$ so that we can use the inequality

(2.11), A_3^ρ becomes:

$$
\begin{aligned}
A_3^\rho = & \int_{B_R(z)} a_{ij}(u-\nu)_i (G_\rho)_j \tau^2 X_1 (u-\nu)^+ dx = \\
= & \frac{1}{2M} \int_{B_R(z)} a_{ij}[X_2\tau^2]_i (G_\rho)_j dx - \frac{1}{M} \int_{B_{tR}(z)\setminus B_{sR}(z)} a_{ij} X_2 \tau_i \tau (G_\rho)_j dx \geq \\
\geq & \frac{1}{2Mw(B_\rho(z))} \int_{B_\rho(z)} X_2\tau^2 w dx - \frac{\Lambda^3\varepsilon}{M\lambda^2} \int_{B_{tR}(z)\setminus B_{sR}(z)} G_\rho^2 |\nabla(\tau X_2^{1/2})|^2 w dx + \\
& - \frac{\Lambda}{\varepsilon M} \int_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla \tau|^2 X_2 w dx
\end{aligned}
\tag{5.63}
$$

Now we shall use the fact that

$$
|\nabla(\tau X_2^{1/2})|^2 \leq 2\left(|\nabla\tau|^2 X_2 + \tau^2 |\nabla(u-\nu)^+|^2((u-\nu)^+)^2 \frac{X_1^2}{X_2}\right)
$$

and decompose $\varepsilon = \overline{\varepsilon}\eta$ choosing η such that $\eta = Cw(B_{sR}(z))R^{-2}$. Hence for $\rho \to 0$, using τ estimates (see also (2.9)), (5.63) becomes

$$
\begin{aligned}
A_3^0 \geq & \frac{C}{2}((u(z)-\nu(z))^+)^2 - \frac{C\Lambda}{\overline{\varepsilon} w(B_{sR}(z))} \int_{B_{tR}(z)\setminus B_{sR}(z)} ((u-\nu)^+)^2 w dx + \\
& - \frac{2\Lambda^3 \overline{\varepsilon} C}{\lambda^2} \int_{B_{tR}(z)\setminus B_{sR}(z)} G^z_{B_{2R}(z)} ((u-\nu)^+)^2 w dx + \\
& - \frac{2\overline{\varepsilon}\Lambda^3 C}{\lambda^2} \int_{B_{tR}(z)\setminus B_{sR}(z)} X_1^2 \tau^2 |\nabla(u-\nu)^+|^2 (G^z)_{B_{2R}(z)} w dx
\end{aligned}
\tag{5.64}
$$

We are only left to estimate A_4^ρ, as before we let ρ converge to zero and use inequality (3.12) of [27]:

$$
|A_4^0| \leq C \int_{B_{tR}(z)} G^z_{B_{2R}} w dx \leq CR^2.
$$

Considering all these estimates we derive from (5.59) for $\tilde{\varepsilon}$ sufficiently small:

$$\lambda \int\limits_{B_{tR}(z)} \tau^2 |\nabla(u-\nu)^+|^2 G^z_{B_{2R}(z)} w dx + [\frac{C}{2}(u(z)-\nu(z))^+]^2 \leq$$

$$\leq C\lambda \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} |(u-\nu)^+|^2 G^z_{B_{2R}(z)} w dx + Cg(R)^2.$$

Clearly the constant in front of the first term on the right hand side is the sum of the constants in front of all the terms of the same kind.

We finally obtain (see also (2.9) and (2.6)):

$$\lambda \int\limits_{B_{sR}(z)} |\nabla(u-\nu)^+|^2 G^z_{B_{2R}(z)} w dx + [(u(z)-\nu(z))^+]^2 \leq$$

(5.65)

$$\leq \frac{C}{w(B_{tR}(z))} \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} |(u-\nu)^+|^2 w dx + Cg(R)^2.$$

As in the proof of theorem 4.4, we choose $z \in B_{qR}(x_0)$ with $q < 1/3$, $s = 2q$ and $t = (1-q)$ in order to have $B_{(1-q)R}(z) \setminus B_{sR}(z) \subset B_R(x_0) \setminus B_{qR}(x_0)$.

