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Jackson type estimates in the approximation of random functions by random polynomials

S.G. GAL

Riassunto: *Il problema dell'approssimazione delle funzioni aleatorie mediante polinomi aleatori è stato studiato in [1-2], [6-8], [10]. Lo scopo di questo lavoro è di ottenere stime dell'ordine di convergenza delle approssimazioni.*

Abstract: *Some theorems of Weierstrass type concerning the uniform approximation in mean of random functions by random polynomials are obtained in [1-2], [6-8], [10] and Korovkin type estimates are given in [11]. Using a suitable modulus of continuity* $\Omega(f;\delta)$*, the main purpose of this paper is to construct random polynomials of degree* $\leq n$ *which approximate the random function* f *uniformly in mean, with the approximation order* $\mathcal{O}\big(\Omega(f;1/n)\big)$*. The cases of both periodic and non-periodic random functions are considered.*

1 – Introduction

Let $(E,\mathcal{B},P)$ be a probability space, where E is a nonempty set, $\mathcal{B}$ a σ-field of parts on E and P a probability on $\mathcal{B}$.

Let us denote by $L(E,\mathcal{B},P)$ the set of the real random variables

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defined on E, a.e. finite and for each $p \geq 1$ let us define

$$L^p(E,\mathcal{B},P) = \left\{ g \in L(E,\mathcal{B},P)\,; \quad \int_E |g(\omega)|^p dP(\omega) < +\infty \right\}.$$

Section 2 of this paper contains some definitions and auxiliary results. Using a suitable modulus of continuity $\Omega(f;\delta)$, in Section 3 we obtain Jackson type estimates for the uniform approximation in mean of 2π-periodic random functions, by random trigonometric polynomials, the approximation order being $\mathcal{O}(\Omega(f;1/n))$.

Finally, Section 4 contains a Timan type estimates for the uniform approximation in mean of random functions defined on a closed interval, by random algebraic polynomials.

Although known arguments used in the approximation of real functions by ordinary polynomials are adapted to our problems and the results are going to give are the analogue of what are well-known in that field, however we believe that they are of some interest in the study of the stochastic processes.

2 – Definitions and auxiliary results

It is known the following

Definition 2.1 (see e.g. [8], p. 46). *Let $f : [a,b] \longrightarrow L^1(E,\mathcal{B},P)$ be a random function. We say that f is continuous in mean (of order one) in a point $t_0 \in [a,b]$, it for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0)$ such that*

$$\int_E |f(t,\omega) - f(t_0,\omega)| dP(\omega) < \varepsilon\,, \quad \textit{for all} \quad t \in [a,b] \quad \textit{with} \quad |t - t_0| < \delta\,.$$

For $a, b \in \mathbb{R}$, $a < b$, let us define

$CM[a,b] = \{f : [a,b] \longrightarrow L^1(E,\mathcal{B},P)\,;\ f$ is continuous in mean in each $t \in [a,b]\}$.

The following result holds.

Lemma 2.2. *If $f \in CM[a,b]$ then*

$$\sup\left\{ \int_E |f(t,\omega)|dP(\omega)\,; \quad t \in [a,b] \right\} < +\infty\,.$$

Proof. Define $F : [a,b] \longrightarrow \mathbb{R}$ by $F(t) = \int_E |f(t,\omega)|dP(\omega)$ and let suppose that F is unbounded on $[a,b]$. This means that there exists a sequence $x_n \in [a,b]$, $n \in \mathbb{N}$, such that $F(x_n) > n$, $n = 1, 2, \ldots$, that is

$$\int_E |f(x_n,\omega)|dP(\omega) > n\,, \quad \forall\, n \in \mathbb{N}\,. \tag{1}$$

Since $x_n \in [a,b]$, $n \in \mathbb{N}$, there exists a subsequence of x_n, $y_k = x_{n_k}$, $k = 1, 2, \ldots$, such that $y_k \to y_0 \in [a,b]$.

