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Conserved quantities of the gravitational field in tetrad notation

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RIASSUNTO: *I superpotenziali gravitazionali per la Lagrangiana del prim'ordine che genera le equazioni di Einstein, dipendenti dalla scelta di un "background" e determinati in un precedente lavoro, vengono qui riespressi nel formalismo delle tetradi. Utilizzando risultati sulla controimmagine della forma di Poincaré-Cartan per Lagrangiane del second'ordine, si dimostra che questi superpotenziali corrispondono a una famiglia di Lagrangiane del prim'ordine nelle tetradi, invarianti sotto trasformazioni di Lorentz.*

ABSTRACT: *We reexpress in tetrad form the background dependent family of superpotentials for the first-order Lagrangians which generate Einstein's field equations, found in an earlier paper. Using a result on the pull-back properties of the Poincaré-Cartan forms for second-order Lagrangian field theories, we show that these superpotentials are the superpotentials corresponding to a Lorentz invariant family of first-order tetrad Lagrangians.*

– Introduction

Among the intriguing problems still open in General Relativity there is the issue of a coherent definition of mass for the solutions of Einstein equations, and more generally the systematic theory of conserved quan-

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tities. In particular, for the mass of the gravitational field there exist in literature a number of different formulae, which often give different results when applied to families of space-times more general than the asymptotically flat ones (for which is commonly agreed that the mass should reduce to the ADM's formula [1]).

In the framework of calculus of variations on fiber bundles there is a general method to construct conserved quantities and Nöther currents starting from the so-called Poincaré-Cartan form associated to a given Lagrangian (see, e.g., [4] and [7]). This is based on the generation of a $(m-1)$-form $E(w)$, where m is the dimension of the base manifold (usually $m = 4$ for relativistic field theories) and w is a vectorfield in the base. It is known that $E(w)$ reduces 'on shell', i.e. along critical sections of the variational principle ensuing from the given Lagrangian, to the differential of a $(m-2)$-form $U(w)$, called the superpotential (see, e.g., [11]). This method, applied to the standard Hilbert Lagrangian for the gravitational field generates a classical superpotential, known as the KOMAR superpotential [13]. However, it is also known that KOMAR's superpotential does not generate at once the mass and the angular momentum of a rotating black hole, because of an extra factor 2. The superpotential of KOMAR, in fact, has to be replaced by a more general superpotential containing the previous one as a term, which apparently was the first derived by KATZ [12].

The earlier result of KATZ found a complete explanation in the papers [5] and [6] by two of us. In these papers it was shown that a whole family of covariant first-order Lagrangians equivalent to the second-order Hilbert Lagrangian exists, each one of them being parametrized by the choice of an arbitrary linear connection which has no dynamical role. This *background connection* may be fixed case by case on the basis of possible physical requirements, such as a priori asymptotic conditions (e.g., asymptotic flatness) or explicit perturbative expansions around a given space-time (e.g., cosmological models based on FRW solutions).

To each one of these first-order Lagrangians there corresponds a new superpotential, which is the sum of KOMAR's term plus an extra term depending of the background connection. Under the asymptotic flatness condition one recovers the splitting by KATZ.

This new class of superpotentials is stimulating not only because it explains the anomalous factor 2 in KOMAR's expression for the mass and angular momentum of a black hole, but also because it allows to reproduce for the equivalent mass of a charged black hole a result which can be obtained by applying the mass formula due to Penrose [16]. As it was shown in [18], Penrose's mass of a Reissner-Nordström solution with mass parameter M and charge e is

$$E_P(r) = M - \frac{\mathrm{e}^2}{2r}$$

which has a more convincing physical interpretation than the different result

$$E_K(r) = M - \frac{\mathrm{e}^2}{r}$$

obtained by a direct evaluation through the Komar superpotential.

Since Penrose's formula and the new superpotential found in [5], [6] reproduce the same result of charged black holes, it is interesting to look for a way to compare the two expressions and seek their possible relations. To this purpose, it seems to be more convenient to reexpress in NP formalism [15] the metric dependent superpotentials generated by the first-order Lagrangians. Hopefully, if not in general, one may try to compare the relevant mass formulae in wide classes of space-times, e.g. in the asymptotically flat cases or in some case of fixed symmetry (like for instance spherical symmetry).

To this end, one needs first to reexpress all the result of [5], [6] in the tetrad formalism, which is the basis for the NP formalism. In this paper we shall thence reexpress in tetrad form the family of (covariant) first-order metric Lagrangians and obtain a family of first-order tetrad Lagrangians which are covariant and invariant under Lorentz transformations of the tetrad fields. Moreover, we show that the corresponding superpotentials are the tetrad expressions of the relevant metric superpotentials with background; these results were first obtained in the Dissertation [14]. For the sake of simplicity and in order to let it be easily understood by the physical audience, the calculations are here performed in direct way; however, they rely on a fairly general result of calculus of variations in fiber bundles, which roughly states the covariance of Poincaré-Cartan forms under arbitrary bundle morphisms and, as a consequence, the functoriality of the ensuing superpotentials.

The application of these new tetrad superpotentials to the case of spherically symmetric space-times gives rise to a new larger class of allowable asymptotic conditions, which will form the subject of a forthcoming paper [8]. The transition from the tetrad version to the NP form of there superpotentials and the comparison with PENROSE's mass will be further investigated and we hope to obtain promising indications.

1 – Preliminaries and notation

In this section we shall specify the mathematical notation and recall some of the objects which will be used in the rest of this paper. For a more thorough discussion see [2] and [3].

Throughout this section M is a manifold of dimension m, with local coordinates (x^μ) $\mu = 0, \dots, m-1$; (B, M, π) is a fibered manifold, with fibered coordinates (x^μ, y^i) i, j, $k = 0, \dots, \dim(B) - m - 1$; $J^k B$ is the k^{th}-order jet prolongation of B, with fibered coordinates $(x^\mu, y^i, y^i_\alpha, \dots)$; $\pi^k_h : J^k B \to J^h B$ is the natural projection $(k > h)$; we set $J^0 B \equiv B$. For a generic manifold N: $\chi(N)$ is the space of all vectorfields on N, $\Omega^p(N)$ is the space of all p-forms on N, $\mathcal{F}(N)$ is the space of all real differentiable functions of N and we set $\Omega^0(N) \equiv \mathcal{F}$. We denote by $\rfloor$ the interior product.

1.1 – Horizontal and vertical operators

A *contact form* in (B, M, π) is a form $\omega \in \Omega^p(J^k B)$ satisfying the property $(j^k\sigma)^*\omega = 0$ for every section σ of (B, M, π). Contact forms are generated by exterior products of the *base contact forms* (or *structural forms*) of B:

$$\vartheta^i_{\alpha_1 \dots \alpha_s} \equiv dy^i_{\alpha_1 \dots \alpha_s} - y^i_{\alpha_1 \dots \alpha_s \lambda} dx^\lambda \,. \tag{1.1}$$

A *vertical vectorfield* on B is a vectorfield $V \in \chi(B)$ which satisfies $T\pi \circ V = 0$. A *horizontal form* on B is a form $\omega \in \Omega^s(B)$ satisfying the property

$$X \rfloor \omega = 0 \,,$$

if X is a vertical vectorfield on B. Horizontal s-forms constitute a $\mathcal{F}(B)$-linear subspace $\Omega^s_H(B)$ of $\Omega^s(B)$. In coordinates, a horizontal form ω is expressed by

$$\omega = \omega_{\alpha_1 \ldots \alpha_s} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_s} \,. \tag{1.2}$$