Applying inequality (2.6) in (5.65) we derive:

$$\sup_{B_{qR}(x_0)} ((u-\nu)^+)^2 \leq C\Big\{\frac{1}{w(B_R(x_0))} \int\limits_{B_R(x_0)\setminus B_{qR}(x_0)} ((u-\nu)^+)^2 w dx\Big\} + Cg(R)^2$$

While to estimate the gradient term we simply choose $z = x_0$, $t = 1$ and $s = q$:

$$\frac{\lambda}{2} \int\limits_{B_{qR}} |\nabla(u-\nu)^+|^2 G^{x_0}_{B_{2Rq^{-1}}(x_0)} w dx \leq$$

$$\leq \frac{C}{w(B_R(x_0))} \frac{\lambda}{2} \int\limits_{B_R(x_0)\setminus B_{qR}(x_0)} |(u-\nu)^+|^2 G^{x_0}_{B_{2Rq^{-1}}(x_0)} w dx + Cg(R)^2.$$

Summing these inequalities we have obtained the desired (4.44) for the positive part. For the negative part we can of course proceede in an analogous way.

The proof of proposition 4.2 is similar to the proof of proposition 4.5 hence, to avoid superfluous repetitions, we only stress the differences and we refer to the previous proof for details.

PROOF OF PROPOSITION 4.2. Let $X_+ = \cosh\{M((u-d_1)^+)^2\}$ and $X_- = \cosh\{M((u-d_2)^-)^2\}$ where:

$$d_1 = \sup_{B_R} \psi_1 \vee d_R \text{ and } \ d_2 = \inf_{B_R} \psi_2 \wedge d_R$$

and M is a positive large constant that will be chosen later.

Consider as a test function in (4.35)

$$v = u + \varepsilon G^z_\rho \tau^2[(u-d_2)^-]X_- - \varepsilon G^z_\rho \tau^2[(u-d_1)^+]X_+$$

with τ a cut off function as in the previous proof.

Proceeding as in the previous proof, we easily obtain (see in particular (5.65) with $\nu = d_1$ and $\nu = d_2$):

$$\begin{aligned}&\sup_{B_{sR}}(u-d_1)^+ + \sup_{B_{sR}}(u-d_2)^- \le \\ &\le C\Big\{\Big(\frac{1}{w(B_R(x_0))}\int_{B_R\setminus B_{sR}}(|(u-d_1)^+|^2+|(u-d_2)^-|^2)w\,dx\Big)^{\frac{1}{2}}+R\Big\}\end{aligned} \tag{5.66}$$

Now observing that $\sup_{B_R}\psi_1 \wedge \inf_{B_R}\psi_2 \le d_R \le \sup_{B_R}\psi_1 \vee \inf_{B_R}\psi_2$ we have that

$$0 \le d_1 - d_2 \le [\sup_{B_R}\psi_1 - \inf_{B_R}\psi_2]^+.$$

Furthermore

$$\operatorname*{osc}_{B_{sR}} u \le d_1 - d_2 + \sup_{B_{sR}}(u-d_1)^+ + \sup_{B_{sR}}(u-d_2)^-$$

and

$$\begin{aligned}(u-d_1)^+ &\le (u-d_R)^+\\ (u-d_2)^- &\le (u-d_R)^-\end{aligned}$$

Thus (4.37) is an immediate consequence of (5.66).

From theorem 2.1 we derive in particular that if $\nu \in H^1(B_{R_1}, w)$,and $L_0\nu \in K(B_{R_1})$ then $\tilde{H}(\nu) \in K(B_{R_1})$. Moreover the following proposition is true:

Proposition 5.1. *If $\nu \in H^1(B_{R_1}, w) \cap L^\infty$ and $L_0\nu + \tilde{H}(\nu) = \mu$, $\mu \in K(B_{R_1})$ then $L_0\nu$ belongs to $K(B_{R_1})$ and the following estimate holds:*

$$\|\nu\|_{L^\infty(B_R)} \le \sup_{\partial B_R} \nu + C(\|\mu\|_{K(B_{R_1})} + R^\alpha). \tag{5.67}$$

where α and C are positive constants independent of R and x_0.

Sketch of the proof.

