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Embedding dual nets in affine and projective spaces

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Riassunto: *Si determina la struttura delle reti finite o infinite le cui duali possono essere immerse in uno spazio proiettivo o in uno spazio affine. Le reti le cui duali possono essere immerse in uno spazio proiettivo sono reti di pseudo-regoli; le reti le cui duali possono essere immerse in uno spazio affine sono quelle estendibili a reti di pseudo-regoli.*

Abstract: *This article determines the structure of arbitrary nets (finite or infinite) whose duals may be embedded into affine or projective space. The main results are that nets whose duals may be embedded into projective space are pseudo-regulus nets and nets whose duals may be embedded into affine space are 1-parallel class retractions of pseudo-regulus nets.*

1 – Introduction

In the 60's, T.G. Ostrom conceived of and developed the theory of finite derivable affine planes. These are finite planes of order q^2 which contain a distinguished set D of parallel classes of cardinality $q+1$ which can be covered by a set of $q^2(q+1)$ affine subplanes of order q so that given any two distinct points which lie on a line of D then there is a unique

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subplane of order q with lines in D which contains these points. Such a set D together with the points of the affine plane is called a derivable net.

Given a derivable net D in a finite affine plane, then Ostrom proved that by replacing the lines of D by the subplanes of D then a new affine plane is constructed called the derived plane.

More generally, it is possible to consider derivable nets without the superstructure of an affine plane containing the net. Furthermore, it is also possible to consider infinite derivable affine planes and infinite derivable nets.

Using certain ideas of Cofman, one of the authors completely characterized arbitrary derivable nets (see JOHNSON [8], [9]). It turns out that there is always a corresponding projective space S of dimension 3 with a designated line N so that the lines skew to N, points of $S - N$, planes of S that intersect N is a point, and planes of S that contain N are the points, lines, Baer subplanes, and parallel classes of the derivable net. Using this structure, it is possible to show that a finite derivable net is always a regulus net; corresponds to a regulus in some corresponding finite 3-dimensional projective space.

More generally, considering arbitrary nets which are covered by subplanes which are not necessarily Baer, we may ask whether similar results are valid for such subplane covered nets.

A finite dual subplane covered net is a partial geometry which satisfies the axiom of Pasch. Using characterization results of THAS and DE CLERCK [16] on such partial geometries, DE CLERCK and JOHNSON [3] recently characterized subplane covered nets in the finite case.

THEOREM (DE CLERCK, JOHNSON [3]). *Every finite subplane covered net order q^n and degree $q+1$ is a $(n-1)$-regulus net.*

It is an open question whether there are similar characterizations for infinite subplane covered nets other than the derivable nets.

COFMAN [1] actually connects an arbitrary derivable net to a 3-dimensional affine space and used this structure to show that the Baer subplanes of the net are Desarguesian extending PROHASKA [14] who showed the same result under the assumption of finiteness. Actually, the affine 3-space corresponds to a derivable net minus a parallel class. Hence, in this way, there are nets whose duals can be embedded into affine space.

De Clerck and Thas [4] also characterize those partial geometries which can be embedded into projective space. There are also various results on the embedding of partial and semi-partial geometries into affine space but here the results are less complete. The reader may want to read the known results in this direction in Thas and Hirschfeld [17] (see chapter 26). In particular, we note here a slight improvement of a result of Thas who completely determined the affine embeddable partial geometries.

Note that in a finite derivable net, the points and subplanes determine a semi-partial geometry which satisfies the diagonal axiom (dual of Pasch). Debroey [2] has classified the semi-partial geometries which satisfy the diagonal axiom. Further, Wilbrink and Brouwer [18] classify certain semi-partial geometries which are not partial geometries by their parameters.

Consider $PG(n+1,q)=S$ with distinguished codimension two subspace $N=PG(n-1,q)$. Then using Thas and De Clerck [16] (and/or De Clerck and Johnson [3]), the points and lines of any finite subplane covered net as a dual partial geometry may be viewed as the sets of lines skew to N and points of $S-N$ respectively. And, the points and subplanes of the subplane covered net as a semi-partial geometry may be seen as the lines of S skew to N subplanes of S which intersect N in a point respectively.

Roughly speaking, once a semi-partial geometry is known to have parameters which one can determine from the above example of lines planes, it is shown by Wilbrink and Brouwer that the semi-partial geometry is exactly this example. This is proven by showing that the diagonal axiom becomes valid and then appealing to Debroey.

Since the arguments and methods of study of partial and semi-partial geometries are expressly finite, it is not clear whether similar results on infinite structures resembling semi-partial or partial geometries continue to be valid.

In particular, we mention the following problems:

Determined the structure of an arbitrary subplane covered net.

Determine the arbitrary nets whose duals can be embedded in a projective space.

Determine the arbitrary nets whose duals can be embedded into an affine space.

We show that any net whose dual may be embedded in an affine or projective space leads to the following structure: Let Σ be a projective space of dimension at least two and let N be a projective subspace of codimension two (a point if the dimension is two). Then the lines skew to N and points of $\Sigma - N$ form a subplane covered net (see section 3) which we call a "projective - codimension two net" (see (3.1)). If Σ is 3-dimensional the corresponding net is derivable. In this case, JOHNSON [9] determined the abstract structure of this net which in the finite case becomes a regulus net. In the more general setting and represented vectorially, we call such a net a pseudo-regulus (see definition (3.3)).