Every 1-form $\omega \in \Omega^1(B)$ can pulled-back to J^1B and rewritten as follows:

$$\begin{aligned} \omega &= \omega_\mu dx^\mu + \omega_i dy^i = \omega_\mu dx^\mu + \omega_i (dy^i - y^i_\mu dx^\mu + y^i_\mu dx^\mu) = \\ &= (\omega_\mu + \omega_i y^i_\mu) dx^\mu + \omega_i \vartheta^i \end{aligned}$$

so that $(\pi^1_0)^*\omega$ can be decomposed into the sum of a contact form $\omega_i \vartheta^i$ and of a horizontal form

$$\mathrm{Hor}(\omega) \equiv (\omega_\mu + \omega_i y^i_\mu) dx^\mu \in \Omega^1_H(J^1B) \,, \tag{1.3}$$

called the *horizontal part* of ω. The definition of horizontal part is extended to forms $\omega \in \Omega^1(J^kB)$ by setting:

$$\mathrm{Hor}(\omega) \equiv (\omega_\mu + \omega_i y^i_\mu + \omega^\alpha_i y^i_{\alpha\mu} + \ldots + \omega^{\alpha_1 \ldots \alpha_k}_i y^i_{\alpha_1 \ldots \alpha_k \mu}) dx^\mu \in \Omega^1_H(J^{k+1}B) \,, \tag{1.4}$$

for every 1-form

$$\omega = \omega_\mu dx^\mu + \omega_i dy^i + \omega^\alpha_i dy^i_\alpha + \ldots + \omega^{\alpha_1 \ldots \alpha_k}_i dy^i_{\alpha_1 \ldots \alpha_k} \in \Omega^1(J^kB)$$

on J^kB.

Finally the definition of the operator Hor can be extended to p-forms on J^kB by setting

$$\mathrm{Hor}(f) = f \tag{1.5}$$

for $f \in \mathcal{F}(J^kB)$ and requiring:

$$\mathrm{Hor}(\varphi \wedge \vartheta) = \mathrm{Hor}(\varphi) \wedge \mathrm{Hor}(\vartheta) \tag{1.6}$$

for generic forms φ and ϑ. In fact, this defines uniquely a linear operator

$$\mathrm{Hor} : \Omega^p(J^kB) \to \Omega^p_H(J^{k+1}B)$$

which coincides with the previous ones for $p = 0$ and $p = 1$ and satisfies (1.6).

The *horizontal differential* d_H of a horizontal p-form $\omega \in \Omega^p_H(J^k B)$ is defined by

$$d_H\omega = \mathrm{Hor}(d\omega)\,. \tag{1.7}$$

The horizontal differential satisfies the property

$$d_H(\varphi \wedge \vartheta) = d_H\varphi \wedge \vartheta + (-1)^p \varphi \wedge d_H\vartheta \tag{1.8}$$

where $\varphi \in \Omega^p_H(J^k B)$ and $\vartheta \in \Omega^q_H(J^k B)$ are both horizontal. Defining the *formal derivative* $d_\mu \equiv d/dx^\mu$ by means of

$$\frac{df}{dx^\mu} \equiv \frac{\partial f}{\partial x^\mu} + \frac{\partial f}{\partial y^i_\alpha} y^i_\mu + \ldots + \frac{\partial f}{\partial y^i_{\alpha_1 \ldots \alpha_k}} y^i_{\alpha_1 \ldots \alpha_k \mu} \in \mathcal{F}(J^{k+1} B)$$

where $f \in \mathcal{F}(J^k B)$, we have that

$$d_H f \equiv \mathrm{Hor}(df) = (d_\mu f) dx^\mu\,. \tag{1.9}$$

In general, if $\omega = \omega_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$ is a horizontal p-form, we have

$$d_H\omega = (d_\mu \omega_{\alpha_1 \ldots \alpha_p}) dx^\mu \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}\,. \tag{1.10}$$

From $d_\mu d_\nu f = d_\nu d_\mu f$ it follows easily $d_H d_H \omega = 0$ for every horizontal form ω.

The definition of d_H can now be extended to all p-forms by recalling that p-forms can be expressed as linear combinations of exterior products of horizontal forms and base contact forms. Setting in fact:

$$d_H \vartheta^i_{\mu\ldots} \equiv d\vartheta^i_{\mu\ldots} = -\vartheta^i_{\mu\ldots\alpha} \wedge dx^\alpha \tag{1.11}$$

for every base contact form $\vartheta^i_{\mu\ldots}$ and requiring d_H to be linear and to satisfy the property (1.8) for every pair of forms, an operator

$$d_H : \Omega^p(J^k B) \to \Omega^{p+1}(J^{k+1} B)$$

is uniquely defined and it satisfies the property

$$d_H d_H \varphi = 0 \tag{1.12}$$

for every p-form $\varphi \in \Omega^p(J^k B)$.

The horizontal differential and the formal derivative satisfy also the relations

$$\begin{aligned} (j^{k+1}\sigma)^* d_H \varphi = (j^k \sigma)^* d\varphi = d\big[(j^k\sigma)^*\varphi\big] \\ (d_\mu f) \circ j^{k+1}\sigma = \partial_\mu (f \circ j^k \sigma)\,, \end{aligned} \tag{1.13}$$

where $\varphi \in \Omega^p(J^k B)$, $f \in \mathcal{F}(J^k B)$ and σ is a section of the fibered manifold (B, M, π).

The vertical differential d_V of a p-form $\omega \in \Omega^p(J^k B)$ is defined as the difference between the standard differential and the horizontal differential; namely:

$$d_V \omega \equiv d\omega - d_H \omega \in \Omega^{p+1}(J^{k+1}B)\,. \tag{1.14}$$

Vertical differentials of the base space coordinates x^μ are always zero, while vertical differentials of the coordinates $y^i, y^i_\mu \ldots$ are the base contact forms $\vartheta^i, \vartheta^i_\mu \ldots$. The vertical differential of any p-form is a contact form; the horizontal part of a vertical differential is always zero. From the properties of the standard differential and of the horizontal differential it follows that the vertical differential is linear and satisfies

$$d_V(\varphi \wedge \vartheta) = d_V \varphi \wedge \vartheta + (-1)^p \varphi \wedge d_V \vartheta$$

where $\varphi \in \Omega^p(J^k B)$ and $\vartheta \in \Omega^q(J^k B)$. Furthermore the vertical differential of a vertical differential is always zero, i.e.:

$$d_V d_V \varphi = 0\,. \tag{1.15}$$

There is a link between the vertical differential and the Lie derivative. If (B, M, π) is a bundle of geometric objects with coordinates (x^μ, y^i), X is a vectorfield on M and X_B is its canonical lift to B, then the Lie derivative $\pounds_X \sigma$ along X of a section σ of the bundle (see [9], [10]), defined by

$$\pounds_X \sigma = T\sigma \circ X - X_B \circ \sigma\,,$$

satisfies

$$\pounds_X \sigma = -\big[(X_{J^1B} \rfloor d_V y^i) \circ j^1\sigma\big] \frac{\partial}{\partial y^i} \,. \tag{1.16}$$

For the notions of Lagrangian $\mathcal{L}$, of Poincaré-Cartan form $\Theta(\mathcal{L})$, of energy-density flow $E^\lambda(\mathcal{L}, w) ds_\lambda$ and of the corresponding superpotentials we refer the reader to the previous papers [3], [4], [11]. We just recall that the Poincaré-Cartan form associated with a Lagrangian $\mathcal{L}$ of order k is an m-form $\Theta(\mathcal{L}) \in \Omega^m(J^{2k-1}B)$ (where m is the dimension of the base space M of the bundle (B, M, π) of the theory) satisfying the following characteristic properties:

(a) the interior product of $\Theta(\mathcal{L})$ with a vectorfield vertical with respect to the projection π^{2k-1}_{k-1} of $J^{2k-1}B$ on $J^{k-1}B$ is always zero (i.e. $\Theta(\mathcal{L})$ does not contain $dy^i_{\mu_1 \ldots \mu_j}$ with $j > k-1$);

(b) the interior product of $\Theta(\mathcal{L})$ with two vertical vectorfields is always zero (i.e. the contact part of $\Theta(\mathcal{L})$ can be written as a linear combination of the exterior products of a contact 1-form and a horizontal form);

(c) the horizontal part of $\Theta(\mathcal{L})$ is the Lagrangian $\mathcal{L}$;

(d) the differential of $\Theta(\mathcal{L})$ generates the field equations; in fact the *critical sections* σ (solutions of the field equations) satisfy

$$(j^{2k-1}\sigma)^* \big[W \rfloor d\Theta(\mathcal{L})\big] = 0$$

where W is a vectorfield on $J^{2k-1}B$.