Step one: Proceeding as in the proof of Proposition 4.5 with the test function $v = \tau^2 G_\rho(\nu - \nu_R)\cosh M(\nu - \nu_R)^2$ we obtain:

$$\int_{B_{qR}(x_0)} |\nabla\nu|^2 G_{2R}^{x_0} dx \le C(\frac{1}{w(B_R)})\|\nu - \nu_R\|^2_{L^2(B_R - B_{qR})} + \|\mu\|^2_{K(B_{R_1})} + R^2 \tag{5.68}$$

using with the same notations as in the proof of proposition 4.5.

Step two: We apply the Poincaré inequality to the right hand side of inequality (5.68) in order to use the “hole filling” technique (see e.g. [14]), thus we obtain that there exist $\alpha \in (0, 1)$ and $C > 0$ such that:

$$\|\tilde{H}(\nu)\|_{K(B_R)} \le C(R^\alpha + \|\mu\|_{K(B_R)})$$

By the conditions (3.24) on H, from the previous inequality and remark 2.2 we can deduce that $\tilde{H}(\nu) \in K(B_{R_1})$ and the first part of proposition 5.1 is proved.

Final step: As ν satisfies $L_0\nu = \mu - \tilde{H}(\nu) \in K(B_{R_1})$ we can apply proposition 5.3 of [10] and thus

$$\|\nu\|_{L^\infty(B_R)} \le \sup_{\partial B_R} |\nu| + C(\|\mu\|_{K(B_R)} + \|\tilde{H}(B_R)\|_{K(B_R)})$$

$$\le \sup_{\partial B_R} |\nu| + C(\|\mu\|_{K(B_R)} + R^\alpha).$$

This completes the proof.

In the proof of lemma 4.3 we are going to use a comparison result, which is due to BARLES and MURAT see [2]. Our version is slightly different from theirs because suited for our purposes but can be proved in the same way. Precisely let u_1, u_2 be solutions of the following problems for $i = 1$ and $i = 2$ respectively:

$$\begin{cases} u_i \in H^1(\Omega, w) \cap L^\infty(\Omega) \;\; \phi_i \le u_i \le \chi_i \\ a(u_i, \varphi - u_i) + \int\limits_\Omega (\tilde H(u_i) - f_i)(\varphi - u_i)dx \ge 0 \\ \forall \varphi \in H^1(\Omega, w) \cap L^\infty(\Omega)\; \phi_i \le \varphi \le \chi_i; \;\; (u_i - \varphi) \in H^1_0(\Omega, w). \end{cases}$$

LEMMA 5.2. *Assume the previous notations and conditions in particular* (4.34) *and* $\phi_1 \le \phi_2$, $\chi_1 \le \chi_2$, $f_1 \le f_2$ *and* $(u_1 - u_2)^+ \in H^1_0(\Omega, w)$ *then*

$$u_1 \le u_2 w\text{-}q.e. \;\; in\; \Omega.$$

PROOF OF LEMMA 4.3. We shall prove (4.39) the other being analogous. For $t := \Psi_2(\varepsilon, R) = +\infty$ (4.39) is trivial, so we shall suppose that t is finite. Let $E_t = \{x \in B_{R/2} : \psi_2 \le t\}$. Let v be a solution of the following problem

$$\begin{cases} v \in H^1(B_R, w) \cap L^\infty(B_R) \;\; v \le t \;\; w - \text{q.e. in } E_t \;\; v = u \vee t \text{ in } \partial B_R \\ a(v, \varphi - v) + \int\limits_\Omega (\tilde H(v))(\varphi - v)dx \ge \int\limits_{B_R} \tilde H(t)^+(\varphi - v)dx \\ \forall \varphi \in H^1(B_R, w) \cap L^\infty(B_R)\; \varphi \le t \;\; w - \text{q.e. in } E_t \;\; \varphi = u \vee t \text{ in } \partial B_R \end{cases}$$

then by lemma 5.2 $v \ge t$ and $v \ge u$ in B_R. Hence

$$\sup_{B_r} u \le \sup_{B_r} v = \inf_{B_r} v + \operatorname*{osc}_{B_r} v$$

Using for v, the results regarding the one obstacle case i.e. inequality (3.54) and corollary 3.1 of [27] and remark 2.2, v satisfies:

$$\operatorname*{osc}_{B_r} v \le \tag{5.69}$$

$$C\left(\left[\frac{1}{w(B_R)} \int\limits_{B_R} |(v-t)|^2 w(x)dx\right]^{\frac12} \exp\left(-\beta \int\limits_r^R \frac{\operatorname{cap}_w(E_t \cap B_\rho, B_{2\rho})}{\operatorname{cap}_w(B_\rho, B_{2\rho})} \frac{d\rho}{\rho}\right) + R + \varepsilon\right).$$

