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An existence result for a non convex problem without upper growth conditions

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RIASSUNTO: *Proviamo un risultato di esistenza di soluzioni di problemi di minimo per funzionali integrali con integrando non convesso e non coercivo. Gli argomenti principali sono la regolarità e le proprietà geometriche delle soluzioni di opportuni problemi approssimanti.*

ABSTRACT: *We deal with existence of solutions of minimum problems for integral functionals with non convex, non coercive integrands. The result is obtained by using regularity and geometrical properties of the solutions of suitable approximating problems.*

– Introduction

In this paper we consider a minimum problem for a non convex functional of the Calculus of Variations of the type

$$F(u) = \int_0^1 f(x, u(x), u'(x))dx \quad u \in W^{1,1}(0,1)$$

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without any growth condition from above for f. It is well known that if $f(x,s,\xi)$ does not satisfy the growth condition

$$\Phi(|\xi|) \leq f(x,s,\xi) \quad \forall (x,s,\xi) \in (0,1) \times R \times R$$

where Φ is a superlinear function at infinity,then it is not possible to state the compactness of the minimizing sequences for $F(u)$, since $W^{1,1}$ is not a reflexive Sobolev space. Sometime this leads to investigate minimum problem for $F(u)$ in BV spaces (see [11]). Here we prove an existence result for the constrained problem

$$\text{(Pb.1)} \ \min\Big\{F(u) : u \in W^{1,1}(0,1), u(0) \geq 0, u(1) \leq \lambda, u' > 0 \ a.e., I(u) < +\infty\Big\}$$

essentially under linear or sublinear growth condition from below for $f(x,s,\xi)$ (see (A.1)), by assuming that the greatest convex function f^{**} less then or equal to f has derivatives satisfying some sign condition (see (A.4)). The difficulty of finding compact minimizing sequences is overcome by applying direct methods of the Calculus of Variations to approximating convex problems whose solutions u_ν verify Euler's equation in a weak form. The arguments to show that solutions u_ν verify Euler's equation are very much alike to the ones used in [1] and they also give $u_\nu \in W^{1,\infty}_{\text{loc}}(0,1)$. This enable us to prove that $u_\nu \in W^{2,2}_{\text{loc}}(0,1)$ (see lemma in sec. 2). Then Euler's equation holds almost everywhere in a strong form and, by using the sign condition (A.4), we obtain some convexity or concavity properties for u_ν. This is enough to get a uniform estimate for u_ν in $W^{1,\infty}_{\text{loc}}(0,1)$; by passing to the limit, we get a solution u of the convexified problem of (Pb.1). The argument to deduce compactness from geometrical properties has been introduced by P.Marcellini in [9] and has been used in [4] and [6]. Our main existence result is achieved since (A.4) leads to the equality $f(x,u(x),u'(x)) = f^{**}(x,u(x),u'(x))$ for almost every $x \in (0,1)$. This paper has been motivated in the framework of non linear elasticity problems. Indeed the well known functional

$$\int_\Omega |Du|^2 + h(\det Du)dx$$

where Ω is the n-dimensional unit ball centered at the origin,in the class of the radial deformations $u(x) = v(r)\dfrac{x}{r}, r = |x|$, becomes

$$\text{(0.1)} \qquad \omega_n \int_0^1 r^{n-1}\big[v'^2 + (n-1)\frac{v^2}{r^2} + h(v'(\frac{v}{r})^{n-1})\big]dr,$$

here ω_n is the $(n-1)$-measure of the surface of Ω. A minimum problem for (0.1), when h is a convex and superlinear function such that $\lim_{t\to 0^+} h(t) = \lim_{t\to+\infty} \dfrac{h(t)}{t} = +\infty$ has been considered in [1]. Our main result (theorem 1) may be applied to the convex,linear or sublinear case (see example 1). Motivated by the so called Blatz-Ko materials (see [8]), non convex integrands with a linear asymptotic behaviour have been treated in [9], where, under some upper growth conditions, an existence theorem is given for a problem of the type :

$$\text{(Pb.2)} \ \min\Big\{I(v) : v \in W^{1,1}(0,1), v(0) \geq 0, v(1) \leq \lambda, v' > 0 \ \ a.e., I(v) < +\infty\Big\}$$

with

$$I(v) = F(v) + \tilde{h}\Big[\int_0^{v(0)} a(s)ds + \int_{v(1)}^{\lambda} a(s)ds\Big] .$$

In that paper it is assumed, on one hand,that the function φ in (A.4) is continuous,on the other hand,that $\lim_{\xi\to 0^+} f(x,s,\xi) \in R$. We point out that this condition on φ is satisfied under particular assumptions,for example when $f(x,s,\xi) = a(x,s)h(\xi), a, h \in C^1$, but in general φ is not a continuous function (see remark 1). Our corollary in section 1 gives an existence result for (Pb.2) without upper growth conditions with regularity assumptions on φ less restrictive than continuity, by assuming $\lim_{\xi\to 0^+} f(x,s,\xi) = +\infty$ which is more natural in the framework of the non linear elasticity theory.

