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Almost complex manifolds with holomorphic distributions

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Riassunto: *Una distribuzione* $2q$*-dimensionale su di una varietà orientabile* M *di dimensione* $2n$ *si dice olomorfa se il suo fibrato tangente* TM *ammette una riduzione del gruppo strutturale ad un prodotto* $U(n-q)\times U(q)$ *di due gruppi unitari. Si dimostra che una varietà con una distribuzione olomorfa ammette due classi di metriche* g, h, *due classi di strutture quasi-complesse* J, J' *ed una struttura quasi-prodotto* Q. *Sono analizzate le condizioni d'integrabilità delle strutture* J, J' *e* Q. *Sono inoltre trattate le condizioni (quasi) Kähler di* J *e* J' *con rispetto a* g *e* h.

Abstract: *A* $2q$*-dimensional distribution on an orientable manifold* M *of dimension* $2n$ *is called holomorphic if its tangent bundle* TM *admits a reduction of the structure group to a product* $U(n-q)\times U(q)$ *of two unitary groups. It is shown that a manifold with a holomorphic distribution admits two kinds of metrics* g, h, *two kinds of almost complex structures* J, J', *and an almost product structure* Q. *The integrability conditions of the structures* J, J' *and* Q *are analysed. The (almost) Kähler conditions of* J, J' *with respect to* g *and* h *are also treated.*

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1 – Introduction

In his book [17], Steenrod proved that a compact manifold admits a pseudo-Riemannian metric of certain signature, say k, if and only if the manifold admits a k-dimensional distribution (or equivalently, a nonsingular field of tangent k-planes). If the manifold is of dimension m, then such a situation is just a reduction of the structure group of its tangent bundle to $O(m-k)\times O(k)$, the maximal compact subgroup of the pseudo-orthogonal group $O(m-k,k)$. We now restrict our attention to an orientable manifold M of even dimension $2n$, which admits an orientable distribution τ of even dimension $2q$. That is, the tangent bundle TM of M admits a reduction of the structure group to $SO(2n-2q)\times SO(2q)$, the maximal compact subgroup of $SO_0(2n-2q,2q)$. We shall call such a $2q$-dimensional distribution τ *holomorphic* if the structure group $SO(2n-2q)\times SO(2q)$ further reduces to a product group $U(n-q)\times U(q)$ of two unitary groups.

The purpose of the present paper is to study such even-dimensional orientable manifolds with holomorphic distributions. It is elementary to recognize that a manifold M with a holomorphic distribution τ is necessarily an almost complex manifold since $U(n-q)\times U(q)\subset U(n)$, and moreover that M is a pseudo-Riemannian manifold of signature $2q$, since $U(n-q)\times U(q)\subset SO(2n-2q)\times SO(2q)\subset SO_0(2n-2q,2q)$. We denote an almost complex structure and a pseudo-Riemannian metric on M by J and g, respectively.

We shall show that a manifold M with a holomorphic distribution τ admits another almost complex structure J' which commutes with J, an almost product structure Q which is written in terms of J and J' as $Q=-JJ'$, and also a Riemannian metric h which is related to g by $h=Qg$. Therefore, a manifold M with τ carries four kinds of almost Hermitian structures

$$(g,J),\qquad (g,J'),\qquad (h,J),\qquad (h,J')$$

and two kinds of almost product metric structures

$$(g,Q),\qquad (h,Q).$$

At the first stage of the paper, we shall study the mutual relations among these six structures.

In Section 2, we shall observe the basic relations among the metrics g, h, the almost complex structures J, J', and the almost product structure Q, which are needed for the later analysis of the various almost Hermitian structures and product metric structures. In Section 3, the holomorphic distributions are characterized in terms of two kinds of Kähler forms, and their integrability are analysed. In Section 4, using the Lie brackets of two $(1,1)$ tensors defined by FRÖHLICHER and NIJENHUIS [8] we study the integrability of J, J' and Q. A special attention will be paid to 4-dimensional manifolds, in relation to the work of BEAUVILLE [2]. In the last section (§5), indefinite Kähler manifolds and opposite Kähler manifolds are treated.