Our main results are as follows:

THEOREM I. *Let R be a projective - codimension two net. Then there exists a left vector space W over a skewfield K such that the points of the net may be identified with $W \oplus W$, and the lines of the net may be represented in the form $\{(0,y)\}$, $\{(x,y)|y = \delta x\} \equiv (y = \delta x)$ where x and y are vectors of W, for all $\delta \in K$ and where $\delta x = (\delta x_i)$ where x is represented as the tuple (x_i) for $i \in \lambda$; that is, a projective - codimension two net is a pseudo-regulus net.*

A projective - codimension two net is a regulus net if and only if the associated skewfield is commutative.

THEOREM II. *The dual of an arbitrary net can be embedded into a projective space if and only if it is a pseudo-regulus net.*

THEOREM III. *The dual of an arbitrary net can be embedded into an affine space if and only if there is a one parallel class extension of the net to a pseudo-regulus net.*

Before proceeding to the main results, we recall the definition of the various embeddings.

DEFINITION 1.1. *Let $R = (P, L, C, I)$ be a net where P denotes the set of points, L denotes the set of lines, C denotes the set of parallel classes, and I denotes the incidence relation. We shall say that the dual of R may be embedded into an affine or projective space Σ if and only if each element of L is a point of Σ, each element of P is a line of Σ and* *we assume that each point of a line of Σ corresponding to an element of P then corresponds to an element of L.*

We also recall the definition of the diagonal axiom (see also Thas and De Clerck [16] or [3] in the finite case).

Definition 1.2. *Let M may be any incidence structure finite of infinite of points and lines where each pair of distinct points are incidence with at most one line and each pair of distinct lines are incidence with at most one point. We shall say that M satisfies the diagonal axiom if and only if for distinct collinear points x and y and points z and w which are collinear to both x and y then it is necessarily the case that z and w are collinear.*

Proposition 1.3. (i) *A subplane covered net satisfies the diagonal axiom and conversely any net which satisfies the diagonal axiom is a subplane covered net.*

(ii) *Any net whose dual can be embedded into a projective space satisfies the diagonal axiom.*

Proof. (i) Let x and y be distinct collinear points of the net. Assume the net is subplane covered. Then there is a subplane $\pi_{P,Q}$ containing P and Q. Moreover, since the subplane has all of the parallel classes as infinite points we must have $P\alpha$ and $Q\beta$ as lines of the subplane where $P\alpha$ is the unique line thru P of the parallel class α. If α and β are distinct parallel classes then $P\alpha \cap Q\beta$ is a point of the subplane. Let R be any point of the subplane which is not on the line PQ. Then we may form PR and QR. If PR is a line of the parallel class γ and QR is a line of the parallel class ρ then $R = P\gamma \cap Q\rho$. Similarly, taking any such intersection point of the subplane to take the place of P in the argument then any point of the subplane may be formed by taking intersections as above. That is, there is a unique subplane containing P and Q. Hence, if Z and W are points that are collinear to both P and Q then Z and W are points of subplanes $\pi_{P,Q}$ and hence are collinear.

Now assume that we have a net which satisfies the diagonal axiom. Let X and Y be any two distinct collinear points. Assume that Z and W are points that are collinear to both X and Y so that Z and W are collinear. We assert that all such points belong to an affine subplane which has all of the parallel classes as its infinite points.

The lines of the structure are defined to be the lines of the net which are incident with at least two points of $\langle X, Y\rangle$. For example, given a point Z of $\langle X, Y\rangle$ which is not incident with the line XY then there is at most one line thru Z which is parallel to XY. Consider the line of the net $Z\alpha$ incident with Z (if XY is in α) and form the intersection with $X\beta$ where a and β are distinct parallel classes to form the point W. Then W and Y are mutually collinear to Z and X so that by the diagonal axiom W and Y are collinear. Hence, any line of the net which is incident with a point of $\langle X, Y\rangle$ is a line of the structure.

Note that the structures $\langle X, Y\rangle$ are the diagonal cliques in THAS and DE CLERCK [16] in the finite case.

To prove (ii), let X and Y be two concurrent lines in a projective space Σ and let Z and W be lines which are concurrent to both X and Y but not concurrent with $X \cap Y$.

Then clearly Z and W intersect as this forces the intersections into a projective subplane. The dual of this statement (called the axiom of Pasch) becomes the diagonal axiom and is hence satisfied in a net whose dual can be embedded into a projective space.

Initially, we shall be concerned with nets whose duals may be embedded into affine space. Clearly such nets are not subplane covered as they do not satisfy the diagonal axiom.

2 – Embedding dual nets in affine spaces

In the following let $R = (P, L, C, I)$ be an arbitrary net whose dual can be embedded into an affine space Σ.

LEMMA 2.1. *Let α and β be distinct parallel classes of the net and for P in α and Q in β considered as points in Σ, let PQ denote the line in Σ joining P and Q. Then the points of PQ intersect each set γ in C (as a set of points in Σ) in a unique point and $PQ \subset \bigcup_C \rho$.*

Proof. P and Q are lines of the net in distinct parallel classes so that PQ in the affine space is $P \cap Q$ in the net. Since each point is on exactly one line of each parallel class and by assumption the points on the line PQ in Σ are all lines of the net, (2.1) follows immediately.