Let us observe that, if f is a convex function with respect to ξ and satisfies some upper control (see [8]), then $I(v)$ is the restriction to $W^{1,1}(0,1)$ of the lower semicontinuous extension to $\Big\{v \in W^{1,1}_{\text{loc}}(0,1), v(0) \geq 0, v(1) \leq \lambda, v' > 0 a.e.\Big\}$ of $F(v)$ in the class $\Big\{v \in W^{1,1}(0,1), v(0) = 0, v(1) = \lambda, v' > 0 a.e.\Big\}$.

1 – Assumptions and main results

Let us consider a Carathedory function $f : (x, s, \xi) \in (0, 1) \times R_+ \times R_+ \to R$, such that:

$$f(x, s, \xi) \geq a(x, s)h(\xi) - K \tag{A.1}$$

where $K \geq 0$, $h : \xi \in R_+ \to h(\xi) \in R_+$ is a convex function such that $\lim_{\xi \to +\infty} \frac{h(\xi)}{\xi} = \tilde{h} \in [0, +\infty)$ and $\lim_{\xi \to 0^+} h(\xi) = +\infty$, $a(x, s)$ is locally bounded from below by a positive constant.

Denoted by $f^{**}(x, s, \cdot)$ the greatest convex function with respect to ξ less than or equal to f, f^{**} satisfies

(A.2) $f^{**}(x, \cdot, \xi)$ is locally lipschitz uniformly with respect to (x, ξ) varying in a compact set of $(0, 1) \times R_+$.

$f^{**}_\xi(x, s, \xi)$ is differentiable and locally lipschitz on $(0, 1) \times R_+ \times R_+$

(A.3) there exists $\delta_0 > 0$ such that

$$|f^{**}_s(x, \sigma s, \xi)| \leq M(x_0, r)|f^{**}(x, s, \xi)| \quad \forall \xi \in R_+$$

for $x \in [x_0, 1), x_0 > 0, s \in [r, \lambda], 0 < r < \lambda, |\sigma - 1| < \delta_0$, where M is a constant only dependent on x_0 and r.

The function $\varphi(x, s, \xi) = f^{**}_s - f^{**}_{\xi x} - \xi f^{**}_{\xi s}$ defined almost everywhere in $(0, 1) \times R_+ \times R_+$ satisfies

$$\varphi(x, s, \xi) \geq 0 \qquad (\text{resp.} \varphi(x, s, \xi) \leq 0)\,. \tag{A.4}$$

REMARK 1. Let us observe that if f is not convex, for x, s fixed, there exists an interval $J = [\xi_0(x, s), \xi_1(x, s)]$ such that $f(x, s, \xi_i(x, s)) = f^{**}(x, s, \xi_i(x, s))$,for $i = 0, 1$ and $f^{**}(x, s, \xi)$ is linear for $\xi \in J$. Therefore

$$f^{**}(x, s, \xi) = f(x, s, \xi_0(x, s)) + f_\xi(x, s, \xi_0(x, s))(\xi - \xi_0(x, s))$$

for $\xi \in J$ and, even if much regularity is assumed on f^{**}, an easy computation gives

$$\begin{aligned}\varphi(x, s, \xi) = \varphi(x, s, \xi_0(x, s)) - f_{\xi\xi}(x, s, \xi_0(x, s))[(\xi_0)_x(x, s)+ \\ + (\xi_0)_s(x, s)\xi_0(x, s)]\end{aligned}$$

for $\xi \in I$. Then,in general, φ is not continuous.

We consider the functional

$$F(v) = \int_0^1 f(x, v(x), v'(x))dx \tag{1.1}$$

and the constrained minimum problem

$$\text{(Pb.1)}\quad \min\Big\{F(v) : v \in W^{1,1}(0,1), v(0) \geq 0, v(1) \leq \lambda, v' > 0 \ \ a.e., F(v) < +\infty\Big\}$$

where λ is a fixed positive number.

Our aim is to prove the following results.

THEOREM 1. *Under assumptions* (A.1),...,(A.4), *there exists a solution $u \in W^{1,\infty}_{\text{loc}}(0,1)$ of* (Pb.1) *and it satisfies the estimate*

$$|u'(x)| \leq \frac{2\lambda}{\delta} \qquad \forall x \in (\delta, 1-\delta) \qquad \forall \delta \in (0, \frac{1}{2})\,. \tag{1.2}$$

THEOREM 2. *Assume* (A.1) *with $\tilde{h} = +\infty$,* (A.2), (A.3) *and, if $f(x, s, \cdot)$ is not convex, $\varphi \neq 0$ instead of* (A.4), *the conclusion of theorem 1 holds true.*

COROLLARY. *Assume* (A.1), ... ,(A.4) *with the function $a(x, s)$ in* (A.1) *independent of x, bounded from above on each interval* $(0, \lambda), \lambda \in R_+$ *and locally lipschitz. Then the problem*

$$\text{(Pb.2)}\quad \min\Big\{I(v) : v \in W^{1,1}(0,1), v(0) \geq 0, v(1) \leq \lambda, v' > 0 \ \ a.e., \ I(v) < +\infty\Big\}$$

with

$$I(v) = F(v) + \tilde{h}\Big[\int_0^{v(0)} a(s)ds + \int_{v(1)}^{\lambda} a(s)ds\Big]$$

admits a solution $u \in W^{1,\infty}_{\text{loc}}(0,1)$ which satisfies (1.2).