By (1) we get

$$k < \int_E |f(y_k,\omega)|dP(\omega) \le \int_E |f(y_k,\omega) - f(y_0,\omega)|dP(\omega) + \\ + \int_E |f(y_0,\omega)|dP(\omega)\,, \qquad k \in \mathbb{N}$$

and passing to limit with $k \to +\infty$, taking into account that f is continuous in mean in y_0 too, we obtain the contradiction

$$\int_E |f(y_0,\omega)|dP(\omega) = +\infty\,.$$

Let us define

$$L^1([a,b]\times E) = \{f \;:\; [a,b] \longrightarrow L^1(E,\mathcal{B},P);\;\; f(t,\omega) \text{ is measurable } \mathcal{L}[a,b]\times\mathcal{B} \text{ and } \int_a^b |f(t,\omega)|dt < +\infty;\text{ a.e. } \omega \in E\}$$

and

$$L^1_{2\pi}(\mathbb{R}\times E) = \{f \;:\; \mathbb{R} \longrightarrow L^1(E,\mathcal{B},P);\;\; f(t,\omega) \text{ is measurable } \mathcal{L}(\mathbb{R})\times\mathcal{B} \text{ and there exists } A \subset E \text{ with } P(A) = 1, \text{ such that } f(t+2\pi,\omega) = f(t,\omega), \text{ for all } t \in \mathbb{R}, \omega \in A \text{ and } \int_{-\pi}^{\pi} |f(t,\omega)|dt < +\infty, \text{ for all } \omega \in A\}, \tag{2}$$

where $\mathcal{L}[a,b]$ and $\mathcal{L}(\mathrm{I\!R})$ denote the classes of Lebesgue measurable sets on $[a,b]$ and on $\mathrm{I\!R}$, respectively.

REMARK. In [3], $f : \mathrm{I\!R} \longrightarrow L^1(E, \mathcal{B}, P)$ is called 2π-periodic if satisfies the weakened condition

$$\int_E |f(t+2\pi,\omega) - f(t,\omega)| dP(\omega) = 0\,, \quad \forall\, t \in R\,.$$

Denote

$$\Delta_h^n f(x) = \sum_{i=0}^{n} (-1)^{n-i} \cdot \binom{n}{i} \cdot f(x+ih)\,.$$

DEFINITION 2.3 (see e.g. [3], p. 460). *Let* $f \in L^1_{2\pi}(\mathrm{I\!R} \times E)$ *and* $n \in \mathrm{I\!N}$. *The* n*-th order modulus of smoothness of* f *is given by*

$$(3) \quad \Omega_n(f;\delta) = \sup\left\{ \sup\left\{ \int_E |\Delta_h^n f(t,\omega)| dP(\omega)\,; \quad t \in \mathrm{I\!R} \right\};\; 0 \le h \le \delta \right\},$$

where $\Delta_h^n f(t,\omega)$ *is applied to the variable* t.

Also, for $f \in L^1([a,b] \times E)$, *the* n*-th order modulus of smoothness of* f *is given by*

$$(4) \quad \Omega_n(f;\delta) = \sup\left\{ \sup\left\{ \int_E |\Delta_h^n f(t,\omega)| dP(\omega)\,;\; t \in [a, b-nh] \right\};\; 0 \le h \le \delta \right\}.$$

The modulus $\Omega_1(f;\delta)$ will be denoted by $\Omega(f;\delta)$.

Concerning the above moduli of smoothness, the following property will be used in Section 3.

LEMMA 2.4. *For all* $\lambda > 0$ *and all* $\delta > 0$ *we have*

$$(5) \qquad \Omega_n(f;\lambda\delta) \le (\lambda+1)^n \Omega_n(f;\delta)\,,$$

where Ω_n *is given by* (4) *or* (3).