Lemma 2.2. *Let α be any parallel class considered as a set of points in Σ. Given any pair of distinct $P_\alpha, Q_\alpha \in \alpha$ then all points on the line $P_\alpha Q_\alpha$ are in α.*

Proof. Let T be any point of $P_\alpha Q_\alpha$ and assume that $T \notin \alpha$. Let $P_\alpha Q_\alpha \dots \alpha$, $P_\delta \in \delta \neq \alpha$. Form the affine plane π_0 generated by the points $P_\delta, P_\alpha, Q_\alpha$ noting that these cannot all be collinear. Let $\bigcup_{\rho \in C} \rho = \Sigma^*$. If the line $P_\delta T \cap \Sigma^*$ contains a point R say in $\beta \neq \delta$ then this forces T to be in Σ^*.

If T is in $\gamma \neq \alpha$ then $P_\alpha\ T$ is incident with a unique point of α by (2.1) which is a contradiction since P_α and Q_α are incident with this line.

Thus, $P_\delta T \cap \Sigma^* \subset \delta$.

Let $P_\beta = P_\delta P_\alpha \cap \beta$ and $Q_\beta = P_\delta Q_\alpha \cap \beta$. Consider $P_\beta T$ and note that $P_\beta T$ cannot be parallel to both $P_\alpha Q_\beta$ and $Q_\beta Q_\alpha$. Since any intersection must in β by the above argument as it is in Σ^*, it follows that $P_\beta T$ intersects $P_\alpha Q_\beta$ or $Q_\alpha Q_\beta$ in a point of β and by uniqueness this intersection must be Q_β. Now let u be the line in π_0 which is parallel to $P_\beta T$ and is incident with P_α. Now since u is parallel to $P_\beta T$ then u must intersect $Q_\alpha Q_\beta$ as otherwise $Q_\alpha Q_\beta$ would be parallel to $P_\beta T$ which by above contains Q_β. Note that P_α is not incident with $Q_\alpha Q_\beta$ so if u intersects $Q_\alpha Q_\beta$ in Q_α then T is a point of u. Hence, u intersects $Q_\alpha Q_\beta$ in a point of Σ^* not in α. Since P_α is also on this line, it follows that u intersects β in a point R_β. Now since $\langle R_\beta, P_\alpha, Q_\alpha \rangle = \pi_0$, we may repeat the above argument with R_β in place of P_β to obtain R_β, T and Q_β collinear and hence that P_β, R_β, T and Q_β are collinear which is a contradiction as R_β is on a line parallel to and distinct from $P_\beta T$. This completes the proof of (2.2).

Proposition 2.3. (i) *For any parallel class $\alpha \in C$ of the net, α is an affine subspace of Σ.*

(ii) $\bigcup_{\rho \in C} \rho = \Sigma^*$ *is an affine subspace of Σ.*

Proof. To prove (i), we note that since given any two points P and Q of α, the line joining these points PQ contains only points of α by (2.2) so that α is closed under joining by points and hence α is an affine subspace of Σ.

Let P and R be points of Σ^*. If both are in some class β then (i) applies to show that the points on PR are in Σ^*. If both are not in the same class then since they are each in some class, it follows from (2.1) that the points on PR are in Σ^* (actually this is an assumption). Hence, Σ^* is an affine space.

For the formal definition of projective - codimension two net see (3.1).

Theorem 2.4. *Let R be an arbitrary net whose dual may be embedded in an affine space Σ. Then R may be extended to a projective - codimension two net R^+ by the addition of one parallel class and the dual of R^+ may be embedded into the projective extension of Σ.*

Proof. Let Π^* denote the projective extension of the affine space $\Sigma^* = \bigcup_{\rho \in C} \rho$ and let H denote the hyperplane at infinity extending Σ^*.

We note that in Σ^* the affine space α is parallel to the affine space β.

Pf: Let P_α, Q_α be distinct points on a line u in α. Let $P_\delta \in \delta$ where δ is distinct from α or β. Then on $P_\alpha P_\delta$ and $Q_\alpha P_\delta$ are distinct points P_β and Q_β of β respectively. Form the affine plane $\langle P_\delta, Q_\alpha, P_\alpha\rangle$. We know that $P_\alpha Q_\alpha$ and $P_\beta Q_\beta$ are disjoint since α and β are disjoint (see (2.1)). Hence, $P_\alpha Q_\alpha \| P_\beta Q_\beta$ so that α is parallel to β.

Extend α to a projective subspace α^+ in Π^* and let N_α denote the hyperplane at infinity of α^+ in H. Note that $N_\alpha = N_\beta = N$ for all $\beta \in C$ since α is parallel to β.

Now let $(\infty) = H$. Note that every line of Σ^* either is parallel to a line of α or intersects α. Hence, α^+ is a hyperplane of Π^* and N is a projective subspace of codimension two (a hyperplane of a hyperplane of Π^*). Thus, $\{\alpha^+, (\infty) | \alpha \in C\}$ is the set of hyperplanes of Π^* which contains the codimension two subspace N. We will note in section 3 that it is possible to construct a net with "points", "lines", "parallel classes" as the lines skew to N, points of $\Pi^* - N$, and hyperplanes containing N respectively. We also note that this is a subplane covered net and provided an algebraic characterization. Since, this constructed net is a one parallel class extension of the net in question and is what we call a projective - codimension two net, we have the proof to the the theorem.

The proof of this result has an application to a result of THAS [15] who classifies the partial geometries which can be embedded in a finite affine space. We shall consider only dimension $d \geq 4$.