It is known (see e.g. [4], [5] and ref.s quoted therein) that Poincaré-Cartan forms always exist globally for any Lagrangian of any order. Uniqueness is lost for Lagrangians of order higher than one, for which an infinite family exists depending upon extra parameters. In any case, for second-order Lagrangians (which is the case of interest in this paper) there is a unique preferred form in the family, which shall be hereafter called the "canonical Poincaré-Cartan form"; it is defined as the only form of the family having symmetric coefficients (see later for an expression; (2.3)).

1.2 – Linear frames and their prolongations

Let us know take $B = L(M)$, the bundle of linear frames in M.

We assume that M can be given a Lorentzian metric $g_{\mu\nu}$, with signature $(-, +, \dots, +)$. In this case, Greek indices label coordinate bases $\partial/\partial x^\mu$ and dx^μ in M; while Latin lowercase indices $a, b, c = 0, \dots, m-1$ label generic (usually orthonormal) bases e_a and θ^a.

Denoting by η^{ab} the Minkowski metric, an orthonormal basis e_a with its dual θ^a are given by the orthonormality condition

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}\,,$$

which implies

$$g_{\mu\nu} = \eta_{ab} \theta^a_\mu \theta^b_\nu\,. \tag{1.17}$$

Indices with respect to an orthonormal basis will be raised and lowered using η_{ab} and η^{ab}.

Starting from an orthonormal basis we introduce the volume element

$$\xi \equiv \theta^0 \wedge \ldots \wedge \theta^{m-1} = \sqrt{g} ds \tag{1.18}$$

where $g \equiv \left|\det(g_{\mu\nu})\right|$ and $ds \equiv dx^0 \wedge \ldots \wedge dx^{m-1}$. We define also the forms

$$\begin{aligned} &\xi_a \equiv e_a \rfloor \xi = \sqrt{g} e^\mu_a ds_\mu \\ &\xi_{ab} \equiv e_b \rfloor \xi_a = \sqrt{g} e^\mu_a e^\nu_b ds_{\mu\nu} \\ &\xi_{abc} \equiv e_c \rfloor \xi_{ab} = \sqrt{g} e^\mu_a e^\nu_b e^\rho_c ds_{\mu\nu\rho} \end{aligned} \tag{1.19}$$

where

$$ds_\mu \equiv \frac{\partial}{\partial x^\mu} \rfloor ds\,, \quad ds_{\mu\nu} \equiv \frac{\partial}{\partial x^\nu} \rfloor ds_\mu\,, \quad ds_{\mu\nu\rho} \equiv \frac{\partial}{\partial x^\rho} \rfloor ds_{\mu\nu}\,.$$

Setting locally $e_a = e^\alpha_a \partial_\alpha$ and $\theta^a = \theta^a_\mu dx^\mu$, (x^α, e^α_a) are natural (fibered) coordinates for $L(M)$.

If $\Gamma^\alpha_{\beta\mu}$ is any symmetric connection then we define the connection forms $\hat{\omega}^c_a \in \Omega^1\big(L(M)\big)$ associated to $\Gamma^\alpha_{\beta\mu}$ by means of

$$\hat{\omega}^c_a \equiv \theta^c_\alpha (de^\alpha_a + \Gamma^\alpha_{\beta\mu} e^\beta_a dx^\mu) \in \Omega^1\big(L(M)\big)\,. \tag{1.20}$$

We denote by $\omega^c_a \equiv \omega^c_{a\mu} dx^\mu \in \Omega^1_H\big(J^1L(M)\big)$ the horizontal parts of the connection forms $\hat{\omega}^c_a$, namely

$$\omega^c_a \equiv \mathrm{Hor}(\hat{\omega}^c_a) = \theta^c_a(e^\alpha_{a,\mu} + \Gamma^\alpha_{\beta\mu} e^\beta_a) dx^\mu \tag{1.21}$$

where $(x^\alpha, e^\alpha_a, e^\alpha_{a,\mu})$ are natural coordinates in $J^1L(M)$. If $\sigma : M \to L(M)$ is any section, then

$$(j^1\sigma)^* \omega^c_a = \theta^c_\alpha(\nabla_\mu e^\alpha_a) dx^\mu\,, \tag{1.22}$$

where ∇_μ is the covariant derivative associated to the connection.

We remark that often in the literature the objects ω^c_a are called connection forms instead of the objects $\hat{\omega}^c_a$.

The horizontal parts of the connection forms can be expressed as follows:

$$\omega^c_a = \omega^c_{am}\theta^m\,,$$

where the coefficients ω^c_{am} are given by $\omega^c_{am} \equiv \omega^c_{a\mu} e^\mu_m$.

The 'contact part' of the connection forms is:

$$\hat{\omega}^c_a - \omega^c_a = \theta^c_a d_V e^\alpha_a = \theta^c_\alpha(de^\alpha_a - e^\alpha_{a,\mu} dx^\mu)\,. \tag{1.23}$$

Using the symmetry of the connection, which implies that $\Gamma^\alpha_{\beta\mu} dx^\beta \wedge dx^\mu = 0$, we get

$$d\theta^a = -\hat{\omega}^a_b \wedge \theta^b\,.$$

Taking the horizontal part of this relation we obtain

$$d_H\theta^a = -\omega^a_b \wedge \theta^b = \omega^a_{bm}\theta^b \wedge \theta^m\,.$$

If $w \in \chi(M)$ is a vectorfield and we set $\nabla_a w^c \equiv e^\mu_a \theta^c_\nu \nabla_\mu w^\nu$, we get

$$\nabla_a w^c = e_a(w^c) + \omega^c_{ma} w^m \tag{1.24}$$

where $e_a(w^c) \equiv e^\mu_a \partial_\mu(\theta^c_\nu w^\nu)$.

If the basis e_a is orthonormal for the metric $g_{\mu\nu}$, then the connection forms associated to the Levi-Civita connection of the metric $g_{\mu\nu}$ satisfy

$$\omega_{ba} = -\omega_{ab} \tag{1.25}$$

where $\omega_{ba} \equiv \eta_{bc}\omega^c_a$.

The following properties of the connection forms are valid only for Levi-Civita connections and orthonormal bases:

i) Being the horizontal parts of the connection forms skew-symmetric, the symmetric parts of the connection forms are contact forms, namely

$$\hat{\omega}_{ca} + \hat{\omega}_{ac} = \theta_{c\alpha} d_V e^\alpha_a + \theta_{a\alpha} d_V e^\alpha_c \tag{1.26}$$

where $\theta_{c\alpha} \equiv \eta_{cb}\theta^b_\alpha$.

ii) The horizontal differentials of the forms (1.19) are

$$\begin{aligned} d_H\xi_a &= \omega^b_a \wedge \xi_b \\ d_H\xi_{ab} &= \omega^c_a \wedge \xi_{cb} + \omega^c_b \wedge \xi_{ac} \\ d_H\xi_{abc} &= \omega^d_a \wedge \xi_{dbc} + \omega^d_b \wedge \xi_{adc} + \omega^d_c \wedge \xi_{abd}\,, \end{aligned} \tag{1.27}$$

and

$$d_H\xi^a_b = -\omega^a_c \wedge \xi^c_b + \omega^c_b \wedge \xi^a_c\,. \tag{1.28}$$

iii) Being $d\xi_{ab} = -\xi_{abm} \wedge \hat{\omega}^m_d \wedge \theta^d$ we obtain

$$d\xi_{ab} - \hat{\omega}^m_a \wedge \xi_{mb} - \hat{\omega}^m_b \wedge \xi_{am} = -\hat{\omega}^m_m \wedge \xi_{ab}\,. \tag{1.29}$$

For the differential of ξ^a_b we get thence

$$d\xi^a_b + \hat{\omega}^a_n \wedge \xi^n_b - \hat{\omega}^n_b \wedge \xi^a_n = \hat{\omega}^{(mn)} \wedge (-\eta_{mn}\xi^a_b + \delta^a_m \xi_{nb} + \delta^a_n \xi_{mb})\,, \tag{1.30}$$

where $(\dots)$ denotes symmetrization.