Proof. Let Ω_n be given by (3), for example. Firstly, we will prove that for any $m \in \mathbb{N}$ we have

$$\Omega_n(f; m\delta) \leq m^n \cdot \Omega_n(f;\delta)\,. \tag{6}$$

Indeed, by [5]. p. 47 we get

$$\Delta^n_{mh} f(t,\omega) = \sum_{k_1=0}^{m-1} \dots \sum_{k_n=0}^{m-1} \Delta^n_h f(t + k_1 h + \ldots + k_n h, \omega)\,, \quad \text{a.e.} \quad \omega \in E\,.$$

Hence passing to absolute value and integrating on E, we obtain

$$\int_E |\Delta^n_{mh} f(t,\omega)| dP(\omega) \leq \sum_{k_1=0}^{m-1} \dots \sum_{k_n=0}^{m-1} \int_E \left| \Delta^n_h f\left(t + h\Big(\sum_{i=1}^n k_i \Big), \omega \right) \right| dP(\omega) \leq$$
$$\leq \sum_{k_1=0}^{m-1} \dots \sum_{k_n=0}^{m-1} \Omega_n(f;\delta) = m^n \cdot \Omega_n(f;\delta)\,,$$

and passing to supremum with $0 \leq h \leq \delta$ and with $t \in \mathbb{R}$, we get (6).

Now, since by (3) obviously $\Omega_n(f;\delta)$ is monotone as function of δ and taking into account that $\lambda < [\lambda] + 1 \leq \lambda + 1$, we immediately get (5).

The proof for $\Omega_n(f;\delta)$ defined by (4) is analogous.

Another important result which will be used in Section 3 and 4 is indicated below.

Lemma 2.5 (i). *If $f \in L^1([a,b] \times E)$, then we have*

$$\int_E \left[\int_a^b |f(t,\omega)| dt \right] dP(\omega) = \int_a^b \left[\int_E |f(t,\omega)| dP(\omega) \right] dt\,.$$

(ii) *If $f \in L^1_{2\pi}(R \times E)$ then*

$$\int_E \left[\int_{-\pi}^{\pi} |f(t,\omega)| dt \right] dP(\omega) = \int_{-\pi}^{\pi} \left[\int_E |f(t,\omega)| dP(\omega) \right] dt\,.$$

PROOF. (i) By $f \in L^1([a,b]\times E)$, obviously $|f| \in L^1([a,b]\times E)$. Then our equality is an immediate consequence of a Fubini type result (see e.g. Theorem 10.2.2 in [9], p. 187).

The proof of (ii) is entirely analogous.

Now, if we define
$CM(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow L^1(E, \mathcal{B}, P);\ f$ is continuous in mean on $\mathbb{R}\}$, the following result holds.

THEOREM 2.6 (i). *If* $f \in L^1([a,b]\times E) \cap CM[a,b]$ *then*

$$\int_E \left[\int_a^b f(t,\omega)dt\right] dP(\omega) = \int_a^b \left[\int_E f(t,\omega)dP(\omega)\right] dt\,. \tag{7}$$

(ii) *If* $f \in L^1_{2\pi}(\mathbb{R}\times E) \cap CM(\mathbb{R})$ *then*

$$\int_E \left[\int_{-\pi}^{\pi} f(t,\omega)dt\right] dP(\omega) = \int_{-\pi}^{\pi} \left[\int_E f(t,\omega)dP(\omega)\right] dt\,. \tag{8}$$

PROOF. (i) Let $f \in L^1([a,b]\times E) \cap CM[a,b]$ be and define $F : [a,b] \longrightarrow R$ by

$$F(t) = \int_E |f(t,\omega)| dP(\omega)\,, \quad t \in [a,b]\,.$$

Since f is measurable $\mathcal{L}[a,b]\times\mathcal{B}$, then by e.g. [9, Theorem 10.2.2, p. 187], F is measurable $\mathcal{L}[a,b]$. Also, by Lemma 2.2, F is bounded on $[a,b]$ and therefore F is Lebesgue integrable on $[a,b]$.

As conclusion, there exists finite the integral $\int_a^b \left[\int_E |f(t,\omega)| dP(\omega)\right] dt$, and by the Fubini's theorem (see e.g. [9], p. 189), we get (7).

The proof of (ii) is entirely analogous.

Finally, let us consider two known concepts by

Definition 2.7 (see [8]). *A finite sum of the form* $P_n(t,\omega) = \sum\limits_{k=0}^{n} a_i(\omega)\cdot t^i$, $t \in [a,b]$, *where* $a_k \in L(E,\mathcal{B},P)$, $k = 0,1,\dots,n$, *will be called random algebraic polynomial od degree* $\le n$.