First we recall the result of Thas:

THEOREM (THAS [15] p. 7). *Suppose that the partial geometry $S = (P, B, I)$ with parameters s, t, α $(\alpha > 1)$ is embedded in $AG(d, s+1)$, where $d \geq 4$ and that P is not contained in an $AG(d', s+1)$ with $d' < d$ then the following cases can occur.*

(a) *$s = 1$, $\alpha = 2$, $t \in \{d-1, d, \dots, 2^d\}$ and that S is a $2-(t+2, 2, 1)$ design (P is an arbitrary pointset of $AG(d, 2)$ which is not contained in an $AG(d', 2)$, $d' < d$).*

(b) *S is the design of points and lines of $AG(d, s+1)$.*

(c) *P is the pointset of $AG(d, s+1)$, and B is the set of all lines of $AG(d, s+1)$ whose points at infinity constitute the complement of a hyperplane $PG(d-2, s+1)$ of the space at infinity of $AG(d, s+1)$.*

Note that (2.4) applies to characterize case (c). We combine (2.4) with section 3 to obtain:

COROLLARY 2.5. *If a partial geometry with parameters s, t, α is embedded into a finite affine geometry of dimension 4 and $\alpha > 2$ then either the partial geometry is the design of points and lines of a finite affine space or is the dual of a net which has a one parallel class extension to a regulus net.*

PROOF. We prove in section 3 that a projective - codimension two net is a regulus net in the case where the associated skewfield is a field. Also, this corollary may be proved similarly by appealing to the work of DE CLERCK and JOHNSON [3] since we are now dealing with a finite case.

A similar argument to that of the above shows that any dual net which may be embedded in a projective space is also a projective - codimension two net.

THEOREM 2.6. *Let M be an arbitrary net whose dual can be embedded into a projective space. Then M is a projective - codimension two net.*

PROOF. Let $P_\alpha, Q_\alpha \in \alpha$ and let $P_\delta \in \delta \neq \alpha$ where α and δ are parallel class of the net. Let π_0 denote the projective plane generated by P_α, Q_α, and P_δ. Let T by any point of $P_\alpha, Q_\alpha - \alpha$. We may use the exact argument as given in the affine case to show that $P_\delta T \cap \Big(\bigcup_{\rho\in C} \rho = \Sigma^*\Big)$ is contained in δ where C denotes the set of parallel classes of the net. Let $P_\beta, Q_\beta \in \beta$ and incident with $P_\delta P_\alpha$ and $P_\delta Q_\alpha$ respectively (note the analogue of (2.1) for embeddings into projective spaces).

Similarly, $P_\beta T \cap \Sigma^* \subset \beta$. Hence, $P_\beta T \cap Q_\alpha Q_\beta$ must be Q_β so that P_β, T and Q_β are collinear. Suppose there is a point S of $P_\alpha Q_\alpha - \alpha \cup \{T\}$. Form $P_\delta S$ and argue as above to note that P_β, S and Q_β must be collinear so that $ST = P_\alpha Q_\alpha = P_\beta Q_\beta$ which is a contradiction since when there are points of differential parallel classes of the net on a line of the projective space there is a unique point from each parallel class on this line.

Thus, there exists at most one point of $P_\alpha Q_\alpha$ which is not in α. Note that $P_\beta Q_\beta \cap P_\alpha Q_\alpha$ is a point R which cannot be in β or α since if it is in α then RP_β contains a unique point on α. Hence, there exists exactly one point on $P_\alpha Q_\alpha$ which is not in α. And, it follows that this point is in the intersection of lines $P_\beta Q_\beta$ for all $\beta \in C$ for certain points $P_\beta Q_\beta$ of β.

Let $\langle\alpha\rangle$ denote the projective subspace generated by α. Note that we have proved that every line of $\langle\alpha\rangle$ contains exactly one point not in α and this point is in $\cap\langle\alpha\rangle = N$.

We assert that any plane $\pi_0 = \langle P_\delta, P_\alpha, Q_\alpha\rangle$ as above is "punctured" by N; contains exactly point of N. To see this, we note that $P_\beta Q_\beta$ is the only line of π_0 in $\langle\beta\rangle$. If there were two lines of π_0 in β then π_0 is a subspace of $\langle\beta\rangle$ and we have seen that lines of $\langle\beta\rangle$ contain at most one point which is not in β where as $P_\delta Q_\alpha$ contains points of each parallel class of the net. Two lines of π_0 intersect within Σ^* unless one is the unique such line in say $\langle\alpha\rangle$ and the other is is a different parallel class β space $\langle\beta\rangle$ and hence there is a unique point of intersection of all such lines. Hence, π_0 is punctured by N.

So, any line joining two points of α intersects N and any line joining two points of N lies in N so that N is a hyperplane of $\langle\alpha\rangle$.

Note that $\alpha \cap N = \emptyset$ since any point of N may be obtained as the intersection of lines say $P_\alpha Q_\alpha$ and $P_\beta Q_\beta$. Clearly, the intersection point cannot be in α.

Hence, $\langle\alpha\rangle = \alpha \cup N$.

Now form $\bigcup_{\alpha \in C} \langle \alpha \rangle = \bigcup_{\alpha \in C} \alpha \cup N = \Sigma^* \cup N$. We assert that this is a projective subspace. We note that given two points P and Q of this set if P and Q are both in Σ^* then the preceding shows that the points of PQ are in $\Sigma^* \cup N$. If P is in α and Q is N then PQ is a line of $\langle \alpha \rangle = \alpha \cup N$. Hence, $\Sigma^* \cup N$ is a projective space and note that $\langle \alpha \rangle$ is a hyperplane of the projective space since every lines intersects $\langle \alpha \rangle$ nontrivially. And, N is a codimension two subspace.