REMARK 2. If $\tilde{h} = +\infty$ the existence theorem for a convex integrand is given in [1].

2 – Preliminar lemma

The following lemma will be used to prove our theorem

LEMMA. *Let* $u \in W^{1,\infty}_{\rm loc}(0,1)$. *Assume that* $\forall \epsilon$ *there exists* $m(\epsilon) > 0$ *such that:*

$$[m(\epsilon)]^{-1} \le u'(x) \le m(\epsilon) \qquad \forall x \in (\epsilon, 1) \tag{2.1}$$

and let u *be a solution of the equation*

$$A(x,u,u') = \int_1^x B(t,u(t),u'(t))dt \ + const \qquad \textit{for a.e. } x \in (0,1) \tag{2.2}$$

where A *is a differentiable and locally lipschitz function on* $(0,1) \times R_+ \times R_+$; B *is bounded on the compact subsets of* $(0,1) \times R_+ \times R_+$; $A_\xi \ge \nu > 0$. *Then* $u \in W^{2,2}_{\rm loc}(0,1)$.

PROOF. We consider a cut-off function $\eta \in C_0^\infty(0,1), 0 \le \eta \le 1, \eta = 1$ in$[a,b] \subset (0,1)$ and for $h > 0$ we set:

$$\tau_h u(x) = \frac{u(x+h) - u(x)}{h}$$

and

$$\phi(x) = \tau_{-h}(\eta^2 \tau_h u)\,.$$

The function $\phi(x)$ has a support in some interval $[\epsilon, r] \subset (0,1)$. Moreover

$$\phi'(x) = \tau_{-h}(2\eta\eta'\tau_h u + \eta^2 \tau_h u')\,.$$

Now we multiply for $\phi'(x)$ in (2.2) and, integrating by parts, we get:

$$\int_0^1 A(x,u(x),u'(x))\phi'(x)\ dx = -\int_{supp\phi} B(x,u(x),u'(x))\phi(x)\ dx \tag{2.3}$$

which is possible because of the assumption on A and B taking into account (2.1).

Let us consider the first member in (2.3)

$$\int_0^1 A(x,u(x),u'(x))\phi'(x)\,dx = \tag{2.4}$$
$$\int_0^1 \tau_h A(x,u,u')[2\eta\eta'\tau_h u + \eta^2\tau_h u']\,dx\,.$$

On the other hand we get

$$\tau_h A(x,u,u') = \frac{1}{h}\int_0^1 [\frac{d}{dt}A(x+th,u(x)+th\tau_h u(x),u'(x)+th\tau_h u'(x))]\,dt \tag{2.5}$$
$$= \int_0^1 (A_x + A_s\tau_h u + A_\xi \tau_h u')\,dt$$

and the functions A_x, A_s, A_ξ are calculated in $(x+th, u(x)+th\tau_h u(x), u'(x)+th\tau_h u'(x))$.

Let us observe that for $x \in \operatorname{supp}\phi = [\epsilon, r] \subset (0,1)$, for $t \in [0,1]$ and h small enough, $x+th \in [\epsilon, r_1] \subset (0,1)$.

Moreover, by the assumptions, we get

$$u(\epsilon) = (1-t)u(\epsilon) + tu(\epsilon) \le u(x) + th\tau_h u(x) =$$
$$= (1-t)u(x) + tu(x+h) \le u(1) \le \lambda$$

and

$$(1-t)[m(\epsilon)]^{-1} + t[m(\epsilon)]^{-1} \le u'(x) + th\tau_h u'(x) =$$
$$= (1-t)u'(x) + tu'(x+h) \le (1-t)m(\epsilon_1) + tm(\epsilon_1)\,.$$

We can conclude that the arguments, where the functions A_x, A_s, A_ξ are calculated, belong to a compact subset of $(0,1)\times R_+ \times R_+$ and (2.5) make sense because of the locally lipschitz assumption on A.

From (2.4) and (2.5) we get:

$$\int_0^1 A(x,u,u')\phi'(x)\,dx =$$
$$= \int_0^1 \left[\int_0^1 (A_x + A_s\tau_h u + A_\xi\tau_h u')\,dt\right](2\eta\eta'\tau_h u + \eta^2\tau_h u')\,dx\,.$$

We estimate the second member in the previous equality by proceedings as in [5] and we get:

$$\int_0^1 A(x,u,u')\phi'(x)\ dx \geq \nu \int_0^1 \eta^2 |\tau_h u'|^2\ dx - c(\eta, \max_{supp\ \eta} |u|)\,. \tag{2.6}$$

Let us estimate the second member in equality (2.3) recalling that B is bounded by our assumptions.

$$|\int_0^1 B(x,u,u')\phi(x)\ dx| \leq \int_{supp\phi} |B(x,u,u')|\ |\tau_{-h}(\eta\ \eta\tau_h u)|\ dx \tag{2.7}$$

$$\leq M \int_0^1 |\tau_{-h}(\eta\ \eta\tau_h u)|\ dx$$

where M is a constant depending on supp η.