We define the curvature forms $\widehat{\Omega}^c_a \in \Omega^2(L(M))$ associated with any given connection by means of

$$\widehat{\Omega}^c_a = d\hat{\omega}^c_a + \hat{\omega}^c_b \wedge \hat{\omega}^b_a\,; \tag{1.31}$$

moreover we define the horizontal parts Ω^c_a of these forms by

$$\Omega^c_a \equiv \mathrm{Hor}(\widehat{\Omega}^c_a) \in \Omega^2(J^1L(M))\,.$$

Denoting the 1-forms $\Gamma^\alpha_{\beta\mu}dx^\mu$ by Γ^α_β, we have

$$\widehat{\Omega}^c_a = \theta^c_\alpha(d\Gamma^\alpha_\beta + \Gamma^\alpha_\mu \wedge \Gamma^\mu_\beta)e^\beta_a \tag{1.32}$$

so that

$$\Omega^c_a = \theta^c_\alpha(d_H\Gamma^\alpha_\beta + \Gamma^\alpha_\mu \wedge \Gamma^\mu_\beta)e^\beta_a = \frac{1}{2}\theta^c_\alpha e^\beta_a R^\alpha_{\beta\mu\nu}dx^\mu \wedge dx^\nu \tag{1.33}$$

where $R^\alpha_{\beta\mu\nu}$ is the Riemann tensor associated with the connection. The contact parts of curvature forms are

$$\widehat{\Omega}^c_a - \Omega^c_a = \theta^c_\alpha d_V \Gamma^\alpha_\beta e^\beta_a\,, \tag{1.34}$$

while the horizontal parts Ω^c_a of the curvature forms satisfy the property $\Omega_{ca} = -\Omega_{ac}$ where $\Omega_{ca} \equiv \eta_{cb}\Omega^b_a$.

2 – Change of field variables and Poincaré-Cartan form

In this section we prove a result about the behaviour of the Poincaré-Cartan form of a second-order field theory when the field variables are changed in an arbitrary way. The proof of this result was first given in the thesis [14]. This result will be useful in General Relativity in order to change variables from metrics to tetrads.

The result is the following: *Let (B', M, π') and (B, M, π) be two fibered manifolds with the same base space M. A second-order field theory with Lagrangian $\mathcal{L}$ is given on (B, M, π); the change of variables is specified by a fibered morphism*

$$F : B' \to B\,,$$

over a diffeomorphism f of M. On B we have the canonical Poincaré-Cartan form $\Theta(\mathcal{L})$; on B' we have the pull-back Lagrangian $F^*\mathcal{L}$, depending on the new field variables, and the canonical Poincaré-Cartan form $\Theta(F^*\mathcal{L})$. Then we have:

$$\Theta(F^*\mathcal{L}) = F^*\Theta(\mathcal{L})\,. \tag{2.1}$$

Proof. Let (x^μ, y^i) be coordinates of B and (t^α, z^A) be coordinates of B' (lowercase Latin indices label coordinates in B, uppercase Latin indices label coordinates in B' and Greek indices label coordinates in M); the morphism F is denoted by

$$\begin{cases} y^i = y^i(z^A, t^\alpha) \\ x^\mu = x^\mu(t^\alpha)\,. \end{cases}$$

Furthermore let $(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu})$ be coordinates of J^2B and $(t^\alpha, z^A, z^A_\alpha, z^A_{\alpha\beta})$ be coordinates of J^2B'.

We have:

$$y^i_\mu = \left(\frac{\partial y^i}{\partial z^A} z^A_\alpha + \frac{\partial y^i}{\partial t^\alpha}\right) X^\alpha_\mu$$

$$y^i_{\mu\nu} = \left(\frac{\partial y^i}{\partial z^A} z^A_\alpha + \frac{\partial y^i}{\partial t^\alpha}\right) X^\alpha_{\mu\nu} + \left(\frac{\partial^2 y^i}{\partial z^A \partial z^B} z^A_\alpha z^B_\beta + 2\frac{\partial^2 y^i}{\partial z^A \partial t^\beta} z^A_\alpha + \right.$$

$$\left. + \frac{\partial y^i}{\partial z^A} z^A_{\alpha\beta} + \frac{\partial^2 y^i}{\partial t^\alpha \partial t^\beta}\right) X^\alpha_\mu X^\beta_\nu\,,$$

where

$$X^\alpha_\mu = \frac{\partial t^\alpha}{\partial x_\mu}\,, \qquad X^\alpha_{\mu\nu} = \frac{\partial^2 t^\alpha}{\partial x^\nu \partial x^\mu}\,.$$

From these relations it follows that

$$\frac{\partial y^i_\mu}{\partial z^A_\alpha} = \frac{\partial y^i}{\partial z^A} X^\alpha_\mu\,, \qquad \frac{\partial y^i_\mu}{\partial z^A} = \left(\frac{\partial^2 y^i}{\partial z^A \partial z^B} z^B_\alpha + \frac{\partial^2 y^i}{\partial z^A \partial t^\alpha}\right) X^\alpha_\mu$$

$$\tag{2.2} \frac{\partial y^i_{\mu\nu}}{\partial z^A_{\alpha\beta}} = \frac{\partial y^i}{\partial z^A} X^\alpha_\mu X^\beta_\nu\,, \quad \frac{\partial y^i_{\mu\nu}}{\partial z^A_\beta} = \frac{\partial y^i}{\partial z^A} X^\beta_{\mu\nu} + $$

$$+ 2\left(\frac{\partial^2 y^i}{\partial z^A \partial z^B} z^B_\alpha + \frac{\partial^2 y^i}{\partial z^A \partial t^\alpha}\right) X^\beta_\mu X^\alpha_\nu\,.$$

The canonical Poincaré-Cartan form associated to $\mathcal{L} = L ds$ is

$$\tag{2.3} \Theta(\mathcal{L}) = L ds + (f^\nu_i d_V y^i + f^{\nu\mu}_i d_V y^i_\mu) \wedge ds_\nu$$

where

$$f^\nu_i \equiv \frac{\partial L}{\partial y^i_\nu} - \frac{d}{dx^\mu}\frac{\partial L}{\partial y^i_{\mu\nu}}\,, \quad f^{\nu\mu}_i \equiv \frac{\partial L}{\partial y^i_{\nu\mu}}\,.$$

Now we have

$$F^*(d_V y^i) = \frac{\partial y^i}{\partial z^A} d_V z^A \,, \qquad F^*(d_V y^i_\mu) = \frac{\partial y^i_\mu}{\partial z^A} d_V z^A + \frac{\partial y^i_\mu}{\partial z^A_\alpha} d_V z^A_\alpha$$

and

$$F^*(ds) = \frac{1}{X} dr \,, \qquad F^*(ds_\nu) = X^\alpha_\nu \frac{1}{X} dr_\alpha$$

where $X = \det(X^\alpha_\mu)$ and

$$dr = dt^0 \wedge \ldots \wedge dt^{m-1} \,, \qquad dr_\alpha = \frac{\partial}{\partial t^\alpha} \rfloor dr \,.$$