Hence, we have shown that the points of the net are the lines of a projective space which are skew to a codimension two subspace and the lines of the net are the points of the projective space which do not lie on the codimension two subspace. Hence, the net is a projective - codimension two net. This completes the proof of (2.6).

3 – Projective - codimension two nets are pseudo-regulus nets

In [9], Johnson determined the structure of a derivable net by embedding the dual net into a projective space as in section 2. Then it is possible to use the structure of the projective space and its collineation groups to show that the net is a translation net which has a particular representation. In the finite case, it may be seen that the net is a regulus net. More generally, if the projective space is Pappian then the derivable net is a regulus net.

In this section, we see that it is possible to modify the arguments of Johnson [9] to establish similar results for arbitrary pseudo regulus nets. Recall, it is noted in De Clerck and Johnson [3] that finite projective - codimension two nets (those obtained via a projective space and a codimension two subspace as above) are regulus nets by appealing to some work of Debroey [2] on semi-partial geometries. Here, we do not use finiteness so we cannot be certain that similar results are valid in the infinite or arbitrary cases.

In this section, we consider the so-called pseudo-regulus nets and establish justifications for the terminology. First we note that projective - codimension two nets are subplane covered nets.

Definition 3.1. *Let Σ be any projective space and let N be any fixed projective subspace of codimension two. Define a structure $R =$*

(P, L, B, C, I) of points P, lines L, subplanes B, parallel classes C, and incidence I as follows:

P is the set of lines of Σ skew to N,

L is the set of points of $\Sigma - N$,

B is the set of planes of Σ which intersects N is a point,

C is the set of hyperplanes of Σ which contains N, and

I is the incidence induced from Σ.

It is well-known in the finite case that (P, L, I) is a net but this may be established in the arbitrary order situation as below. Note that if the provided (P, L, I) is a net then by (1.3)*, it satisfies the diagonal axiom so that it is a subplane covered net. In fact, the subplanes in question correspond to the set B above. We shall call such a subplane covered net a projective - codimension two net.*

Note 3.2. *The structure constructed in* (3.1) *is a subplane covered net.*

Proof. Let P be a point of R which is then a line of Σ skew to N. Then there is a unique intersection with a given hyperplane H that contains N in a point of $\Sigma - N$. The lines of the net are the points of $\Sigma - N$ so this translates to the following statements:

(i) there is a partition of the lines of the net into disjoint equivalence classes (two lines are equivalent if and only if they are points of the same hyperplane in Σ in $\Sigma - N$),

(ii) each point of the net is indicent with a unique line of each equivalent class.

Note also that two points of the net are incident with at most one line at two lines skew to N intersect in at most one point of $\Sigma - N$.

It now follows that R is a net and hence a subplane covered net.

Definition 3.3. *Let W be a left vector space over a skewfield K. Let $V = W \oplus W$. Let $(x = 0) \equiv \{(0, y) | y \in W\}$ and $(y = \delta x) \equiv \{(x, \delta x) | y \in W$ and δ a fixed element of $K\}$. Note that $y = \delta x$ is a left $Z(K)$ (center of K)-subspace. Then V defines a translation subplane covered net with points the vectors of V and lines the translates of $x = 0$, $y = \delta x$ for all $\delta \in K$. The subplanes are the translates of the K-subspaces $\{(\alpha w, \beta w) \in K$ for w fixed in $W - \{0\}\}$.*

This net is called a pseudo-regulus net.

In order to obtain an algebraic representation for a projective - codimension two net, we consider the underlying left vector space V over a skewfield K such that the associated projective space Σ is the lattice of (left) vector subspaces. Since N is a projective subspace of codimension two, there is a basis W for N which extends to a basis $W \cup \{e_1, e_2\}$ for V so that representing V with respect to this basis, we have $V = M_0 \oplus N = \Big\{((x_1, x_2), (y_i))|x_i \in K$ for $i = 1, 2$, $N = \{(y_i)|y_i \in K$ where $i \in \lambda\}\Big\}$ where M_0 is a complement of N. Note that we are considering scalar multiplication on the left so for v a vector with representation $((x_1, x_2), (y_i))$ then $\alpha\, v$ for $\alpha \in K$ is simply $((\alpha x_1, \alpha x_2), (\alpha y_i))$.

Since we will be considering the lines skew to N as points of the projective - codimension two net, which will turn out to be a translation net, we want to determine a setting wherein these lines are vectors of an appropriate vector space. For this, we begin by a determination of a unique representation for a basis of a line of Σ skew to N.

Lemma 3.4. *Any line of Σ skew to N has a unique basis of the form $\{e_1 + v_1, e_2 + v_2\}$ where v_1 and v_2 are vectors of N.*

Proof. Suppose the line M as a 2-dimensional subspace has basis $\{ae_1 + be_2 + w_1, ce_1 + de_2 + w_2\}$ where a, b, c, d are in K and w_1 and w_2 are vectors in N. Note that none of the pairs (a, b), (a, c), (c, d) of (b, d) can be $(0, 0)$. For example, suppose $(a, b) = (0, 0)$ then w_1 must be 0 so that one of the basis vectors is zero. Similarly, (c, d) is not $(0, 0)$. If $(b, d) = (0, 0)$ then there exists a nonzero vector w_1 or w_2 in the subspace.