Since $\tau_{-h}(fg) = (\tau_{-h}f)g + f(x+h)\tau_{-h}g$, by using Young inequality and proposition 3.3 in [5], for ϵ small enough we get:

$$\int_0^1 |\tau_{-h}(\eta \cdot \eta\tau_h u)|\ dx \leq \int_0^1 |\eta(x+h)|\ |\tau_{-h}\eta\tau_h u)|\ dx + \tag{2.8}$$

$$+ \int_0^1 |\eta(x)|\ |\tau_h u|\ |\tau_{-h}(\eta)|\ dx \leq \frac{\epsilon}{2} \int_0^1 |(\eta\tau_h u)'|^2\ dx\ +\ c(\eta, \max_{supp\eta} |u'|)\,.$$

From (2.7) and (2.8) we have

$$|\int_0^1 B(x,u,u')\phi(x)\ dx| \leq c(\eta, \max_{supp\eta} |u'|) + \frac{\epsilon}{2} \int_0^1 \eta^2 |\tau_h u)'|^2\ dx\,. \tag{2.9}$$

Finally (2.3), (2.6) and (2.9) give

$$\int_0^1 \eta^2 |\tau_h u'|^2\ dx \leq c(\eta, \max_{supp\eta} |u'|)$$

and recalling that $\eta = 1$ in $[a,b]$

$$\int_a^b |\tau_h u'|^2 \, dx \le c(\eta, \max_{supp\eta} |u'|)\,.$$

Since, by our assumptions, $u \in W^{1,\infty}_{\mathrm{loc}}(0,1)$, the previous inequality implies that $u \in W^{2,2}_{\mathrm{loc}}(0,1)$.

3 – Proofs

Proof of theorem 1.

We proceed in four steps:

Step 1. Assumption (A.1) implies that $\lim_{\xi \to 0+} f^{**}(x,s,\xi) = +\infty$ so we can extend the definition of f^{**} to $(0,1) \times R \times R$ by setting $f^{**}(x,s,\xi) = +\infty$ if $s \le 0$ and $\xi \le 0$. This extension will be convex in ξ and lower semicontinuous in $s \in R$.

Now we consider the functional

$$F_\nu(w) = \int_0^1 [f^{**}(x,w,w') + \nu(1 + |w'|^2)] \, dx \qquad \text{for } \nu > 0 \tag{3.1}$$

and minimize $F_\nu(w)$ on the set $W = \{w \in W^{1,1}(0,1) : w(1) \le \lambda, F_\nu(w) < +\infty\}$ that is equivalent to minimize $F_\nu(w)$ on the set

$$\{w \in W^{1,1}(0,1) : w(0) \ge 0, w(1) \le \lambda, w'(x) > 0 \text{ a.e.}, F_\nu(w) < +\infty\}\,.$$

Let $\{u^n_\nu\}_{n \in N}$ be a minimizing sequence, i.e.

$$\lim_n F_\nu(u^n_\nu) = \inf\{F_\nu(w) : w \in W\}\,.$$

By assumption (A.1) we get:

$$\nu \int_0^1 (1 + |(u^n_\nu)'|^2) \, dx \le F_\nu(u^n_\nu) + K \le \text{ const.}$$

Then there exists a subsequence of $\{u_\nu^n\}_{n\in N}$ weakly convergent to some function $u_\nu \in W^{1,1}(0,1)$. Moreover by th.1 in [7] and standard arguments in direct methods of the Calculus of Variations, we get existence of a minimizer $u_\nu \in W^{1,1}(0,1)$ for (3.1).

Such function u_ν satisfies the following properties:

$$u_\nu(0) \geq 0, u_\nu(1) \leq \lambda, u'_\nu(x) > 0 \qquad \text{for a.e. } x \in (0,1)\,. \tag{3.2}$$

STEP 2. For simplicity we set $u = u_\nu$. We prove, following [1], that u satisfies the Euler's equation of (3.1).

For $k = 2, 3, \ldots.$, set

$$S_k = \{x \in (\frac{1}{k}, 1) : \frac{1}{k} \leq u'(x) \leq k\}$$

and denote by χ_k the characteristic function of S_k.

For $v \in L^\infty(0,1)$ such that $\int_{S_k} v(t)\ dt = 0$ and ϵ small enough define

$$u_\epsilon(x) = u(x) + \epsilon \int_0^x \chi_k(t) v(t)\ dt\,.$$

It follows that $u'_\epsilon(x) = u'(x)$ for $x \leq \dfrac{1}{k}$ or $u'(x) \notin \left[\dfrac{1}{k}, k\right]$ and $u_\epsilon(0) = u(0)$.

Now we estimate the first variation of the functional:

$$\begin{aligned}
&|\frac{1}{\epsilon}[f^{**}(x, u_\epsilon(x), u'_\epsilon(x)) + \nu(1 + |u'_\epsilon|^2) + \\
&- f^{**}(x, u(x), u'(x)) - \nu(1 + |u'|^2)]| \leq \\
&\leq \frac{1}{\epsilon}|f^{**}(x, u_\epsilon(x), u'_\epsilon(x)) - f^{**}(x, u(x), u'_\epsilon(x))| + \\
&+ \frac{1}{\epsilon}|f^{**}(x, u(x), u'_\epsilon(x)) - f^{**}(x, u(x), u'(x))| + \\
&+ \frac{\nu}{\epsilon}|(u'_\epsilon)^2 - (u')^2|\,.
\end{aligned} \tag{3.3}$$

If $x \leq \dfrac{1}{k}$ since $\int_0^x \chi_k(t) v(t)\ dt = 0, u_\epsilon(x) = u(x)$ and $u'_\epsilon(x) = u'(x)$ then the first member in (3.3) is equal to zero.

