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Some examples of manifolds with hyperbolic structures

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Riassunto: *Le varietà quasi Hermitiane iperboliche (quasi parahermitiane) sono state introdotte in [8]. Una classifica di queste varietà è stata fatta in [1]. In questo lavoro si costruiscono degli esempi di [1], fra i quali vengono menzionati i casi delle varietà di Hopf e di Calabi-Eckmann iperboliche.*

Abstract: *Introduced by P. Libermann, and studied by several authors, the almost hyperbolic Hermitian (parahermitian) manifolds were classified in [1] with respect to the covariant derivative of the second fundamental form. We give here some specific examples to show the existence and to distinguish between certain classes. Among them we construct in the hyperbolic case the correspondent of the Hopf and Calabi-Eckmann manifolds.*

1 – Introduction

An almost hyperbolic Hermitian (abbreviated a.h.H., called also almost parahermitian, [8]) structure (P, g) on an m-dimensional manifold M consists in a pseudo-Riemannian structure g and an almost product structure P which is skew-symmetric with respect to g, i.e. $P^2 = I$, $P \neq \pm I$, $g(PX, Y) + g(X, PY) = 0$, $\forall\, X, Y \in \chi(M)$.

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Therefore a manifold M can be endowed with an a.h.H. structure if and only if its structural group can be reduced to ([1])

$$G = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \Big/ A \in GL(k, R) \right\}.$$

Remark that any a.h.H. manifold M has an almost symplectic structure Φ which is the associated 2-form (called the second fundamental form), $\Phi(X,Y) = g(PX,Y)$, $\forall\, X,Y \in \chi(M)$ and moreover the isomorphic distributions V and H corresponding to the eigenvalues ± 1 of P are two Lagrangian distributions with respect to Φ. The a.h.H. manifolds were classified in [1] with respect to the covariant derivative of Φ. The converse of the above remark holds good: any almost symplectic manifold M that admits a Lagrangian distribution, [7] reduces its structural group to G [16] and therefore it assures the existence of an a.h.H. structure on M.

There are even some symplectic manifolds, (e.g. the sphere S^2) that do not admit any a.h.H. structure. The existence problem of such structures is not trivial and there are few examples in the literature. Our aim is to give some new examples and to use them to distinguish between classes and subclasses in [1].

2 – The hyperbolic Hopf and Calabi-Eckmann manifolds

We construct in the hyperbolic case the correspondent of the Hopf and Calabi-Eckmann manifolds and use them to give some examples of a.h.H. structures.

An almost paracontact structure (F, ξ, η) on an m-dimensional manifold M consists in a tensor field F of type $(1,1)$, a vector field ξ and a 1-form η such that ([15])

$$F^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1\,. \tag{2.1}$$

Definition. (F, ξ, η, g) *is an almost paracontact hyperbolic metric structure with spacelike (resp. timelike) field on the almost paracontact* *manifold* (M, F, ξ, η) *if*

$$(2.2)\qquad \begin{array}{r} g(FX, FY) = -g(X,Y) + \eta(X)\eta(Y)\,, \\ (\textit{resp. } g(FX, FY) = g(X,Y) - \eta(X)\eta(Y)) \end{array} \qquad \forall\, X, Y \in \chi(M)\,.$$

It follows that ξ is a spacelike (resp. timelike) unit vector field, i.e. $g(\xi,\xi) = 1$ (resp. $g(\xi,\xi) = -1$). Similar structures appear in [2], [3], [13], [14].

Let $(R^{2k}_k, \langle,\rangle)$ be the pseudo-Euclidean space of $2k$-dimension and index k. Then the pseudosphere $S^{2k-1}_k(r) = \{x \in R^{2k}_k / \langle x,x\rangle = r^2\}$, (resp. the pseudohyperbolic space $H^{2k-1}_{k-1}(r) = \{x \in R^{2k}_k / \langle x,x\rangle = -r^2\}$) of $(2k-1)$-dimension, of index k (resp. $k-1$), $k = 1, 2, \dots$, and radius $r > 0$ is endowed with an almost paracontact hyperbolic metric structure with timelike (resp. spacelike) field.