Now M must intersect the subspace $N \oplus \langle e_1\rangle$ in a nonzero vector say $\alpha(ae_1 + be_2 + w_1) - \beta(ce_1 + de_2 + w_2) = (\alpha a - \beta c)e_1 + (\alpha w_1 - \beta w_2)$ for some nonzero elements α and β of K so that $\alpha b = \beta d$. Now $\alpha a - \beta c \neq 0$ for otherwise, $\alpha w_1 - \beta w_2 = 0$ since M does not nontrivially intersect N which contradicts the fact that the vector is nonzero.

Hence, there is a point of the form $e_2 + \bar{w}_2$ in M where $\bar{w}_2 \in N$ and similarly, there is a point of the form $e_1 + \bar{w}_1$ in M which are clearly linearly independent.

To show uniqueness, let $\langle e_1 + w_1, e_2 + w_2\rangle = \langle e_1 + v_1, e_2 + v_2\rangle$ where w_i, $v_i \in N$ for $i = 1, 2$. Let $e_1 + w_1 = \rho(e_1 + v_1) + \gamma(e_2 + v_2)$ where

$\rho, \gamma \in K$. Then $(\rho - 1)e_1 + \gamma e_2 \in N$ so that it must be that $\rho = 1$ and $\gamma = 0$ so that $w_1 = v_1$. Similarly, $v_2 = w_2$.

NOTATION 3.5. *If M is a line of Σ skew to N with unique basis of the form $\{e_1 + v_1, e_2 + v_2\}$ we denote M by the tuple (v_1, v_2) where $v_1, v_2 \in N$. We now find a suitable vector space associated with the projective space containing such tuples.*

We first note the following proposition:

PROPOSITION 3.6. *Any projective - codimension two net of Σ associated to a vector space over a skewfield K with fixed codimension two subspace N admits $P\Gamma L(V, K)_N$ as a collineation group.*

PROOF. Simply note that the stabilizer of N leaves invariant the set of lines skew to N, the set of parallel classes with contain N, the set of points of $\Sigma - N$, and the set of planes which intersect N in a point.

At this point, it might be noted that HIRAMINE and JOHNSON [6] have shown that any finite net of order q^n and degree $q + 1$ is a $(n - 1)$-regulus net if and only if the net admits $PSL(n + 1, q)_N$ as a collineation group where N is a $(n - 1)$-dimensional projective subspace.

The proof heavily relies on finite group theory and in particular the minimal degrees of certain finite simple groups. However, it is still possible to at least consider the following question: If an arbitrary net admits a collineation group isomorphic to $PSL(V, K)_N$ where V is a vector space over a skewfield K and N is a codimension two projective subspace of the corresponding lattice of (left) K-subspaces, is the net a pseudo-regulus net?

DEFINITION 3.7. *An element τ of $\Gamma L(V, K)$ is a transvection if and only if there is a hyperplane H such that $\tau | H = 1_H$ and $\tau(v) - v\epsilon\langle d\rangle$ for some fixed vector $d \in H$. We call the hyperplane H, the axis of τ and the vector d the direction of τ.*

PROPOSITION 3.7. *The subgroup T of $GL(V, K)_N$ which is generated by the transvections with axis a hyperplane containing N and in the directions d in N may be represented as follows: $T = \langle \tau | \tau(x_1, x_2, (y_i)) = (x_1, x_2, (y_i) + x_1(d_i) + x_2(g_i))\rangle$ for all (d_i), $(g_i) \in N$. Note that not every* *element in T is a transvection; in order that τ be a transvection, it must be that (d_i) and (g_i) generate the same 1-dimensional K-space. Note that T induces an isomorphic subgroup in $PGL(V,K)_N$ as there are no nontrivial elements in T which fix all 1-dimensional K-subspaces.*

Proof. First consider the transvection τ with axis $N \oplus \langle e_1 \rangle$ with direction (d_i) on N. If τ maps e_2 onto $e_2 + a(d_i)$ for $a \in K$ then $\tau(x_1, x_2, (y_i)) = (x_1, x_2, (y_i) + x_2 a(d_i))$.

Similarly, the transvections τ with axis $N \oplus \langle e_2 \rangle$ and direction (g_i) have the form $\tau(x_1, x_2, (y_i)) = (x_1, x_2, (y_i) + x_1 \beta(g_i))$. Let T_1, T_2 denote the two groups of transvections listed above with axes $N \oplus \langle e_1 \rangle$ and $N \oplus \langle e_2 \rangle$ respectively. We shall show that the full transvection group is $T_1 T_2$.

Let ρ be any transvection with the axis a hyperplane $H = N \oplus \langle a e_1 + b e_2 \rangle$ containing N and direction $(d_i) \in N$. Then if ρ maps e_1 onto $e_1 + (a d_i)$ and maps e_2 onto $e_2 + (\beta d_i)$ then $\rho(x_1, x_2, (y_i)) = (x_1, x_2, (y_i) + x_1(a d_i) + x_2(\beta d_i))$ and is in $T_1 T_2 = T$. Hence, we also have that an arbitrary element τ of T is a transvection if and only if (d_i) and (g_i) generate the same 1-dimensional K-space.

As a direct application, we obtain:

Corollary 3.9. *If M is a line of Σ skew to N with basis $\{e_1 + w_1, e_2 + w_2\}$ considered as a vector subspace where w_1, w_2 are vectors in N considered as a vector subspace then the group T of* (3.8) *maps $\{e_1 + w_1, e_2 + w_2\}$ onto the set of all lines with bases of the form $\{e_1 + w_1 + v_1, e_2 + w_2 + v_2\}$ for all $v_1, v_2 \in N$.*

With the notation adopted above where the lines of the projective space skew to N with basis represented by $\{e_1 + w_1, e_2 + w_2\}$, we obtain the following:

We consider the points of the net represented as ordered pairs of elements of N. That is, the points set of the net is $N \oplus N$. This set admits a "translation group" T which is the group generated by the transvection with axes the hyperplanes of Σ containing N. T acts on $N \oplus N$ in the standard manner, $(x, y) \to (x + a, y + b)$ where all entries are in N.