Using these relations we can compute the pull-back of the Poincaré-Cartan form, which is:

$$\begin{aligned} F^*\Theta(\mathcal{L}) = F^*\mathcal{L} + \Bigg[&\frac{1}{X} \left(f^\nu_i \frac{\partial y^i}{\partial z^A} + f^{\mu\nu}_i \frac{\partial y^i_\mu}{\partial z^A} \right) X^\beta_\nu d_V z^A + \\ &+ \frac{1}{X} f^{\mu\nu}_i \frac{\partial y^i_\mu}{\partial z^A_\alpha} X^\beta_\nu d_V z^A_\alpha \Bigg] \wedge dr_\beta \,. \end{aligned} \tag{2.4}$$

For the Lagrangian $F^*\mathcal{L} = L' dr$ we get

$$F^*\mathcal{L} = F^*(L ds) = \frac{L}{X} dr \,,$$

so that $L' = L/X$. The Poincaré-Cartan form associated to $F^*\mathcal{L}$ is

$$\Theta(F^*\mathcal{L}) = L' dr + (f'^{\beta}_A d_V z^A + f'^{\beta\alpha}_A d_V z^A_\alpha) \wedge dr_\beta$$

where

$$f'^{\beta}_A = \frac{\partial L'}{\partial z^A_\beta} - \frac{d}{dt^\alpha} \frac{\partial L'}{\partial z^A_{\alpha\beta}} \,, \quad f'^{\alpha\beta}_A = \frac{\partial L'}{\partial z^A_{\alpha\beta}} \,.$$

Using the formulae (2.2), we get finally

$$\begin{aligned} f'^{\beta}_A &= \frac{1}{X} \left[\frac{\partial L}{\partial y^i_{\mu\nu}} \frac{\partial y^i_{\mu\nu}}{\partial z^A_\beta} + \frac{\partial L}{\partial y^i_\mu} \frac{\partial y^i_\mu}{\partial z^A_\beta} - X^\alpha_\mu X^\beta_\nu \frac{d}{dt^\alpha} \left(f^{\mu\nu}_i \frac{\partial y^i}{\partial z^A} \right) \right] = \\ &= \frac{1}{X} \left(f^\nu_i \frac{\partial y^i}{\partial z^A} + f^{\mu\nu}_i \frac{\partial y^i_\mu}{\partial z^A} \right) X^\beta_\nu \\ f'^{\alpha\beta}_A &= \frac{1}{X} f^{\mu\nu}_i \frac{\partial y^i_{\mu\nu}}{\partial z^A_{\alpha\beta}} = \frac{1}{X} \frac{\partial y^i_\mu}{\partial z^A_\alpha} X^\beta_\nu \,. \end{aligned}$$

Thus (2.1) follows as claimed. □

3 – Energy-density and superpotentials from the Hilbert Lagrangian

In this section, starting from the tetrad version of the usual Hilbert Lagrangian for the gravitational field and applying the method based on the Poincaré-Cartan form as in [5] and [6], we deduce the formula for the superpotential associated to a generic vectorfield. Throughout this section and the following one M will be four dimensional.

The bundle of the fields is therefore $L(M)$, the bundle of linear frames, instead of $\mathrm{Lor}(M)$, the bundle of all the metrics of signature $(-,+,+,+)$. Between these bundles there is a morphism

$$F : L(M) \to \mathrm{Lor}(M) \tag{3.1}$$

given by

$$F(\theta^a) = \eta_{ab}\theta^a \otimes \theta^b \,.$$

This morphism, which is surjective but not injective, specifies the change of variables that we are performing.

The Hilbert Lagrangian expressed using the metric is

$$L_H ds = \frac{c^4}{16\pi G}\sqrt{g}R\, ds = \frac{c^4}{16\pi G}\sqrt{g}g^{\mu\nu}R_{\mu\nu}ds \,. \tag{3.2}$$

For the sake of simplicity in this section and in the next one we rescale the volume form ξ as $(c^4/16\pi G)\theta^0 \wedge \ldots \wedge \theta^3$, so that the factor $c^4/16\pi G$ that appears in (3.2) remains hidden in the forms ξ^a_c. The pull-back of $L_H ds$ under the morphism F is thence

$$\mathcal{L}_H = -\xi^a_c \wedge \Omega^c_{\ a} \,, \tag{3.3}$$

which is the standard Hilbert Lagrangian in tetrad form (cfr. [17]).

We compute the variation of $\mathcal{L}_H$ using (1.28), and we find:

$$\delta\mathcal{L}_H = -\delta\theta^b \wedge \xi^a_{cb} \wedge \Omega^c_{\ a} - d_H(\xi^a_c \wedge \delta\omega^c_{\ a}) \,. \tag{3.4}$$

Accordingly, the field equations for the tetrad are (cfr. [17])

$$\xi^a_{cb} \wedge \Omega^c_a = 0\,. \tag{3.5}$$

Since the tetrad Lagrangian (3.3) is the pull-back of the metric Lagrangian (3.2), we can apply the theorem of section 2 to conclude that the Poincaré-Cartan form dependent on the tetrad must be the pull-back of the Poincaré-Cartan form dependent on the metric, which (cfr. [5] and [6]) is known to be:

$$\Theta(L_H) = L_H ds + \pi^{\mu\nu} d_V u^\alpha_{\mu\nu} \wedge ds_\alpha \tag{3.6}$$

where $\pi^{\mu\nu} \equiv (c^4/16\pi G)\sqrt{g} g^{\mu\nu}$ and

$$u^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \frac{1}{2}(\delta^\alpha_\mu \Gamma^\sigma_{\sigma\nu} + \delta^\alpha_\nu \Gamma^\sigma_{\sigma\mu})\,,$$

$\Gamma^\alpha_{\mu\nu}$ being the Levi-Civita connection of the space-time metric $g_{\mu\nu}$.

The Poincaré-Cartan form dependent on the tetrad is thence

$$\Theta(\mathcal{L}_H) = -\xi^a_c \wedge \widehat{\Omega}^c_a\,. \tag{3.7}$$

In fact, rewriting this form in coordinates one finds:

$$\Theta(\mathcal{L}_H) = -\xi^a_c \wedge \Omega^c_a - \xi^a_c \wedge (\widehat{\Omega}^c_a - \Omega^c_a) = \mathcal{L}_H - \xi^a_c \wedge \theta^c_\alpha e^\beta_a d_V \Gamma^\alpha_{\beta\mu} \wedge dx^\mu \tag{3.8}$$

so that

$$\Theta(\mathcal{L}_H) = L_H ds + \pi^{\beta\rho} d_V u^\alpha_{\beta\rho} \wedge ds_\alpha\,;$$

$\Theta(\mathcal{L}_H)$ is thence the pull-back of $\Theta(L_H)$.

The expression given for $\Theta(\mathcal{L}_H)$ can be justified also verifying that it satisfies the characteristic properties of the Poincaré-Cartan form recalled in section 1. In fact, $\Theta(\mathcal{L})$ satisfies trivially the first three properties: ξ^a_c is a horizontal 2-form, $\widehat{\Omega}^c_a$ is a 2-form containing differentials of e^μ_a and $e^\mu_{a,\alpha}$, so that $\Theta(\mathcal{L}_H)$ is a 4-form satisfying the property (a); (b) holds true since the contact part

$$-\xi^a_c \wedge (\widehat{\Omega}^c_a - \Omega^c_a) = -\xi^a_c \wedge \theta^c_\alpha e^\beta_a (d_V \Gamma^\alpha_{\beta\mu}) \wedge dx^\mu$$

is the exterior product of a contact 1-form with a horizontal form; the horizontal part of $\Theta(\mathcal{L}_H)$ is $-\xi^a_c \wedge \Omega^c_a$, which implies (c). It is possible to show that also condition (d) is fulfilled. In fact, computing $d\Theta(\mathcal{L}_H)$ with the aid of the definition of Ω^c_a and the formula (1.30) we obtain:

$$d\Theta(\mathcal{L}_H) = -d\xi^a_c \wedge \widehat{\Omega}^c_a - \xi^a_c \wedge d\widehat{\Omega}^c_a = -\hat{\omega}^{(mn)} \wedge (-\eta_{mn}\xi^a_c + \delta^a_m \xi_{nc} + \delta^a_n \xi_{mc}) \wedge \widehat{\Omega}^c_a \,. \tag{3.9}$$