By straightforward computations, we obtain now a family of a.h.H. structures:

Proposition 1. *Let* $(M_i, F_i, \xi_i, \eta_i)$, $i = 1, 2$ *be two almost paracontact manifolds. Then the tensor field* P *defined on the manifold* $M = M_1 \times M_2$ *by*

$$(2.3)\qquad \begin{aligned} P(X_1, X_2) = (&aF_1^2X_1 + bF_1X_1 + c\eta_2(X_2)\xi_1\,, \quad dF_2^2X_2 + eF_2X_2 + \\ &+ c\eta_1(X_1)\xi_2)\,, \qquad \forall\, X = (X_1, X_2) \in \chi(M)\,, \end{aligned}$$

with $c = \pm 1$ *and* $a, b, d, e \in R$, *is an almost product structure on* M *if and only if* $ab = de = 0$ *and* $a^2 + b^2 = d^2 + e^2 = 1$.

Proposition 2. *Let* (M_i, F_i, ξ_i, g_i), $i = 1, 2$ *be two almost paracontact hyperbolic metric manifolds, both of spacelike (resp. timelike) field. If* P *is given by (2.3) and* $G = \begin{pmatrix} -g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, *then* (P, G) *is an a.h.H. structure on* $M_1 \times M_2$ *if and only if* $a = d = 0$ *and* $b^2 = e^2 = 1$. *If either* M_1 *or* M_2 *is* 1*-dimensional, then* M *still carries a family of a.h.H. structures.*

When $a = d = 0$ and $b = c = e = 1$, we find the structures given in [6].

Corollary. *The manifolds* $S^{2k_1-1}_{k_1}(r_1)\times S^{2k_2-1}_{k_2}(r_2)$ *and* $H^{2k_1-1}_{k_1}(r_1)\times H^{2k_2-1}_{k_2-1}(r_2)$, $k_1, k_2 = 1, 2, \ldots,$ *with* $r_1, r_2 > 0$ *can be endowed by some a.h.H. structures.* (*They are lying in the class* U *of a.h.H. manifolds, [1]*).

When $k_1 = 1$, $k_2 \geq 2$ (resp. $k_1, k_2 \geq 2$) we call the above manifolds, hyperbolic Hopf (resp. hyperbolic Calabi-Eckmann) by the analogy with the known Hopf and Calabi-Eckmann manifolds.

3 – An almost hyperbolic Kaehler manifold which is not hyperbolic Kaehler

The class $U_4 \oplus U_6$ of almost hyperbolic Kaehler manifolds contains the a.h.H. manifolds with closed fundamental 2-form and its subclass PK of hyperbolic Kaehler (or parakaehler) manifolds contains only those with integrable product structure, [1].

We give now an example of a manifold lying in $U_4 \oplus U_6$ but not in PK.

Let M be an m-dimensional manifold, endowed with a symmetric linear connection $\overline{V}$ and let V (resp. H) be the vertical distribution (resp. the horizontal distribution induced by $\overline{V}$) on the total space of the cotangent bundle (T^*M, π, M). Now, we construct on T^*M an almost product structure P, with V and H as the eigen distributions corresponding to the eigenvalues $+1$ and -1, resp. If G is the Riemann extension of $\overline{V}$, [12], then by a straightforward calculation, we get:

Proposition 3. *Under the above notation,* (P, G) *is an a.h.H. structure on* T^*M *and its fundamental* 2*-form is (up to the sign) the canonical symplectic form on* T^*M*. Moreover,* (P, G) *is a hyperbolic Kaehler structure if and only if* $\overline{V}$ *is flat.*

The above result remains true for the generalization of Riemann extension, [11].

4 – Special a.h.H. manifolds of Vidal type

In this section, let (M, P, g) be an a.h.H. manifold of m-dimension and let $\overline{V}$, V and H be the Levi-Civita connection of g, the vertical and horizontal distributions (i.e. the distributions corresponding to the eigenvalues ± 1 of P), resp. Then $m = 2n$, $\dim V = \dim H = n$.