Let us observe that $E_2^*(\varepsilon_2, \rho) \subset E_t \cap B_\rho$. Consider v_2 solution of

$$\begin{cases} L_0 v_2 + \tilde{H}(v_2) = \tilde{H}^+(t) & \text{in } B_R \\ v_2 = u \vee t & \text{on } \partial B_R \end{cases}$$

Hence from Remark 2.2 and proposition 5.1 with $\mu \equiv \tilde{H}^+(t)$, we derive the following inequality:

$$v - t \le v_2 - t \le \sup_{\partial B_R} u \vee t - t + CR^\alpha \le (\sup_{B_R} u - t)^+ + CR^\alpha \tag{5.70}$$

and we obtain (4.39) putting together inequalities (5.69) and (5.70). This complete the proof.

Proof of lemma 4.7: We only prove (4.47), the other being analogous.
Fix $t = \inf_{B_r}(\psi_2 - \nu)$, $0 < r \le \dfrac{R}{4}$ and $\xi = (L_0\nu + \tilde{H}(\nu + t))^+$, and we recall that $\xi \in K(\Omega)$ by Remark 5.1. Furthermore consider u_2 solution of

$$\begin{cases} u_2 \in H^1(B_{R/2}, w) \cap L^\infty, u_2 \le \psi_2 \text{ w-q.e. on } B_{R/2},\ u_2 = u \vee (\nu + t) \text{ on } \partial B_{R/2} \\ a(u_2, v - u_2) + \displaystyle\int_{B_{\frac{R}{2}}} \tilde{H}(u_2)(v - u_2)dx \ge \int_{B_{\frac{R}{2}}} \xi(v - u_2)dx \\ \forall v \in H^1(B_{R/2}, w) \cap L^\infty, v \le \psi_2 \text{ w-q.e. on } B_{R/2},\ u_2 = u \vee (\nu + t) \text{ on } \partial B_{R/2} \end{cases}$$

By comparison lemma 5.2 we have, as in the proof of lemma 4.2:

$$\psi_1 \le \nu + t \le u_2$$

and hence also $u \le u_2$, therefore:

$$\begin{aligned} \sup_{B_r}(u - \nu) &\le \sup_{B_r} u_2 - \inf_{B_r} \nu \le \inf_{B_r} \psi_2 + \operatorname*{osc}_{B_r} u_2 - \inf_{B_r} \nu \\ &\le \sup_{B_r}(\psi_2 - \nu) + \operatorname*{osc}_{B_r} u_2 + \operatorname*{osc}_{B_r} \nu. \end{aligned} \tag{5.71}$$

So that using the results for one obstacle i.e. inequality (3.31) and corollary 3.1 of [27], u_2 satisfy:

$$\operatorname*{osc}_{B_r} u_2 \le C\left\{ \|u_2 - d\|_{L^2(B_{R/2}, w)} \exp\left(-\beta \int_r^{R/2} \delta_2(\varepsilon, \rho) \frac{d\rho}{\rho} \right) + R + \varepsilon + \|\xi\|_{K(B_{R/2})} \right\} \tag{5.72}$$

for any $d \leq \psi_2$, so that in particular we can choose $d := d_2 := \inf_{B_{R/2}}(\nu + t)$.

We consider now w_2 solution of $L_0 w_2 + \tilde{H}(w_2) = \xi$, in $B_{R/2}$ and $w_2 = u \vee (\nu + t)$ on $\partial B_{R/2}$, which by comparison argument satisfy $u_2 \leq w_2$.