If $x > \dfrac{1}{k}$ and $x \notin S_k$ we already know that $u'_\epsilon(x) = u'(x)$, then the right hand size in (3.3) reduces to:

$$\frac{1}{\epsilon}|f^{**}x, u_\epsilon(x), u'_\epsilon(x)) - f^{**}(x, u(x), u'_\epsilon(x))| =$$

$$= \frac{1}{\epsilon}|f^{**}(x, u_\epsilon(x), u'(x)) - f^{**}(x, u(x), u'(x))|.$$

Let us estimate the last term.

$$\frac{1}{\epsilon}|f^{**}(x, u_\epsilon(x), u'(x)) - f^{**}(x, u(x), u'(x))| \tag{3.4}$$

$$= \frac{1}{\epsilon}\Big|\int_0^1 \frac{d}{dt} f^{**}(x, tu_\epsilon(x) + (1-t)u(x), u'(x))\ dt\Big|$$

$$\leq \frac{1}{\epsilon}\int_0^1 |f_s^{**}(x, tu_\epsilon(x) + (1-t)u(x), u'(x))|\ |u_\epsilon(x) - u(x)|\ dt$$

$$\leq \Big|\int_0^x \chi_k(t)v(t)\ dt\Big| \int_0^1 |f_s^{**}(x, tu_\epsilon(x) + (1-t)u(x), u'(x))|\ dt\,.$$

We observe that $x > \dfrac{1}{k}$ gives $u(x) > u\Big(\dfrac{1}{k}\Big) = \delta > 0$.

Indeed $u\Big(\dfrac{1}{k}\Big)$ cannot be zero because of conditions $(3.2)_1$, $(3.2)_3$ on the minimizing function, then for $x \geq \dfrac{1}{k}$, if δ_0 is the constant in (A.3), for $\epsilon < \delta_0 u(1/k)/||v||_{L^\infty(0,1)}$, we get

$$\Big|\frac{tu_\epsilon + (1-t)u}{u(x)} - 1\Big| < \delta_0$$

and we can apply assumption (A.3) to obtain that $f_s^{**}(x, tu_\epsilon + (1 - t)u, u') \in L^1(0,1)$.

Therefore for every $x \notin S_k$, (3.4) and, consequently, the first member in (3.3) is controlled from above by an L^1-function independent on ϵ, for ϵ small enough.

Suppose now that $x \in S_k$ so that $\chi_k(x) = 1$. We get $u'_\epsilon(x) = u'(x) + \epsilon v(x)$ and the right hand side of (3.3) is bounded by the quantity

$$\left| \int_0^x \chi_k(t) v(t)\, dt \right| \left| \int_0^1 f_s^{**}(x, tu_\epsilon + (1-t)u, u'_\epsilon)\, dt \right| +$$
$$+ \left[2\nu \tilde{\sigma}_\epsilon(x) + \left| f_\xi^{**}(x, u(x), \overline{\sigma}_\epsilon(x)) \right| \right] \; |\chi_k(x) v(x)|$$

where for $\epsilon > 0, \overline{\sigma}_\epsilon(x)$ and $\tilde{\sigma}_\epsilon(x)$ belong to the interval with extrems $u'_\epsilon(x)$ and $u'(x)$ for a.e. $x \in (0,1)$.

Moreover for $\overline{\sigma}_\epsilon = tu'_\epsilon + (1-t)u'$ and $||v||_{L^\infty} \leq \dfrac{1}{k(k+1)}$ we get the estimate

$$\frac{1}{k+1} \leq \frac{1}{k} - t\epsilon ||v||_{L^\infty(0,1)} \leq \overline{\sigma}_\epsilon(x) = u'(x) + t\epsilon v(x) \leq k + \epsilon ||v||_{L^\infty(0,1)} \leq$$
$$\leq k + ||v||_{L^\infty(0,1)} \, .$$

It follows that, for $\dfrac{1}{k} \leq x \leq 1, \delta = u\left(\dfrac{1}{k}\right) \leq u(x) \leq \lambda$, $\dfrac{1}{k+1} \leq \overline{\sigma}_\epsilon(x) \leq k + ||v||_{L^\infty(0,1)}$ and, by assumption (A.2), we get

$$\left| f_\xi^{**}\left(x, u(x), \overline{\sigma}_\epsilon(x)\right) \right| < c(k) \, . \tag{3.5}$$