Definition [1]. *An a.h.H. manifold M is of Vidal type on the vertical (resp. horizontal) distribution if $(\overline{V}_T P)T = 0$, $\forall\, T \in \Gamma(V)$ (resp. $\forall\, T \in \Gamma(H)$).*

The following characterization of the above notion will be useful later on:

Lemma. *An a.h.H. manifold M is of Vidal type on the vertical distribution V if and only if V has a local frame $\{A_i / i = \overline{1, n}\} \subset \Gamma(V)$ on a neighborhood of any point of M such that $g([A_i, A_j], A_k)$ is:*

(1) unchangeable to any cyclic permutation of (i, j, k) when $1 \leq i < j < k \leq n$ and

(2) zero when $1 \leq i < j \leq n$ and $k = i$ or j.

Analogous result for the a.h.H. manifolds of Vidal type on the horizontal distribution. We remark that any vector field $L \in \chi(M)$ belongs to $\Gamma(V)$ if and only if

$$g(L, A) = 0\,, \qquad \forall\, A \in \Gamma(V)\,. \tag{4.1}$$

Proof. The manifold M is of Vidal type on V if and only if

$$\overline{V}_{A_i} A_j + \overline{V}_{A_j} A_i \in \Gamma(V)\,, \quad \forall\, i, j = \overline{1, n}\,. \tag{4.2}$$

The direct part of this assertion is obvious. For the converse, we prove by induction over s that $\overline{V}_A A \in \Gamma(V)$, $\forall\, A = \sum_{i=1}^{s} a_i A_i$, $s = \overline{1, n}$, where a_i are some differential functions. From (4.1) we get that (4.2) is equivalent with $g(\overline{V}_{A_i} A_j + \overline{V}_{A_j} A_i, A_i) = 0$, $i, j = \overline{1, n}$, which is equivalent (from the relation that defines $\overline{V}$) with

$$g\big([A_k, A_i], A_j\big) = g\big([A_j, A_k], A_i\big)\,, \quad i, j, k = \overline{1, n}\,. \tag{4.3}$$

If we distinguish the case when all indices are different one from another, it is obvious that (4.3) is equivalent with:

$$(4.4)\qquad \begin{cases} g\big([A_k,A_i],A_j\big)=g\big([A_j,A_k],A_i\big)\,, & i\neq j\neq k\neq i \quad \text{and} \\ g\big([A_k,A_i],A_j\big)=g\big([A_i,A_k],A_j\big)\,, & \text{for the rest.} \end{cases}$$

By taking all six permutations of (i,j,k) we get (4.4.1) equivalent with (1) and by taking two of the indices (i,j,k) equal, we get (4.4.2) equivalent with (2). ∎

Our aim now is to find an example of an a.h.H. manifold of Vidal type on the verical distribution, but with the vertical distribution nonintegrable.

In the case of the almost product Riemannian manifolds, the first example of a Vidal space with both vertical and horizontal distributions nonintegrable was given in [9] by taking a certain manifold of 4-dimension. The hyperbolic case is different:

Proposition 4. *There are no a.h.H. m-dimensional manifolds of Vidal type on the nonintegrable vertical (resp. horizontal) distribution, when $m<6$.*

Proof. We suppose $m<6$. Then it is 2 or 4. If $m=2$, then both distributions are integrable. If $m=4$, then $\dim V=\dim H=2$. We suppose V nonintegrable, the case of H being similar. Let $\{A,B\}$ be a local frame of V such that $[A,B]\notin\Gamma(V)$. Since M is of Vidal type on V, we use (1) of Lemma and by (4.1) we get $[A,B]\in\Gamma(V)$. ∎

From the above Lemma, we construct now an a.h.H. manifold of Vidal type on the nonintegrable vertical distribution.