To estimate the L^2 norm of u_2, we can procede as in [10] to get:

$$
\begin{aligned}
&\frac{1}{w(B_{R/2})}\|u_2 - d_2\|_{L^2(B_{\frac{R}{2}},w)} \leq \sup_{B_{\frac{R}{2}}}(w_2 - d_2) \\
&\leq \sup_{\partial B_{\frac{R}{2}}}[u \vee (\nu + t)] - \inf_{B_{\frac{R}{2}}}(\nu + t) + C(\|\xi\|_{K(B_R)} + R^\alpha) \\
&\leq [\sup_{B_{\frac{R}{2}}} u - \sup_{B_{\frac{R}{2}}}(\nu + t)]^+ + \operatorname*{osc}_{B_{\frac{R}{2}}} \nu + C(\|\xi\|_{K(B_R)} + R^\alpha) \\
&\leq \operatorname*{osc}_{B_{\frac{R}{2}}} u + \operatorname*{osc}_{B_{\frac{R}{2}}} \nu + C(\|\xi\|_{K(B_R)} + R^\alpha) \\
&\leq \operatorname*{osc}_{B_{\frac{R}{2}}}(u - \nu) + C(\frac{1}{w(B_R)}\|\nu - \nu_R\|_{L^2(B_R)} + \|\xi\|_{K(B_R)} + R^\alpha)
\end{aligned} \tag{5.73}
$$

For $r \in (0, \frac{R}{4}]$ putting together inequalities (5.71) ,(5.72) and (5.73) we get (4.47) for $r \in \left(0, \frac{R}{4}\right]$. For $r \in \left(\frac{R}{4}, R\right)$ then it is enough to notice that:

$$
\exp(-\beta \int_r^R \delta_2(\varepsilon, \rho)\frac{d\rho}{\rho}) \geq 4^{-\beta}
$$

so that

$$
\begin{aligned}
\sup_{B_r}(u - \nu) &= \inf_{B_r}(u - \nu) + \operatorname*{osc}_{B_r}(u - \nu) \\
&\leq \inf_{B_r}(\psi_2 - \nu) + 4^\beta \operatorname*{osc}_{B_R}(u - \nu) \exp(-\beta \int_r^R \delta_2(\varepsilon_2, \rho)\frac{d\rho}{\rho}).
\end{aligned}
$$

which implies (4.47) for every $C \geq 4^\beta$.

This complete the proof.

Proof of proposition 4.6 We procede as in the proof of proposition 4.5 but we choose $\theta = \theta_{sR}$ to be the w-capacitary potential of $B_{sR}(z)$

in $B_{tR}(z)$ with respect to L_0. The proof proceedes analogously so that we emphasize only the differences. In particular A_2^ρ becomes:

$$|A_2^\rho| := |\int_{B_R(z)} a_{ij}(u-\nu)_i\theta_j G_\rho X_1(u-\nu)^+ dx| \leq$$

$$\frac{C\Lambda\varepsilon}{4}\int_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla(u-\nu)^+|^2 G_\rho w dx + \frac{C\Lambda}{\varepsilon}\sup|(u-\nu)^+|^2 \int_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla\theta|^2 G_\rho w dx$$

We let $\rho \to 0$ and we use the definition of potential to obtain for any $\eta > 0$:

$$\begin{aligned} |A_2^0| &\leq \frac{C}{\eta}\int_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla(u-\nu)^+|^2 G^z_{B_{2R}(z)} w dx + \\ &+ \Lambda C\eta \operatorname{cap}^w(B_{sR}(z), B_{tR}(z)) \sup((u-\nu)^+)^2. \sup_{B_{tR}(z)\setminus B_{sR}(z)} G^z_{B_{2R}(z)}. \end{aligned} \tag{5.74}$$

On the other hand (5.63) becomes

$$\begin{aligned} A_3^\rho &= \int_{B_R(z)} a_{ij}(u-\nu)_i (G_\rho)_j \theta X_1 (u-\nu)^+ dx = \\ &= \frac{1}{2M}\int_{B_R(z)} a_{ij}[X_2\theta]_i (G_\rho)_j dx - \frac{1}{2M}\int_{B_{tR}(z)\setminus B_{sR}(z)} a_{ij} X_2 \theta_i (G_\rho)_j dx \geq \\ &\geq \frac{C}{2w(B_\rho)(z)}\int_{B_\rho(z)} [(u-\nu)^+]^2 \theta w dx - \frac{1}{2M} a(\theta, G_\rho X_2) + A_2^\rho \end{aligned}$$