Throughout the paper, we always pay attention to examples in various cases. The analysis on Chern classes determined by J and J' are treated separately [4].

2 – Manifolds with holomorphic distributions

It is well-known (STEENROD [17]) that the existence of a pseudo-Riemannian metric on a manifold is equivalent to the existence of a distribution of certain dimension k. In other words, an m-dimensional manifold admits a pseudo-Riemannian metric of signature k if and only if the Grassmann manifold bundle $G_k(TM)$ over M (associated with its tangent bundle TM), with fibre $G_k(\mathbb{R}^m) = O(m)/O(m-k) \times O(k)$, admits a section.

In the present paper, we shall restrict our attention to even dimensional orientable manifolds, where the orthogonal group in Steenrod's analysis can be replaced by unitary groups. Such manifolds are in fact almost complex manifolds.

Let M be a $2n$-dimensional almost complex manifold, and $G_q^{\mathbb{C}}(TM)$ a bundle over M (associated with TM), with fibre

$$G_q^{\mathbb{C}}(\mathbb{R}^{2n}) = U(n)/U(n-q) \times U(q).$$

We call the bundle $G_q^{\mathbb{C}}(TM)$ a *complex Grassmann manifold bundle* over M.

THEOREM 2.1. *If an almost complex manifold (M, J) admits an almost pseudo-Hermitian metric of signature $2q$, then the metric defines* *a section of the complex Grassmann manifold bundle* $G_q^{\mathbb{C}}(TM)$ *over* M*. Conversely, if there exists a section of* $G_q^{\mathbb{C}}(TM)$*, then* M *admits an almost pseudo-Hermitian metric of signature* $2q$*.*

PROOF. From Steenrod's analysis [17, 40.6], if M admits a pseudo-Riemannian metric g of signature $2q$ (i.e., if the structure group of TM reduces to $SO_0(2n-2q, 2q)$), then there exists a pair (h, Q) of a Riemannian metric h and an almost product structure Q on M such that

$$Qh = hQ = g,$$

where Q, h and g are considered as linear endomorphisms of TM. (Here, an almost product structure Q associated with g is a symmetric bilinear form of signature $2q$, which satisfies $Q^2 = Id$ [9], [17 Ch. XI], [19, p.423]. Q is also a pseudo-Riemannian metric of signature $2q$.) It is elementary to know that Q defines a $2q$-dimensional distribution, denoted by τ, and also its complementary distribution $\tau^{\perp}$ of dimension $(2n-2q)$.

Assume that g is J-invariant:

$$Jg = gJ,$$

which is equivalent to the condition

$$g(JX, JY) = g(X, Y)$$

for $X, Y \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$: the algebra of smooth vector fields on M). Then, the structure group of TM further reduces to

$$G = U(n) \cap SO_0(2n-2q, 2q) = U(n-q) \times U(q).$$

This implies that Q is also J-invariant:

$$JQ = QJ,$$

and hence that the distributions τ and $\tau^{\perp}$ are both J-invariant, i.e.,

$$J\tau \subset \tau, \quad J\tau^{\perp} \subset \tau^{\perp}.$$

Thus, an almost pseudo-Hermitian metric g of signature $2q$ defines a section of $G_q^{\mathbb{C}}(TM)$.