As explained in section 1, $\hat{\omega}^{(mn)}$ is a contact form; rewriting $\widehat{\Omega}^c_a$ as the sum of Ω^c_a and of a contact form, we can hence write

$$\begin{aligned} d\Theta(\mathcal{L}_H) = &-\hat{\omega}^{(mn)} \wedge (-\eta_{mn}\xi^a_c + \delta^a_m \xi_{nc} + \delta^a_n \xi_{mc}) \wedge \Omega^c_a + \\ &- \hat{\omega}^{(mn)} \wedge (-\eta_{mn}\xi^a_c + \delta^a_m \xi_{nc} + \delta^a_n \xi_{mc}) \wedge (\widehat{\Omega}^c_a - \Omega^c_a)\,, \end{aligned} \tag{3.10}$$

where the second addendum is a contact form of order two, i.e. the exterior product of two contact 1-forms with a horizontal form. Using now the skewsymmetry of Ω^{ca} we have

$$\begin{aligned} d\Theta(\mathcal{L}_H) = &-\eta_{ms}\hat{\omega}^{(mn)} \wedge \xi^a_{cn} \wedge \Omega^c_a \wedge \theta^s - \hat{\omega}^{(mn)} \wedge (-\eta_{mn}\xi^a_c + \\ &+ \delta^a_m \xi_{nc} + \delta^a_n \xi_{mc}) \wedge (\widehat{\Omega}^c_a - \Omega^c_a)\,. \end{aligned} \tag{3.11}$$

Starting from this last expression of $d\Theta(\mathcal{L}_H)$ and recalling that the pull-back of a contact form along a section is zero, we obtain

$$(j^3\sigma)^*[W \rfloor d\Theta(\mathcal{L}_H)] = -\eta_{ms}\lambda^{mn}\xi^a_{cn} \wedge \Omega^c_a \wedge \theta^s$$

where σ is a section of $L(M)$, W is a vectorfield on $J^1(L(M))$ and $\lambda^{mn} = (j^3\sigma)^*(W \rfloor \hat{\omega}^{(mn)})$. Then the differential of $\Theta(\mathcal{L}_H)$ gives the field equations, so that $\Theta(\mathcal{L}_H)$ satisfies the last characteristic property of a Poincaré-Cartan form.

From the expression of the Poincaré-Cartan form for the Hilbert Lagrangian, one immediately obtains the formula for the conserved quantity associated to a vectorfield in space-time. If w is a vectorfield and $\hat{w}$ is its canonical lift to the bundle of field variables, in this case $J^1L(M)$, then the associated conserved quantity is

$$E(\mathcal{L}_H, w) = -\operatorname{Hor}\,[\hat{w} \rfloor \Theta(\mathcal{L}_H)]\,,$$

namely

$$
\begin{aligned}
E(\mathcal{L}_H, w) &= -\operatorname{Hor}(\hat{w}\rfloor\mathcal{L}_H) + \operatorname{Hor}\left[\hat{w}\rfloor(\xi^a_c \wedge \theta^c_\alpha e^\beta_a d_V \Gamma^\alpha_{\beta\mu} \wedge dx^\mu)\right] = \\
&= -w\rfloor\mathcal{L}_H - \xi^a_c \wedge \theta^c_\alpha e^\beta_a \pounds_w \Gamma^\alpha_{\beta\mu} dx^\mu
\end{aligned} \tag{3.12}
$$

where we have used equations (1.16) and (3.8) and the relation $\operatorname{Hor}(\varphi \wedge d_V\vartheta) = 0$, which holds for every pair of forms φ and ϑ. Using (3.3) and the formula for the Lie derivative of a symmetric connection:

$$
\pounds_w \Gamma^\alpha_{\beta\mu} = \nabla_\mu \nabla_\beta w^\alpha - R^\alpha_{\beta\mu\sigma} w^\sigma
$$

we obtain

$$
\begin{aligned}
E(\mathcal{L}_H, w) &= w\rfloor(\xi^a_c \wedge \Omega^c_a(-\xi^a_c \wedge \theta^c_\alpha e^\beta_a(\nabla_\mu\nabla_\beta w^\alpha - R^\alpha_{\beta\mu\sigma} w^\sigma) \wedge dx^\mu = \\
&= w^b \xi^a_{cb} \wedge \Omega^c_a - d_H(\nabla_a w^c \xi^a_c)\,,
\end{aligned} \tag{3.13}
$$

where $w^b \equiv \theta^b_\nu w^\nu$ and $\nabla_a w^c \equiv e^\mu_a \theta^c_\nu \nabla_\mu w^\nu$.

Therefore, we see that the expression of $E(\mathcal{L}_H, w)$ is the sum of two terms. The first one contains field equations while the second is the horizontal differential of the 2-form

$$
U(\mathcal{L}_H, w) = -\nabla_a w^c \xi^a_c \tag{3.14}
$$

which is the superpotential for the tetrad version of Einstein's gravitational theory. Then, the horizontal differential of $E(\mathcal{L}_H, w)$ is zero when evaluated on the solutions of the field equations, being the horizontal differential of a horizontal differential. This property justifies the name "conserved quantity" for $E(\mathcal{L}_H, w)$.

The interpretation of $E(\mathcal{L}_H, w)$ as a physical quantity depends of course on the vectorfield w: if w is time-like then $E(\mathcal{L}_H, w)$ will be the energy-density; if w is space-like and his flow is a spatial translation, then $E(\mathcal{L}_H, w)$ is the momentum density in the direction of the translation, while if w is space-like and his flow is a rotation around an axis, then $E(\mathcal{L}_H, w)$ is the density of angular momentum with respect to the axis of rotation.

The coordinate expression of the superpotential $U(\mathcal{L}_H, w)$ is

$$
\begin{aligned}
U(\mathcal{L}_H, w) &= -\nabla_a w^c \xi^a_c = -\frac{c^4}{16\pi G} e^\mu_a \theta^c_\nu \nabla_\mu w^\nu \sqrt{g}\eta^{ab} e^\rho_b e^\sigma_c ds_{\rho\sigma} = \\
&= -\frac{c^4}{16\pi G}\sqrt{g}\nabla^\mu w^\nu ds_{\mu\nu}
\end{aligned}
$$

which is the superpotential one finds starting from the metric Lagrangian (3.2) of Hilbert (see, e.g. [5]); i.e., half of the well known KOMAR superpotential (see [13]):

$$U_K = -\frac{c^4}{8\pi G}\sqrt{g}\nabla^\mu w^\nu ds_{\mu\nu}\,.$$

4 – Energy-density and superpotentials from first-order Lagrangians

In this section we finally show that invariant first-order Lagrangians for general relativity can be written in tetrad form, in full analogy with the result of [5] and [6] for the metric case.

From these first-order Lagrangians we compute conserved quantities and the corresponding superpotentials.