It is easy to check out that the scalar mappings $z \to kz$ of $\Gamma L(V, K)$ are semi-linear mappings and are exactly the semi-linear mappings that fix each 1-dimensional K subspace.

Hence, any semi-linear mapping τ which fixes $M_0 = \langle e_1, e_2 \rangle$ pointwise (as a projective space) must have the form $z \to kz$ for some $k \in K$ acting on M_0. Similarly, if τ fixes N pointwise as a projective space then $\tau|N$ as a vector space must have the form $z \to s\,v$ for some $s \in K$. Considering the associated vector space as $M_0 \oplus N$ then $\tau(x, y) = (ky, sy)$. However, in order that τ be semi-linear, it must be that $k^{-1}s$ is in the center $Z(K)$ of K (see the proof of (2.16) [9]).

Hence we obtain:

Proposition 3.10 (compare with (2.16) and (3.7) of [9]). *The subgroup of $P\Gamma L(V, K)_N$ which fixes the line M_0 and the codimension two subspace N pointwise is isomorphic to the subgroup of $\Gamma L(V, K)_N$ represented as follows: $\langle \tau_\delta | \tau_\delta(x, y) = (\delta x, \delta y)$ where $V = M_0 \oplus N$, $x \in M_0$, $y \in N$ and $\delta \in Z(K) - \{0\} \rangle$. The representation $\langle \tau_\delta^* \rangle$ of this group on the points of the net is as follows: $\langle \tau_\delta^*(w_1, w_2) = (\delta^{-1}w_1, \delta^{-1}w_2) \rangle$.*

Proof. Note that a line of Σ skew to $\mathcal{N}$ with vector basis $\{e_1 + w_1, e_2 + w_2\}$ is mapped to $\{\delta e_1 + w_1, \delta e_2 + w_2\}$ under τ_δ so that the point of the net (w_1, w_2) is mapped to $(\delta^{-1}w_1, \delta^{-1}w_2)$ under τ_δ^*.

We now define a vector space over $Z(K)$ where the points are $N \oplus N$, where vector addition is defined via the group of translations or rather the group generated by the transvections in the direction of N and scalar multiplication is defined via the subgroup of $P\Gamma L(V, K)_N$, which fixes a given line M_0 and the subspace N pointwise and is then defined by $\delta(w_1, w_2) = \tau^*_{\delta^{-1}}(w_1, w_2) = (\delta w_1, \delta w_2)$.

The following result follows immediately:

Theorem 3.11. *Let Σ be a projective space and N a projective subspace of codimension two. Let V denote the corresponding vector space over a skewfield K such that Σ is the lattice of (left) vector K-subspaces. Let M_0 be a fixed line of Σ which is skew to N.*

Then the set of lines skew to N can be made into a vector space over the center of K, $Z(K)$. Furthermore, if Q is any fixed point of M_0 then the set of lines of Σ skew to N which contain Q is a $Z(K)$-subspace.

Proof. It only remains to prove that the indicated set forms a $Z(K)$-subspace.

Let $Q = \langle re_1 + se_2 \rangle$ for some fixed $r, s \in K$. A line with basis $\{e_1 + w_1, e_2 + w_2\}$ that contains Q must satisfy $rw_1 + sw_2 = 0$ since $re_1 + se_2 = \alpha(e_1 + w_1) + \beta(e_2 + w_2)$ forces $r = \alpha, \beta = s$, and $\alpha w_1 + \beta w_2 = 0$. But, $rw_1 + sw_2 = 0$ if and only if $\delta(rw_1 + sw_2) = r\delta w_1 + s\delta w_2 = 0$ where $\delta \in Z(K)$ so that we have the proof of the theorem.

We now may state our main characterization result:

Theorem 3.12. *Let $R = (P, L, B, C, I)$ be any projective - codimension two net with distinguished codimension two subspace N.*

Then R is a translation subplane covered net.

(i) *The points P of R may be identified with the vectors of a vector space $V = N \oplus N$ over a skew-field K. The lines of L which are incident with the zero vector are $Z(K)$-subspaces. The subplanes of B which are incident with the zero vector are K-subspaces.*

(ii) *There is a representation for the projective - codimension two net so that the lines incident with the zero vector may be represented in the form $\{(0, y)\} \equiv (x = 0)$, $\{(x, y) | y = \delta x\} \equiv (y = \delta x)$ for all $x, y \in N$ and $\delta \in K$. The subplanes incident with the zero vector may be represented in the form $\{(\alpha w_1, \beta w_1) | \alpha, \beta \in K\}$ for fixed $w_1 \in N - \{0\}$; a projective - codimension two net is a pseudo-regulus net.*

Proof. We let the line M_0 in (3.10) be the zero vector in question identified with a particular point of the net. In the proof of (3.10) where $Q = \langle re_1 + se_2 \rangle$, let $(r, s) = (1, 0)$ then the lines with bases $\{e_1 + w_1, e_2 + w_2\}$ containing Q satisfy $w_1 = 0$. If $r = \delta$ and $s = -1$ then the lines with bases $\{e_1 + w_1, e_2 + w_2\}$ containing Q satisfy $w_2 = \delta w_1$. Using the notation established above, we have the representation for the lines of the net which are incident with the zero vector.