Analogously, a sum of the form $S_n(t,\omega) = \sum\limits_{k=0}^{n} [a_k(\omega)\cdot \cos kt + b_k(\omega)\cdot \sin kt]$, $t \in \mathbb{R}$, $\omega \in E$, *where* a_k, $b_k \in L(E,\mathcal{B},P)$, $k = 0,1,\dots,n$, *will be called random trigonometric polynomial of degree* $\le n$.

3 – Approximation by random trigonometric polynomials

The approximation of 2π-periodic random functions $f \in L^1_{2\pi}(\mathbb{R}\times E)$ by random Fourier series and by random trigonometric sum of Fejer type is considered in [3], where additional references can be found.

In this section we will consider random trigonometric polynomials of Jackson type, which will approximate $f \in L^1_{2\pi}(\mathbb{R}\times E)$ with the approximation order $\mathcal{O}(\Omega(f;1/n))$.

The main result of this section is

Theorem 3.1. *There exists an absolute constant* $C > 0$ *such that for each* $f \in L^1_{2\pi}(\mathbb{R}\times E)$ *there exists a sequence* $(J_n)_n$ *of random trigonometric polynomials od degree* $\le n$ *which satisfies*

$$\int\limits_E |f(x,\omega) - J_n(f)(x,\omega)|dP(\omega) \le C\cdot\Omega(f;1/n)\,, \quad x \in \mathbb{R}\,, \quad n \in \mathbb{N}$$

and even

$$\int\limits_E |f(x,\omega) - J_n(f)(x,\omega)|dP(\omega) \le C\cdot\Omega_2(f;1/n)\,, \quad x \in \mathbb{R}\,, \quad n \in \mathbb{N}\,.$$

Proof. For $f \in L^1_{2\pi}(\mathbb{R}\times E)$, let us define

$$J_n(f)(x,\omega) = \int\limits_{-\pi}^{\pi} f(x+t,\omega)\cdot K_n(t)dt\,, \quad \omega \in A\,, \quad n \in \mathbb{N}\,,$$

where A is given by (2) and $K_n(t)$ is defined as in [5, p. 55, relation (5)].

Since $f \in L^1_{2\pi}(\mathbb{R} \times E)$, obviously we can write

$$J_n(f)(x,\omega) = \int_{-\pi}^{\pi} f(t,\omega) \cdot K_n(x-t)dt\,, \quad \omega \in A\,, \quad n \in \mathbb{N}\,,$$

which means that $J_n(f)(x,\omega)$ are random trigonometric polynomials.

In the following, we will reason exactly as in the proof of Theorem 2 in [5, p. 56]. We get

$$f(x,\omega) - J_n(f)(x,\omega) = \int_{-\pi}^{\pi} [f(x,\omega) - f(x+t,\omega)] \cdot K_n(t)dt =$$

$$= \int_{0}^{\pi} [2f(x,\omega) - f(x+t,\omega) - f(x-t,\omega)] \cdot K_n(t)dt\,, \quad \omega \in A\,, \quad x \in \mathbb{R}\,.$$

Hence passing to absolute value and integrating on E, we immediately get

$$\begin{aligned}
&\int_E |f(x,\omega) - J_n(f)(x,\omega)| dP(\omega) \leq \\
&\qquad \leq \int_E \left[\int_0^{\pi} |2f(x,\omega) - f(x+t,\omega) - f(x-t,\omega)| \cdot K_n(t)dt \right] dP(\omega) = \\
&\qquad = \int_0^{\pi} \left[\int_E |2f(x,\omega) - f(x+t,\omega) - f(x-t,\omega)| dP(\omega) \right] K_n(t)dt \leq \\
&\qquad \leq \int_0^{\pi} \Omega_2(f;t) \cdot K_n(t)dt = \int_0^{\pi} \Omega_2(f; nt(1/n)) \cdot K_n(t)dt \leq \\
&\qquad \leq \Omega_2(f;1/n) \int_0^{\pi} (nt+1)^2 \cdot K_n(t)dt \leq C \cdot \Omega_2(f;1/n)\,, \quad x \in \mathbb{R}\,,
\end{aligned}$$

where we have applied firstly Lemma 2.5, (ii), and secondly (5).