Using the expression

$$\Omega^c_a = d_H\omega^c_a + \omega^c_d \wedge \omega^d_a \tag{4.1}$$

and (1.28), the Hilbert Lagrangian can be written as follows:

$$\mathcal{L}_H = -\xi^a_c \wedge \Omega^c_a = -d_H(\xi^a_c \wedge \omega^c_a) - \xi^d_c \wedge \omega^a_d \wedge \omega^c_a\,. \tag{4.2}$$

The second term depends only on first derivatives of the field variables and differs from the Hilbert Lagrangian for a horizontal differential, so that it might be considered as a first-order Lagrangian equivalent to Hilbert's one. However, this first-order Lagrangian is not invariant under a tetrad change, because ω^c_a is not invariant. In fact, if we perform the tetrad change

$$a_{a'} = A^a_{a'}e_a\,, \quad \theta^{a'} = B^{a'}_a\theta^a \quad (B^{a'}_c A^c_{b'} = \delta^{a'}_{b'})$$

where $p \mapsto A^a_{a'}(p)$ is a smooth family of Lorentz transformations, we have

$$\begin{aligned}\omega^{a'}_{b'} \equiv \theta^{a'}_\alpha(\partial_\beta e^\alpha_{b'}dx^\beta + \Gamma^\alpha_{\beta\sigma}e^\sigma_{b'}dx^\beta) = B^{a'}_a\theta^a_\alpha[\partial_\beta(A^b_{b'}e^\alpha_b)dx^\beta + \\ + \Gamma^\alpha_{\beta\sigma}A^b_{b'}e^\sigma_b dx^\beta] = B^{a'}_a A^b_{b'}\omega^a_b + B^{a'}_b dA^b_{b'}\end{aligned}$$

and the last term in general is not zero because $A^b_{b'}$ depends on position. (Note that, because $A^b_{b'}$ depends only on position, we have $\partial_\beta A^b_{b'} dx^\beta = d_H A^b_{b'} = dA^b_{b'}$).

Applying to this first-order Lagrangian the procedure of section 3 to obtain conserved quantities would give results dependent on the choice of a tetrad. This situation is not acceptable for quantities that must have a physical meaning; in fact, tetrads connected by a Lorentz transformation describe the same physical situation, so that all physical quantities must be equal whichever 'equivalent' tetrad one chooses.

We shall hence proceed in analogy with [6]. We introduce a fixed symmetric connection $\Gamma^\alpha_{\mu\nu}(x)$, the *background connection*, together with the corresponding horizontal parts of the connection forms

$$\bar{\omega}^a_b \equiv \theta^a_\alpha (e^\alpha_{b,\beta} + \Gamma^\alpha_{\beta\sigma}(x) e^\sigma_b) dx^\beta \,.$$

Hence we can write

$$\begin{aligned} \mathcal{L}_H &= -d_H(\xi^a_c \wedge \omega^c_a) - \xi^d_c \wedge \omega^a_d \wedge \omega^c_a = \\ &= -d_H\big[\xi^a_c \wedge (\omega^c_a - \bar{\omega}^c_a)\big] - d_H(\xi^a_c \wedge \bar{\omega}^c_a) - \xi^d_c \wedge \omega^a_d \wedge \omega^c_a = \\ &= -d_H\big[\xi^a_c \wedge (\omega^c_a - \bar{\omega}^c_a)\big] - \xi^a_c \wedge \big[\overline{\Omega}^c_a + (\omega^d_a - \bar{\omega}^d_a) \wedge (\omega^c_d - \bar{\omega}^c_d)\big] \end{aligned} \tag{4.3}$$

where $\overline{\Omega}^c_a$ are the horizontal parts of the curvature forms associated with the background connection. Defining

$$Q^c_a \equiv \omega^c_a - \bar{\omega}^c_a \tag{4.4}$$

we obtain for the Hilbert Lagrangian the expression

$$\mathcal{L}_H = -d_H(\xi^a_c \wedge Q^c_a) - \xi^a_c \wedge [\overline{\Omega}^c_a + Q^d_a \wedge Q^c_d] \,, \tag{4.5}$$

which gives a family of first-order equivalent Lagrangians:

$$\mathcal{L}_B = -\xi^a_c \wedge [\overline{\Omega}^c_a + Q^d_a \wedge Q^c_d] \,. \tag{4.6}$$

Each one of these first-order Lagrangians is invariant under arbitrary changes of tetrad. In fact, if we consider the same tetrad change as before we have:

$$\omega^{a'}_{b'} = B^{a'}_a A^b_{b'} \omega^a_b + B^{a'}_b dA^b_{b'} \,, \quad \bar{\omega}^{a'}_{b'} = B^{a'}_a A^b_{b'} \bar{\omega}^a_b + B^{a'}_b dA^b_{b'}$$

and

$$Q^{a'}_{b'} \equiv \omega^{a'}_{b'} - \bar{\omega}^{a'}_{b'} = B^{a'}_{a} A^{b}_{b'} Q^{a}_{b} \,.$$

For the curvature forms we have

$$\overline{\Omega}^{a'}_{b'} = d_H \bar{\omega}^{a'}_{b'} + \bar{\omega}^{a'}_{c'} \wedge \bar{\omega}^{c'}_{b'} = B^{a'}_{a} A^{b}_{b'} \overline{\Omega}^{a}_{b} \,,$$

where we used the relations $A^{a}_{c'} B^{c'}_{c} = \delta^{a}_{c}$ and $B^{a'}_{a} dA^{a}_{c'} = -A^{a}_{c'} dB^{a'}_{a}$. Finally, we have

$$\begin{aligned}
\xi' &= \theta^{0'} \wedge \theta^{1'} \wedge \theta^{2'} \wedge \theta^{3'} = \det(B)\xi = \xi \\
\xi^{a'}_{b'} &= \eta^{a'c'} e_{b'} \rfloor (e_{c'} \rfloor \xi') = B^{a'}_{a} A^{b}_{b'} \xi^{a}_{b} \,,
\end{aligned}$$

where we assumed $\det(B) = 1$, which holds for Lorentz transformations which do not modify the spatial orientation. Therefore, the Lagrangian $\mathcal{L}_B$ satisfies the following transformation rule

$$\begin{aligned}
\mathcal{L}'_B = -\xi^{a'}_{c'} \wedge [\overline{\Omega}^{c'}_{a'} + Q^{d'}_{a'} \wedge Q^{c'}_{d'}] = -B^{a'}_{a} A^{c}_{c'} \xi^{a}_{c} \wedge [B^{c'}_{d} A^{e}_{a'} \overline{\Omega}^{d}_{e} + \\
+ B^{d'}_{e} A^{f}_{a'} Q^{e}_{f} \wedge B^{c'}_{g} A^{h}_{d'} Q^{g}_{h}] = \mathcal{L}_B \,,
\end{aligned}$$

which shows that $\mathcal{L}_B$ is invariant.

The variation of $\mathcal{L}_B$ with respect to the tetrad field is:

$$\begin{aligned}
\delta\mathcal{L}_B &= \delta\mathcal{L}_H + d_H(\delta\xi^{a}_{c} \wedge Q^{c}_{a}) + d_H(\xi^{a}_{c} \wedge \delta Q^{c}_{a}) = \\
&= -\delta\theta^{b} \wedge \Omega^{c}_{a} + d_H(\delta\xi^{a}_{c} \wedge Q^{c}_{a} - \xi^{a}_{c} \wedge \delta\bar{\omega}^{c}_{a}) \,.
\end{aligned} \tag{4.7}$$

The variation of $\mathcal{L}_B$ with respect to the background (indicated here with δ_B) is instead:

$$\delta_B \mathcal{L}_B = -d_H(\xi^{a}_{c} \wedge \delta\bar{\omega}^{c}_{a}) \,,$$

where we used $\delta_B \xi^{a}_{c} = 0$ and $\delta_B \omega^{c}_{a} = 0$. Since the variation with respect to the background is a boundary term, there are no field equations for the background, which accordingly has no dynamics. This is in complete agreement with the analogous property of the family of first-order Lagrangians considered in [5] and [6].

The Poincaré-Cartan form associated to $\mathcal{L}_B$ is

$$\Theta(\mathcal{L}_B) = \mathcal{L}_B + [d_V \xi^{a}_{c} - \xi^{a}_{m} \wedge \breve{\omega}^{m}_{c} + \xi^{m}_{c} \wedge \breve{\omega}^{a}_{m}] \wedge Q^{c}_{a} \,, \tag{4.8}$$

where $\check{\omega}^a_c$ are the contact parts of the connection forms, namely

$$\check{\omega}^a_c \equiv \hat{\omega}^a_c - \omega^a_c = \theta^a_\alpha d_V e^\alpha_c \, .$$

The form $\Theta(\mathcal{L}_B)$ so defined satisfies the characteristic properties of a Poincaré-Cartan form. In fact:

$\Theta(\mathcal{L}_B)$ is obviously a 4-form on $J^1L(M)$. The second term contains only differentials of the θ^c_α (or of the e^β_a) along with horizontal parts, so that property (a) is fulfilled.