We take $M=S^7\times V$, where W is a 1-dimensional manifold. Let $N=x^ie_i$ be the normal unit vector field of S^7, where $\{e_i/i=\overline{1,8}\}$ is the canonical basis of R^8. A parallelization of S^7 is given by $\{x_i=N\times e_{i+1}/i=\overline{1,7}\}$ where $\times$ denotes the product in the Cayley algebra. Let $Z\in\chi(M)$ be a nowhere zero vector field and denote $A_1=X_1$; $A_2=X_2$; $A_3=X_5$; $A_4=Z$; $A_5=X_7$; $A_6=X_4$; $A_7=X_3$ and $A_8=X_6$. We define an a.h.H. structure (P,g) on M by $P=\begin{pmatrix} I_4 & 0\\ 0 & -I_4\end{pmatrix}$ and $g=\begin{pmatrix} 0 & I_4\\ I_4 & 0\end{pmatrix}$ with respect to $\{A_1,\dots,A_8\}$. We remark that $\{A_i/i=\overline{1,4}\}$ (resp. $\{A_i/j=\overline{5,8}\}$ is a basis of V (resp. H).

Proposition 5. *The manifold $M = S^7 \times W$ endowed with the a.h.H. structure (P, g) defined above is of Vidal type on the nonintegrable vertical distribution and M is not of Vidal type on the horizontal distribution.*

Proof. Since $[A_1, A_2] = 2A_7$; $[A_1, A_3] = -2A_6$; $[A_2, A_3] = 2A_5$; $[A_1, A_4] = [A_2, A_4] = [A_3, A_4] = 0$ we get from the Lemma that M is of Vidal type on V. As $[A_1, A_2] \notin \Gamma(V)$ then V is nonintegrable. It can be easily verified that M is not of Vidal type on H. ∎

Definition [1]. *A manifold M endowed with an a.h.H. structure (P, G) is hyperbolic nearly-Kaehler if $(\overline{V}_X P) = 0$, $\forall X \in \chi(M)$.*

We remark that any hyperbolic nearly-Kaehler manifold is of Vidal type on both distributions and any hyperbolic Kaehler manifold is hyperbolic nearly-Kaehler. Our aim is to find a hyperbolic nearly-Kaehler manifold which is not hyperbolic Kaehler.

The pseudosphere S^6_3 was endowed in [8] with an a.h.H. structure (P, g) which is not hyperbolic Kaehler. We recall now the notion of the second kind Cayley algebra Q over the real field (which is different of the Cayley algebra), [8]. Each element $x \in Q$ can be written as $x = AI + a$, where $a = t^i e_i \in R^7_3$ and $(I, e_1, \ldots, e_7)$ is the canonical basis of Q. The dot product is defined as follows:

$$\begin{cases} e_i \cdot e_j = -e_j \cdot e_i \quad \text{for} \quad i \neq j\,; \quad e_i \cdot (e_i \cdot e_j) = (e_i)^2 \cdot e_j\,, \quad i, j = \overline{1,7}\,; \\ (e_i)^2 = -1\,,\ i = \overline{1,3}\,; \quad (e_i)^2 = 1\,,\ i = \overline{4,7} \\ e_1 \cdot e_2 = e_3\,; \quad e_1 \cdot e_6 = e_4 \cdot e_2 = e_5 \cdot e_3 = e_7\,; \\ e_4 \cdot e_5 = e_1\,; \quad e_2 \cdot e_5 = e_4 \cdot e_3 = e_6\,. \end{cases}$$

For any $x = AI + a$ and $y = BI + b$, the inner product $\times$ is defined by $x \cdot y = \langle A, B \rangle I + a \times b$, where $\langle , \rangle$ is the pseudo-Euclidean structure of R^7_3. It is easily seen that $\times$ is bilinear, skew-symmetric and $\langle a \times b, a \rangle = \langle a \times b, b \rangle = 0$.

Let g be the pseudo-Riemannian structure induced on S^6_3 by $\langle , \rangle$ from R^7_3. The tangent space $T_a(S^6_3)$ can be identified with the subspace of R^7_3 which is orthogonal to a, $\forall a \in S^6_3$, [10].

Let P be the tensor field of type $(1,1)$ defined on S^6_3 by $P_a(b) = a \times b$, $\forall b \in T_a(S^6_3)$.

As for S^6 in [4] we can prove the following:

PROPOSITION 6. *The pseudosphere S^6_3 with the nonintegrable a.h.H. structure (P, g) is hyperbolic nearly-Kaehler.*

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