In conclusion, this proves the second estimate in Theorem 3.1.

Now, the first estimate is an immediate consequence of the obvious inequality $\Omega_2(f;\delta) \leq 2\Omega(f;\delta)$.

REMARKS. 1) For $f \in L^1_{2\pi}(\mathbb{R} \times E)$ and $p \in \mathbb{N}$, let us define

$$I_n(f)(x,\omega) = -\int_{-\pi}^{\pi} K_{n,r}(t) \cdot \sum_{k=1}^{p+1} (-1)^k \cdot \binom{p+1}{k} f(x+kt,\omega)dt\,, \quad n \in \mathbb{N}\,,$$

where r and $K_{n,r}(t)$ are defined as in [5, p. 57].

Then, reasoning exactly as in the proof of Theorem 3 in [5, p. 57-58] and as in the proof of the previous Theorem 3.1, we get

$$f(x,\omega) - I_n(f)(x,\omega) = (-1)^{p+1} \cdot \int_{-\pi}^{\pi} K_{n,r}(t) \cdot \Delta_t^{p+1} f(x,\omega)dt\,,\ \omega \in A\,,\ n \in \mathbb{N}\,.$$

Hence passing to absolute value and integrating on E, by Lemma 2.5, (ii) and by (5) we obtain

$$\begin{aligned}
&\int_E |f(x,\omega) - I_n(f)(x,\omega)| dP(\omega) \leq \\
&\qquad \leq \int_E \left[\int_{-\pi}^{\pi} K_{n,r}(t) \cdot |\Delta_t^{p+1} f(x,\omega)| dt\right] \cdot dP(\omega) = \\
&\qquad = \int_{-\pi}^{\pi} K_{n,r}(t) \cdot \left[\int_E |\Delta_t^{p+1} f(x,\omega)| dP(\omega)\right] dt \leq \\
&\qquad \leq \int_{-\pi}^{\pi} K_{n,r}(t) \cdot \Omega_{p+1}(f;|t|)dt = 2\int_0^{\pi} \Omega_{p+1}(f; nt(\tfrac{1}{n})) \cdot K_{n,r}(t)dt \leq \\
&\qquad \leq 2 \cdot \Omega_{p+1}(f;\tfrac{1}{n}) \int_0^{\pi} (nt+1)^{p+1} K_{n,r}(t)dt \leq C_{p+1} \cdot \Omega_{p+1}(f;\tfrac{1}{n})\,.
\end{aligned}$$

As conclusion, for every $f \in L^1_{2\pi}(\mathbb{R} \times E)$ and every $p = 1,2,\dots$, there exists a sequence $\big(I_n(f)\big)_n$ of random trigonometric polynomials of degree $\leq n$, such that

$$\int_E |f(x,\omega) - I_n(f)(x,\omega)| dP(\omega) \leq C_{p+1} \cdot \Omega_{p+1}(f;1/n)\,, \tag{9}$$

$$n \in \mathbb{N}\,, \qquad x \in \mathbb{R}\,,$$

where C_{p+1} is an absolute constant.

2) If in addition f is supposed to be in $CM(\mathbb{R})$, then since it is easy to see that $\Omega_{p+1}(f;\delta) \leq 2^p \cdot \Omega(f;\delta)$, and as in the usual case, $f \in CM(\mathbb{R})$ implies $\Omega(f;\delta) \to 0$, as $\delta \to 0$, we get $\Omega_{p+1}(f;\delta) \to 0$, for $\delta \to 0$.

Hence, obviously the random trigonometric polynomials $\big(J_n(f)\big)_n$ in Theorem 3.1 and $\big(I_n(f)\big)_n$ in (9), converge uniformly in mean on $\mathbb{R}$, to the random function f.