The horizontal part of $\Theta(\mathcal{L}_B)$ is the Lagrangian $\mathcal{L}_B$, since the second term is a contact form, being $\check{\omega}^a_c$ and $d_V\xi^a_c$ contact forms; the contact part is of order one. This proves (b) and (c).

To show that the differential of $\Theta(\mathcal{L}_B)$ generates the field equations, we show that $\Theta(\mathcal{L}_B)$ differs from $\Theta(\mathcal{L}_H)$ by an exact differential, so that $d\Theta(\mathcal{L}_B) = d\Theta(\mathcal{L}_H)$. We have in fact:

$$\begin{aligned}\Theta(\mathcal{L}_B) = \mathcal{L}_H + d(\xi^a_c \wedge Q^c_a) - \xi^a_c \wedge \theta^a_\alpha e^\beta_a d_V \Gamma^\alpha_{\beta\mu} \wedge dx^\mu + \\ - \xi^a_c \wedge d_V(\theta^c_\alpha e^\beta_a) q^\alpha_{\beta\mu} \wedge dx^\mu + [-\xi^a_m \wedge \check{\omega}^m_c + \xi^m_c \wedge \check{\omega}^a_m] \wedge Q^c_a\end{aligned} \tag{4.9}$$

where

$$q^\alpha_{\beta\mu} \equiv \Gamma^\alpha_{\beta\mu} - \Gamma^\alpha_{\beta\mu}(x) \, , \quad Q^c_a = \theta^c_\alpha e^\beta_a q^\alpha_{\beta\mu} dx^\mu$$

and we used the relation $d_V\Gamma^\alpha_{\beta\mu}(x) = 0$ (which holds because the background connection depends only on the position and not on fields). Using now the expression (3.8) of the Poincaré-Cartan form for the Hilbert Lagrangian and the formula

$$d_V(\theta^c_\alpha e^\beta_a) q^\alpha_{\beta\mu} \wedge dx^\mu = \check{\omega}^m_a \wedge Q^c_m - \check{\omega}^c_m \wedge Q^m_a$$

we get

$$\Theta(\mathcal{L}_B) = \Theta(\mathcal{L}_H) + d(\xi^a_c \wedge Q^c_a) \, . \tag{4.10}$$

We can now compute the formula for the conserved quantities associated to $\mathcal{L}_B$ starting from $\Theta(\mathcal{L}_B)$. Keeping into account the link between the vertical differential and the Lie derivative, expressed by

$$\mathrm{Hor}(\hat{w} \rfloor d_V e^\alpha_a) = -\pounds_w e^\alpha_a \, ,$$

we obtain

$$\mathrm{Hor}(\hat{w}\rfloor\breve{\omega}^c_a) = -\theta^c_\alpha \pounds_w e^\alpha_a \tag{4.11}$$

and

$$\mathrm{Hor}(\hat{w}\rfloor d_V \xi^a_c) = -\pounds_w \xi^a_c \,. \tag{4.12}$$

Using these relations, we get for the conserved quantities

$$\begin{aligned} E(\mathcal{L}_B, w) &\equiv -\,\mathrm{Hor}\,(\hat{w}\rfloor\Theta(\mathcal{L}_B)) = \\ &= -w\rfloor\mathcal{L}_H - w\rfloor d_H(\xi^a_c \wedge Q^c_a) - \xi^a_c \wedge \pounds_w(\theta^c_\alpha e^\beta_a) q^\alpha_{\beta\mu} dx^\mu + \\ &- \xi^a_c \wedge \theta^c_\alpha e^\beta_a \pounds_w \Gamma^\alpha_{\beta\mu} dx^\mu + \xi^a_c e^\beta_a \pounds_w \Gamma^\alpha_{\beta\mu}(x) dx^\mu + \pounds_w(\xi^a_c \wedge Q^c_a) + \\ &+ [-\xi^a_m \theta^m_\alpha \pounds_w e^\alpha_c + \xi^m_c \theta^a_\alpha \pounds_w e^\alpha_m] \wedge Q^c_a \,, \end{aligned} \tag{4.13}$$

where we used $\mathrm{Hor}(\breve{\omega}^c_a) = 0$. In the expression found for $E(\mathcal{L}_B, w)$ the sum of the first and the fourth terms gives $E(\mathcal{L}_H, w)$. Furthermore, we have

$$\xi^a_c \wedge \pounds_w(\theta^c_\alpha e^\beta_a) q^\alpha_{\beta\mu} dx^\mu = [\xi^m_c \theta^a_\alpha \pounds_w e^\alpha_m - \xi^a_m \theta^m_\alpha \pounds_w e^\alpha_c] \wedge Q^c_a \,. \tag{4.14}$$

Hence

$$\begin{aligned} E(\mathcal{L}_B, w) &= E(\mathcal{L}_H, w) - w\rfloor d_H(\xi^a_c \wedge Q^c_a) + \pounds_w(\xi^a_c \wedge Q^c_a) + \\ &\quad + \xi^a_c \wedge \theta^c_\alpha e^\beta_a \pounds_w \Gamma^\alpha_{\beta\mu}(x) dx^\mu = E(\mathcal{L}_H, w) + \\ &\quad + d_H\big[w\rfloor(\xi^a_c \wedge Q^c_a)\big] + \xi^a_c \wedge \theta^c_\alpha e^\beta_a \pounds_w \Gamma^\alpha_{\beta\mu}(x) dx^\mu \end{aligned} \tag{4.15}$$

where the Lie derivative is computed using the formula

$$\pounds_w(\xi^a_c \wedge Q^c_a) = w\rfloor d_H(\xi^a_c \wedge Q^c_a) + d_H\big[w\rfloor(\xi^a_c \wedge Q^c_a)\big] \,.$$

Substituting in the expression of $E(\mathcal{L}_B, w)$ the value of $E(\mathcal{L}_H, w)$ already calculated in section 3 we finally obtain

$$\begin{aligned} E(\mathcal{L}_B, w) &= w^b \xi^a_{cb} \wedge \Omega^c_a + d_H\big[-\xi^a_c \nabla_a w^c + w\rfloor(\xi^a_c \wedge Q^c_a)\big] + \\ &\quad + \xi^a_c \wedge \theta^c_\alpha e^\beta_a \pounds_w \Gamma^\alpha_{\beta\mu}(x) dx^\mu \,. \end{aligned} \tag{4.16}$$

The conserved quantities associated to $\mathcal{L}_B$ are composed of three terms: field equations, the differential of a 2-form (the superpotential, which depends now on the background) and the term containing the Lie derivative $\pounds_w \Gamma^\alpha_{\beta\mu}(x)$ of the background itself. The third term is zero if we choose the vectorfield w to be a symmetry of the background connection, i.e. a vectorfield which satisfies $\pounds_w \Gamma^\alpha_{\beta\mu}(x) = 0$. In this case $E(\mathcal{L}_B, w)$ assumes an expression similar to (3.13). The superpotential is hence given by:

$$U(\mathcal{L}_B, w) = -\xi^a_c \nabla_a w^c + w \rfloor (\xi^a_c \wedge Q^c_a) \tag{4.17}$$

and the horizontal differential of $E(\mathcal{L}_B, w)$, with w a 'Killing field' for the background, is zero on solutions of field equations. It is easy to show that the superpotential $U(\mathcal{L}_B, w)$ is the tetrad expression of the superpotential found in [5] and [6] for the metric version of the family (4.6) of first-order Lagrangians.

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