By the same arguments we get

$$|f_s^{**}\left(x, tu_\epsilon(x) + (1-t)u(x), u'_\epsilon(x)\right)| \leq c(k) \tag{3.6}$$

and

$$|\tilde{\sigma}_\epsilon(x)|^2 \leq c(k) \, . \tag{3.7}$$

From (3.3), ..., (3.7) we deduce that also if $x \in S_k$ the first member in (3.3) is not greater than some L^1 function. We are able to apply the Lebesgue's theorem of the dominated convergence to get

$$\frac{d}{d\epsilon} F_\nu(u_\epsilon) \bigg|_{\epsilon=0} = \int_{\frac{1}{k}}^1 \left[f_s^{**} \int_0^x \chi_k(\tau) v(\tau) d\tau + (f_\xi^{**} + 2\nu u') \chi_k(x) v(x) \right] dx = 0. \tag{3.8}$$

By using assumption (A.3), $f_s^{**}(x,u,u') \in L^1\Big(\dfrac{1}{k},1\Big)$; in fact if $x \in \Big(\dfrac{1}{k},1\Big)$

$$\int_{\frac{1}{k}}^{1} |f_s^{**}(x,u,u')| \; dx \leq M\left(\frac{1}{k}, u(\frac{1}{k})\right) \int_{\frac{1}{k}}^{1} |f^{**}(x,u,u')| \; dx$$

which is finite since u is a minimizing function.

By proceedings as in [1] we integrate by parts and we get

$$\int_{S_k} \left[f_\xi^{**} + 2\nu u' - \int_1^x f_s^{**} \right] v(x) \; dx = 0 \tag{3.9}$$

for all $v \in L^\infty(0,1)$ such that $||v||_{L^\infty} \leq \dfrac{1}{k(k+1)}$ and $\int_{S_k} v(t) \; dt = 0$. It is not difficult to see that (3.9) holds for all $v \in L^\infty(0,1)$ with $\int_{S_k} v(x) \; dx = 0$. Since, by (A.2), $f_\xi^{**} + 2\nu u'$ is bounded in S_k, we deduce

$$f_\xi^{**} + 2\nu u' + \int_1^x f_s^{**} = c_k \quad \text{for a.e. } x \in S_k.$$

Moreover we have

$$f_\xi^{**} + 2\nu u' - \int_1^x f_s^{**} = \text{const} \quad \text{for a.e. } x \in (0,1) \tag{3.10}$$

because meas $[(0,1)/\cup_k S_k] = 0$

From (3.10), if $x \in (\epsilon, 1]$, we get

$$|f_\xi^{**}(x,u,u') + 2\nu u'| \leq \text{ const } + \int_\epsilon^1 |f_s^{**}| \; dx$$

which means that the first member is in $L^\infty_{\text{loc}}(0,1)$, and, by assumptions (A.1) and (A.2), there exists $m(\epsilon) > 0$ such that

$$[m(\epsilon)]^{-1} \leq u'(x) \leq m(\epsilon) \qquad \forall x \in (\epsilon, 1]\,. \tag{3.11}$$

Step 3. The minimizing function $u = u_\nu$ satisfies equation (3.10) and condition (3.11) so that we can apply the lemma proved in Sec. 2 with

$A(x,u,u') = f^{**}_\xi(x,u,u') + 2\nu u'$

$B(x,u,u') = f^{**}_s(x,u,u')$

since its assumption are verified because of (A.2).

We obtain that $u = u_\nu \in W^{2,2}_{\mathrm{loc}}(0,1)$.

We show that $u = u_\nu$ is convex (resp. concave) for every ν. Indeed, since $u_\nu \in W^{2,2}_{\mathrm{loc}}(0,1)$, for almost every $x \in (0,1)$ we can derive Euler's equation (3.10) to get

$$\left(f^{**}_{\xi\xi} + 2\nu\right) u'' = f^{**}_s - f^{**}_{\xi x} - \xi f^{**}_{\xi s}\,.$$

The second member of the above equation is $\varphi \geq 0$ (resp. $\varphi \leq 0$) by assumption and the quantity $f^{**}_{\xi\xi} + 2\nu$ is never negative so we can conclude that, for almost every $x \in (0,1)$, $u'' \geq 0$ i.e. u is convex (resp. $u'' \leq 0$ i.e. u is concave).

Assume,for example,that u_ν is convex for every ν, we can deduce a uniform estimate for the $W^{1,\infty}_{\mathrm{loc}}$ norm. In fact, by proceeding as in [9],

$$\frac{u_\nu(0) - u_\nu(x)}{-x} \leq u'_\nu(x) \leq \frac{u_\nu(1) - u_\nu(x)}{1-x}$$

and for $x \in (\delta, 1-\delta)$, $0 < \delta < 1/2$ we get

$$|u'_\nu(x)| \leq \frac{2}{\delta}\lambda\,.$$

Then there exists a subsequence which converges to some $u_0 \in W^{1,\infty}_{\mathrm{loc}}(0,1)$ in the weak topology of $W^{1,\infty}_{\mathrm{loc}}(0,1)$. It is easily seen that

$$||u'_0(x)||_{L^\infty_{(\delta,1-\delta)}} \leq \underline{\lim}_{\nu} ||u'_\nu(x)||_{L^\infty_{(\delta,1-\delta)}} \leq \frac{2\lambda}{\delta} \qquad \forall \delta \in \left(0, \frac{1}{2}\right)\,. \tag{3.12}$$

We prove now that u_0 is a minimizing function for the functional

$$F_0(w) = \int_0^1 f^{**}(x,w,w')\,dx$$

in the class $\{w \in W^{1,1}(0,1) : w(0) \geq 0, w(1) \leq \lambda, w'(x) > 0$ a.e., $F_0(w) < +\infty\}$ of the admissible functions.