3) A question which remains to settle is if Theorem 3.1 remains valid by replacing in (2) the condition

$$f(t+2\pi,\omega) = f(t,\omega)\,, \quad \forall\, t \in \mathbb{R}\,, \quad \forall\, \omega \in A\,,$$

with the weakened condition in [3]

$$\int_E |f(t+2\pi,\omega) - f(t,\omega)|\,dP(\omega) = 0\,, \quad \forall\, t \in \mathbb{R}\,.$$

4 – Approximation by random algebraic polynomials

In this section we will extend Theorem 3.1 to the case of approximation of $f \in L^1\big([a,b] \times E\big)$ by random algebraic polynomials.

We set

$$L[a,b] = \{f : [a,b] \longrightarrow \overline{\mathbb{R}}\,; \quad f \text{ is Lebesgue integrable on } \quad [a,b]\}\,,$$

let us consider a positive linear operator $T : L[a,b] \longrightarrow L[a,b]$ satisfying

$$\text{(10)} \qquad \begin{aligned} &T(1_{[a,b]}) = 1_{[a,b]}\,, \\ &\int_E T\big[f(u,\omega)\big](x)\,dP(\omega) = T\left[\int_E f(u,\omega)\,dP(\omega)\right](x)\,, \end{aligned}$$

for all $f \in L^1\big([a,b] \times E\big)$ and all $x \in [a,b]$. $T\big[f(u,\omega)\big]$ means that T acts on f considered as function of the first variable u; by $f \in L^1\big([a,b] \times E\big)$, obviously $f(\cdot,\omega) \in L[a,b]$, a.e. $\omega \in E$.

In the sequel we need the following result

Theorem 4.1. *Let* $L : L^1([a,b]\times E)\cap CM[a,b] \longrightarrow L^1([a,b]\times E)\cap CM[a,b]$ *be the linear operator defined by*

$$L(f)(x,\omega) = T\big[f(u,\omega)\big](x)\,, \quad \forall\, x\in[a,b]\,, \quad a.e. \quad \omega\in E\,,$$

where T *satisfies (10).*

For all $f\in L^1([a,b]\times E)\cap CM[a,b]$ *and all* $x\in[a,b]$*, we have*

$$\int_E |f(x,\omega)-L(f)(x,\omega)|dP(\omega) \le 2\cdot\Omega\big(f;T(|u-x|)(x)\big)\,.$$

Proof. Let $x\in[a,b]$ be fixed and $\delta>0$. By the standard technique and by (10) we immediately get

$$\begin{aligned}
&\int_E |f(x,\omega)-L(f)(x,\omega)|dP(\omega) = \int_E |f(x,\omega)-T\big[f(u,\omega)\big](x)|dP(\omega) = \\
&=\int_E |T\big[f(x,\omega)-f(u,\omega)\big](x)|dP(\omega) \le \int_E T\big(|f(x,\omega)-f(u,\omega)|\big)(x)dP(\omega) = \\
&= T\left[\int_E f(x,\omega)-f(u,\omega)|dP(\omega)\right](x) \le T\big[\Omega(f;\delta\cdot|u-x|/\delta)\big](x) \le \\
&\le T\big[(1+|u-x|/\delta)\cdot\Omega(f;\delta)\big](x) = \big[1+T(|u-x|)/\delta\big]\cdot\Omega(f;\delta)\,.
\end{aligned}$$

We have two possibilities:

$$\text{(i)} \qquad T(|u-x|)(x)=0 \qquad \text{and} \qquad \text{(ii)} \qquad T(|u-x|)(x)>0\,.$$

Case (i). By the above inequality we get

$$\int_E |f(x,\omega)-L(f)(x,\omega)|\cdot dP(\omega) \le \Omega(f;\delta)\,, \quad \text{for all} \quad \delta>0\,.$$

Passing with $\delta\to 0$ and taking into account that $f\in CM[a,b]$ implies $\Omega(f;\delta)\to\Omega(f;0)=0$, we get

$$\int_E |f(x,\omega)-L(f)(x,\omega)|dP(\omega) = 0 = \Omega\big(f;T(|u-x|)(x)\big)\,.$$

Case (ii). Choosing $\delta = T\big(|u-x|\big)(x)>0$, the proof is immediate.