So that we can use the previous estimate for the last term while it is easy to see that:

$$\frac{1}{2M} a(\theta, G_\rho X_2) \leq C \sup_{B_{tR}(z)} ((u-\nu)^+)^2$$

and A_3^0 becomes (see also (2.9), (2.14)) :

$$\begin{aligned} A_3^0 &\geq C((u-\nu)^+)^2(z) - C\sup_{B_{tR}(z)}((u-\nu)^+)^2 - \\ &- \frac{C\lambda}{\eta}\int_{B_{tR}(z)\setminus B_{sR}(z)} |\nabla(u-\nu)^+|^2 G^z_{B_{2R}(z)} w dx \end{aligned} \tag{5.75}$$

Putting these estimates in (5.59) and choosing $\eta = \gamma$, and renaming the constants we obtain:

(5.76)

$$\lambda \int\limits_{B_{sR}(z)} |\nabla(u-\nu)^+|^2 G^z w dx + [(u(z)-\nu(z))^+]^2 \leq$$

$$\leq (c_1+\gamma) \sup |(u-\nu)^+|^2 + \frac{c_2\Lambda}{\gamma} \int\limits_{B_{tR}(z)\setminus B_{sR}(z)} G^z |\nabla(u-\nu)^+|^2 w dx +$$

$$+ g(R)^2.$$

This complete the proof.

Proof of Remark 4.3. We are in the hypotheses that $\overline{\psi}_1(x_0) = \underline{\psi}_2(x_0)$. If we put

$$\nu_R = \frac{1}{w(B_R(x_0))} \int\limits_{B_R(x_0)} \nu(x) w(x) dx$$

and

$$d_{1R} := \sup_{B_R(x_0)} \psi_1 \vee \nu_R, \quad d_{2R} = \inf_{B_R(x_0)} \psi_2 \wedge \nu_R,$$

and proceed as in the proof of lemma 4.2 we obtain:

$$\sup_{B_{tR}(x_0)} ((u-\nu)^+)^2 \leq C \Big\{ \frac{1}{w(B_R(x_0))} \int\limits_{B_R(x_0)\setminus B_{tR}(x_0)} ((u-\nu)^+)^2 w dx + g_1(R)^2 \Big\}$$

(5.77)

$$\sup_{B_{tR}(x_0)} ((u-\nu)^-)^2 \leq C \Big\{ \frac{1}{w(B_R(x_0))} \int\limits_{B_R(x_0)\setminus B_{tR}(x_0)} ((u-\nu)^-)^2 w dx + g_2(R)^2 \Big\}$$

where

$$g_i(R) = R + \|L_0 \nu\|_{K(B_R(x_0))} + \frac{1}{w(B_R(x_0))^{\frac{1}{2}}} \|\nu - \nu_R\|_{L_2(B_R(x_0),w)} + |d_i - \nu_i|.$$

It is clear that if we use inequalities (5.77) to evaluate the terms A_6^ρ, A_7^ρ in the proof of proposition 4.5 then in the inequality (5.65) the term $g(R)^2$

is replaced by $g_i(R)^2$. Of course proposition 4.6 can be modified in the same way. Thus (4.43) becomes:

$$
\begin{aligned}
V(r) \leq C\{& w(B_R(x_0))^{-1/2}\|u-\nu\|_{L^2(B_R,w)}[\omega_{1,\sigma_1}(r,R)+\omega_{2,\sigma_2}(r,R)]^\beta + \\
&+\sigma_1\omega_{1,\sigma_1}(r,R)+\sigma_2\omega_{2,\sigma_2}(r,R)+ \\
&+\frac{1}{w(B_R(x_0))^{\frac{1}{2}}}\|\nu-\nu_R\|_{L^2(B_R,w)}+\|L_0\nu\|_{K(B_R)}+R^\alpha+ \\
&+(\sup_{B_R}\psi_1-\nu_R)^+ + (\nu_R-\inf_{B_R}\psi_2)^+.
\end{aligned} \tag{5.78}
$$

This completes the proof.

Acknowledgements

We are greatful to E. Fabes, C. Kenig and U. Mosco for the stimulating conversations we have had during the development of this work.

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Lavoro pervenuto alla redazione il 17 settembre 1993
ed accettato per la pubblicazione il 2 marzo 1994

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