In fact, by the convexity of $f^{**}(x,s,\xi)$ with respect to ξ, recalling that u_ν is a minimizing function for $F_\nu(v)$, we get:

$$\int_0^1 f^{**}(x,u_0(x),u_0'(x))\,dx \leq \varliminf_\nu \int_0^1 \left[f^{**}(x,u_\nu(x),u_\nu'(x)) + \nu(1+|u_\nu'|^2)\right]\,dx \leq$$
$$\leq \lim_\nu \int_0^1 \left[f^{**}(x,w,w') + \nu(1+|w'|^2)\right]\,dx = \int_0^1 f^{**}(x,w,w')\,dx$$

for w any function in the class of the admissible ones.

STEP 4. First we assume that φ has a strict sign: $\varphi > 0$ (resp. $\varphi < 0$). Now we are able to prove existence for (Pb. 1)

To this aim we show that

$$f(x,u_0(x),u_0'(x)) = f^{**}(x,u_0(x),u_0'(x)) \quad \text{for a.e. } x \in (0,1]\,. \tag{3.13}$$

By the same arguments used in Step 2 we get that $u_0(x)$ satisfies the Euler's equation

$$f_\xi^{**}(x,u_0(x),u_0'(x)) = c + \int_1^x f_s^{**}(t,u_0(t)u_0'(t))\;dt. \tag{3.14}$$

Since $u_0'(x)$ is monotone, then it is differentiable almost everywhere in $(0,1)$. Let $x \in (0,1)$ be a point where $u_0'(x)$ is differentiable and $f(x,u_0(x),u_0'(x)) \neq f^{**}(x,u_0(x),u_0'(x))$; since $f^{**}(x,u_0(x),\xi)$ is linear where $f \neq f^{**}$, by a derivation with respect to x of (3.14) we get

$$f_{\xi x}^{**}(x,u_0(x),u_0'(x)) + f_{\xi s}^{**}(x,u_0(x),u_0'(x))u_0'(x) = f_s^{**}(x,u_0(x),u_0'(x))$$

which contradicts the condition $\varphi > 0$ (resp. $\varphi < 0$).

If φ has not a strict sign, following [9] we consider the function

$$f^\epsilon(x,s,\xi) = f(x,s,\xi) + \epsilon e^{\pm s}$$

with $\epsilon \in (0,1]$ and the sign $\underline{+}$ is chosen in dependence on the sign of φ (sign $+$ if $\varphi \geq 0$). Then

$$\varphi^\epsilon(x,s,\xi) = \varphi(x,s,\xi)\underline{+}\epsilon e^{\underline{+}s}$$

has a strict sign. By the previous part of the proof, for every ϵ, there exists u_ϵ minimizing the functional (1.1) with f replaced by f^ϵ; by similar arguments as in [9], we can conclude the proof of theorem 1.

Proof of theorem 2. In [1] existence of a solution u_0 of the convexified problem of (Pb 1) is proved without the condition $\varphi \neq 0$. It is also proved that u_0 satisfies Euler equation, then we can apply step 3 in the proof of theorem 1 to obtain $u_0 \in W^{2,2}_{\text{loc}}$.

Moreover by the same arguments as in step 4, under the assumption $\varphi \neq 0$, we get existence for the non convex problem.

Proof of the corollary. If $\tilde{h} = 0, I(v) = F(v)$ and existence has been proved in the theorem 1. If $\tilde{h} \in (0,+\infty)$ we set $k_0 = \inf[h(\xi) - \tilde{h}\xi]$, existing because of the convexity of $h(\xi)$ and the limit conditions in (A.1) and

$$\Phi(\xi) = \begin{cases} h(\xi) - \tilde{h}\xi & \text{if} \quad k_0 \geq 0 \\ h(\xi) - \tilde{h}\xi - k_0 & \text{if} \quad k_0 < 0\,. \end{cases}$$

Defining $g(x,s,\xi) = f(x,s,\xi) - a(s)\tilde{h}\xi$ we get

$$g(x,s,\xi) \geq a(s)\Phi(\xi) - \tilde{k}$$

where

$$\tilde{k} = \begin{cases} k & \text{if} \quad k_0 \geq 0 \\ k - k_0 sup\{a(s) : s \in (0,\lambda)\} & \text{if} \quad k_0 < 0\,. \end{cases}$$

Assumption (A.1)is verified by $g(x,s,\xi)$ since $\Phi(\xi)$ is a non negative convex function satisfying

$$\tilde{\Phi} = \lim_{\xi \to +\infty} \frac{\Phi(\xi)}{\xi} = \lim_{\xi \to +\infty} \frac{h(\xi)}{\xi} - \tilde{h} = 0\,.$$

Moreover $\lim\limits_{\xi \to +\infty} \Phi(\xi) = +\infty$ and $g(x,s,\xi)$ satisfies (A.2) because of the assumption on $a(s)$. Finally $g(x,s,\xi)$ does not satisfies exactly assumption (A.3) but some assumption $(A.3)_{bis}$ which is enough for our aims:

$(A.3)_{bis}$. There exists $\delta_0 > 0$ such that

$$|g_s^{**}(x, \sigma s, \xi)| \le N(x_0, r)(|g^{**}(x, s, \xi)| + \tilde{h}\xi) \quad \forall \xi \in R_+$$

for $x \in [x_0, 1), x_0 > 0, s \in [r, \lambda], 0 < r < \lambda, |\sigma - 1| < \delta_0$, where N is a constant only dependent on x_0 and r. Now

$$I(v) = \int_0^1 g(x, v(x), v'(x))dx + \tilde{h} \int_0^1 a(v)v'dx + \tilde{h}\Big[\int_0^{v(0)} a(s)ds + \int_{v(1)}^{\lambda} a(s)ds\Big].$$

By a change of variable the last three terms reduce to $\tilde{h}\int_0^{\lambda} a(s)ds$ which is independent of v. Therefore (Pb2) is equivalent to (Pb1) for the integrand g and we can apply the theorem.

4 – Some examples

Here we present some examples of integrand functions f to which our existence results apply.

Ex.1.

$$f(x, s, \xi) = (\frac{x}{s})^{p-\frac{p}{n}}\xi^p + (n-1)(\frac{s}{x})^{\frac{p}{n}} + g(\xi), \quad 1 < p < n \tag{4.1}$$

where g is a convex function such that $\lim_{\xi \to +\infty} \frac{g(\xi)}{\xi} = \tilde{g} \in [0, +\infty)$ and $\lim_{\xi \to 0^+} g(\xi) = +\infty$. Integrand (4.1) is obtained by the change of variables in (0.1) $r = x^{\frac{1}{n}}$ and $w(x) = v^n(x^{\frac{1}{n}})$. Here an easy computation gives $\varphi \ge 0$. The function (4.1) is considered in [1] under the assumption $\lim_{\xi \to +\infty} \frac{g(\xi)}{\xi} = +\infty$. He proves existence of a minimizing function for the functional $F(u)$. Theorem 1 gives existence for the same functional when

$$\lim_{\xi \to +\infty} \frac{g(\xi)}{\xi} \in [0, +\infty).$$

In [4] existence of a minimizing function for $I(v)$, under suitable assumptions on f, is proved.

We remark that the integrand (4.1) doesn't satisfy their assumptions, since they assume that $f(x, s, 0)$ is finite and some structure condition of the type

$$a(x, s)h(\xi) \leq f(x, s, \xi) \leq a(x, s)h(\xi) + b(x, s).$$

Our corollary gives existence for $I(v)$ with f given by (4.1).

Ex.2.

$$f(x, s, \xi) = a(s)g(\xi)$$

where $a(s)$ and $g(\xi)$ are such that assumption (A.1), ... ,(A.3) are satisfied. In particular (A.1) is satisfied with $h(\xi) = g^{**}(\xi)$ and $k = 0$. Moreover,if $\tilde{h} = 0$ and $a(s)$ is a monotone function assumption (A.4) is satisfied.For example we consider

$$f_1(x, s, \xi) = s^p[\xi^{-1} + \xi^{\frac{1}{2}}]$$
$$f_2(x, s, \xi) = s^{-p}[\xi^{-1} + \xi^{\frac{1}{2}}]\,.$$

Theorem 1 applies to f_1 and f_2.

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REFERENCES

[1] J. Ball: *Discontinuous equilibrium solutions and cavitations in non linear elasticity*, Phil. Trans. R. Soc. London, **306** (1982).

[2] P.J. Blatz – W.L. Ko: *Application on finite elastic theory to the deformation of rubbery materials*, Trans. Soc. Rheology, **6** (1962).

[3] B. Botteron – B. Dacorogna: *Existence and non existence results for non coercive variational problems and applications in ecology*, Journal diff. eq., Vol. 85, N 2 (1990).

[4] B. Botteron – P. Marcellini: *A general approach to the existence of minimizers of one dimensional coercive integrals of the Calculus of Variations*, Ann. I.H.P. Analyse non lineaire, **8** (1991).

[5] E. Giusti: *Equazioni alle derivate parziali*, Quaderni U.M.I.

[6] D. Giachetti – R. Schianchi: *Minima of some non convex non coercive problems*, Ann. Mat. Pura e Appl., (IV), CLXIV, (1993).

[7] A.D. Ioffe: *On lower semicontinuity of integral functionals*, SIAM J. Control and Optimization Vol. 15 N. 4,(1977).

[8] P. Marcellini: *The stored energy in presence of cavitation in non linear elasticity*, Essays in honor of E. De Giorgi, Vol. II; edited by F. Colombini ed al., Birkhäuser (1989).

[9] P. Marcellini: *Non convex integrals of the Calculus of Variations*, Methods of non convex analysis Varenna (1989), Lecture Notes in Math, **1446** Springer, Berlin, (1990).

[10] J.P.Raymond: *Conditions necessaires et suffisantes d'existence de solutions en calcul des variations*, Ann. Inst. Henri Poincaré, Analyse non lineaire. Vol. 4 (1987).

[11] J.P. Raymond – D. Seghir: *Lower semicontinuity and integral representation of functionals in* $BV([a,b];R^m)$, to appear.

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