Remark. A global (and not pointwise) estimate of the previous type was given in [11, Theorem 4.1].

Now, let us consider the Lehnhoff algebraic polynomials introduced in [4] by

$$T_n(f)(x) = (1/\pi) \cdot \int_{-\pi}^{\pi} f\left[\cos(\arccos x + v)\right] \cdot K_{3n-3}(v)dv\,,$$
$$f \in C[-1,1]\,, \qquad x \in [-1,1]\,,$$

where

$$K_{3n-3}(v) = \left[10/\left[n(11n^4 + 5n^2 + 4)\right]\right] \cdot \left[\sin(nv/2)/\sin(v/2)\right]^6 .$$

In [4] it is proved that $T_n(f)(x)$ is an algebraic polynomial of degree $3n-3$ and that is a positive linear operator on $C[-1,1]$, which satisfies

$$\begin{aligned} &T_n(1_{[-1,1]}) = 1_{[-1,1]}\,, \\ &T_n(|u-x|^2)(x) \le \left(\frac{30}{11}\right) \cdot \left[\frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2}\right]^2 . \end{aligned} \tag{11}$$

Let us define $L_n : L^1([-1,1] \times E) \cap CM[-1,1] \longrightarrow L^1([-1,1] \times E) \cap CM[-1,1]$, by

$$\begin{aligned} L_n(f)(x,\omega) &= T_n\left[f(u,\omega)\right](x) = \\ &= (1/\pi) \cdot \int_{-\pi}^{\pi} f\left[\cos(\arccos x + v), \omega\right] K_{3n-3}(v)dv\,. \end{aligned} \tag{12}$$

Since $f \in L^1([-1,1] \times E) \cap CM[-1,1]$, obviously $f \circ \cos$ belongs to the same intersection and therefore we immediately get

$$L_n(f)(x,\omega) = (1/\pi) \cdot \int_{-\pi}^{\pi} f[\cos v, \omega] \cdot K_{3n-3}(\arccos x - v)dv\,,$$

which proves that L_n is a random algebraic polynomial of degree $3n-3$.

On the other hand, by Theorem 2.6, (ii), we obtain

$$\int\limits_E \left[(1/\pi) \cdot \int\limits_{-\pi}^{\pi} f\left[\cos(\arccos x + v), \omega\right] K_{3n-3}(v) dv \right] dP(\omega) =$$
$$= \int\limits_{-\pi}^{\pi} \left[(1/\pi) \cdot \int\limits_E f\left[\cos(\arccos x + v), \omega\right] dP(\omega) \right] K_{3n-3}(v) dv \,.$$

Hence by (11) and by Theorem 4.1, the following estimate of Timan type holds.

Corollary 4.2. *For all* $f \in L^1([-1,1] \times E) \cap CM[-1,1]$ *and all* $x \in [-1,1]$ *we have*

$$\int\limits_E |f(x,\omega) - L_n(f)(x,\omega)| dP(\omega) \leq 4 \cdot \Omega(f; \sqrt{1-x^2}/n + |x|/n^2) \,,$$

where $L_n(f)(x,\omega)$ *is given by* (12).

Remarks. 1) An open problem is the following: what become Theorem 3.1 and Corollary 4.2 if the integral which defines $J_n(f)(x,\omega)$ and $L_n(f)(x,\omega)$ is replaced by the integral in probability defined in e.g. [8, p. 50]?

2) Another problem of some intereset is to find similar results when the convergence in mean is replaced by the convergence in mean of order $p > 1$ (see e.g. [8, p. 46]) and the moduli of smoothness defined by (3) or (4) are replaced by

$$\Omega_{n,p}(f;\delta) = \sup\left\{ \sup\left\{ \left(\int\limits_E \left|\Delta_h^n f(t,\omega)\right|^p dP(\omega) \right)^{1/p} ;\ t \in \mathbb{R} \right\} ;\ 0 \leq h \leq \delta \right\}.$$

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Indirizzo dell'Autore:

S.G. Gal – Department of Mathematics – University of Oradea – Str. Armatei Romane 5 – 3700 Oradea – Romania