The subplanes of the net in question are the planes of the projective space which intersect N in a point. Any such plane $M_0 \oplus \langle w_1 \rangle$ which contains M_0 (the zero vector as a point of the net) must have a basis of the form $\{e_1, e_2, w_1\}$ for some $w_1 \in N$. The intersections of this plane with the hyperplanes $N \oplus \langle e_1 \rangle$ and $N \oplus \langle e_2 \rangle$ are the subspaces $\langle e_1, w_1 \rangle$ and $\langle e_2, w_1 \rangle$ respectively. The lines joining a point from each of the two subspaces have bases of the form $\langle \alpha e_1 + \beta w_1, \delta e_2 + \gamma w_1 \rangle$ for all $\alpha, \beta, \delta, \gamma$

in K. In order that such a line be skew to N, we require $\alpha\delta \neq 0$. Hence, the set of such lines skew to N have bases $\{e_1 + \alpha w_1, e_2 + \beta w_1\}$ for all $\alpha, \beta \in K$. This translates to the set of points of the net having the form $\{(\alpha w_1, \beta w_1) | \alpha, \beta \in K\}$ for fixed $w_1 \in N$. Note that clearly this set is a left K-subspace. This completes the proof of (3.12).

Definition 3.13. *Let Π be a projective space with underlying vector space V over a skew field L such that there is a L subspace W of V such that V is isomorphic to $W \oplus W$. A regulus R of Π is a set of mutually disjoint subspaces L-isomorphic to W which pairwise span Π such that if there is a line u of Π which intersects three mutually distinct subspaces of R then the line intersects all of the elements of R and the points of u lie within these subspaces (see* Grundhöfer [5] *for a slight variation of the definition of regulus).*

Note 3.14 (also see Grundhöfer [5]). *A regulus exists in a projective space whose corresponding vector space $V = W \oplus W$ if and only if the corresponding skewfield K is a field. Any regulus defines a regulus net in V with lines translates of vector subspaces of the form $x = 0$, $y = \delta x$ for all $\delta \in K$ and subplanes translates of the subspaces of the form $\{(\alpha w_1, \beta w_1) | \alpha, \beta \in K\}$ for w_1 a fixed vector of $W - \{0\}$.*

Proof. The assumptions imply that a regulus R produces a partial spread in V. The lines of the projective space become two dimensional affine subplanes of the corresponding translation net obtained by taking lines as translates of the regulus spaces. Choose any two such subspaces and decompose the vector space with these subspaces and identify both with W. Then with $V = \{(x, y) | x, y \in W\}$ we obtain $x = 0$ and $y = 0$ as equations of the two given subspaces. We may choose a third subspace to have the form $y = x$ and the remaining to be of the form $y = \sigma(x)$ where σ is a nonsingular L-linear mapping. It remains to show that $\sigma(x) = \delta x$ for some $\delta \in L$. Note that if this is possible and since σ is linear then $Z(L) = L$ so that L is a field.

Consider the 2-dimensional L-subspaces $\{(\alpha w_1, \beta w_1) | \alpha, \beta \in L\}$ for fixed but various $w_1 \in W$. Clearly, each such subspace corresponds to a line of the projective space and since the subspace intersects $x = 0$, $y = 0$ and $y = x$ nontrivially, it follows that the subspaces intersect each subspace of the regulus nontrivially. Thus, if $y = \sigma(x)$ is a subspace of the regulus then the 2-dimension L subspace above nontrivially intersects $y = \sigma(x)$ if and only if $\sigma(w_1) = \delta w_1$ for some $\delta \in L$. And, given an element $\rho \in L$, there is a regulus subspace $y = \tau(x)$ such that $\tau(w_1) = \rho w_1$ since the subspace (line) is covered. If $\delta \neq \rho$ then $\tau \neq \sigma$ for otherwise, the 2-dimensional subspace would be contained in the regulus subspace. Hence, there is a $1-1$ correspondence between the regulus subspaces different from $x = 0$ and the elements of L. Moreover, it follows that if $y = \sigma(x)$ and $y = \tau(x)$ are regulus subspaces so that, $\sigma, \tau \in GL(V, L)$ then $\sigma - \rho \in GL(V, L)$ (see for example, the argument of Lüneburg [11] ((2.2), (2.3)) which also is valid for partial spreads).

Then, it follows that if $\sigma(w_1) = \delta w_1$ for some nonzero w_1 then $\sigma(x) = \delta x$ for all $x \in W$. This completes the proof of (3.14).

Corollary 3.15 (see De Clerck and Johnson [3] for the finite case). *A projective - codimension two or pseudo regulus net is a regulus net if and only if the associated skew field corresponding to the vector space underlying the projective space containing the net is a field.*

Proof. Both structures can be brought into the same canonical form provided the associated skewfield is a field.

Hence, combining the previous results, we also have the proofs to theorem I, II and III stated in the introduction.

It might be noted that when the dimension of the ambient space of an embedding is two then a codimension two subspace is a point. Then a net whose dual may be embedded projectively is simply a Desarguesian affine plane. In this case, there is exactly one subplane of the net incident with any affine point namely the plane itself. When the dual of a net is embedded in $AG(3, K)$ for some skewfield then the net is a one parallel class retraction of a Desarguesian affine plane.

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