Conversely, if there exists a section of $G_q^{\mathbb{C}}(TM)$, then it uniquely defines a $2q$-dimensional distribution τ and also its complementary distribution $\tau^{\perp}$. It further defines uniquely an almost product structure Q such that $QJ = JQ$. It is well-known that we can construct an almost Hermitian metric h in terms of any Riemannian metric h_0 on M as follows

$$h(X, Y) = h_0(X, Y) + h_0(JX, JY),$$

for $X, Y \in \mathfrak{X}(M)$. That is, J and h commute with each other

$$Jh = hJ.$$

From Steenrod's analysis, we can construct a pseudo-Riemannian metric g such that

$$g = Qh = hQ,$$

or equivalently

$$g(X, Y) = h(QX, Y) = h(X, QY).$$

Since h and Q are both J-invariant, g is also J-invariant: $Jg = gJ$. Therefore, the existence of a section of $G_q^{\mathbb{C}}(TM)$ implies the existence of an almost pseudo-Hermitian metric g. $\square$

In the above theorem, it should be noted that such an almost pseudo-Hermitian metric g is not unique, because an almost Hermitian metric h on M is not unique.

When $G_q^{\mathbb{C}}(TM)$ admits a section (or equivalently, the reduction of the structure group to $G = U(n-q) \times U(q)$), we call the corresponding J-invariant distribution τ a *holomorphic distribution.* The following proposition is also a crucial observation for the present issue.

Proposition 2.2. *Let M be a $2n$-dimensional manifold with an almost complex structure J and a $2q$-dimensional holomorphic distribution τ. Then, M admits an almost complex structure J' such that*

$$J' = JQ = QJ,$$

where Q is the almost product structure on M associated with τ.

Proof. From the proof of Theorem 2.1, we already know that $JQ = QJ$. It is easy to see that QJ is an almost complex structure, since $(QJ)^2 = QJQJ = Q^2J^2 = -Id$. Clearly, $J' \neq J$. ∎

Definition 2.3. *We call J' an opposite almost complex structure on M.*

Thus, we have observed that M with a holomorphic distribution τ admits two kinds of almost complex structures J, J', and an almost product structure Q. Moreover, M admits an almost pseudo-Hermitian metric g and an almost Hermitian metric h. In summary, these structures are mutually related to each other as follows:

$$JJ' = J'J = -Q, \quad JQ = QJ = J', \quad J'Q = QJ' = J, \tag{1}$$

and for $X, Y \in \mathfrak{X}(M)$,

$$\begin{cases} g(JX, JY) = g(X, Y), \quad h(JX, JY) = h(X, Y) \\ g(J'X, J'Y) = g(X, Y), \quad h(J'X, J'Y) = h(X, Y) \\ g(X, Y) = h(QX, Y) = h(X, QY) \\ h(X, Y) = g(QX, Y) = g(X, QY). \end{cases} \tag{2}$$

Therefore, we have the following.

Theorem 2.4. *A manifold M with a holomorphic distribution τ admits four kinds of almost Hermitian structures:*

(g, J) (*almost pseudo-Hermitian*)

(g, J') (*opposite almost pseudo-Hermitian*)

(h, J) (*almost Hermitian*)

(h, J') (*opposite almost Hermitian*),

and moreover, two kinds of almost product metric structures:

(g, Q) (*almost pseudo-product metric*)

(h, Q) (*almost product metric*).

These six structures in the above theorem are the main concern of the present paper. In what follows, we shall call the $2q$-dimensional (resp. $(2n-2q)$-dimensional) holomorphic distribution τ (resp. $\tau^{\perp}$) *spacelike* (resp. *timelike*), since vectors in τ (resp. $\tau^{\perp}$) are positive (resp. negative) definite with respect to g.

3 – Indefinite Kähler Forms and opposite Kähler Forms

Let M be a manifold with a holomorphic distribution τ. Associated with the four kinds of almost Hermitian structures in Theorem 2.4, we have four kinds of Kähler forms:

$$F_J(X,Y) = g(X,JY), \quad F_{J'}(X,Y) = g(X,J'Y)$$

$$F^h_J(X,Y) = h(X,JY), \quad F^h_{J'}(X,Y) = h(X,J'Y).$$

The following proposition shows the relations among them.

PROPOSITION 3.1. *The following relations hold for $X, Y \in \mathfrak{X}(M)$,*

$$F_J(X,Y) = F^h_{J'}(X,Y), \quad F_{J'}(X,Y) = F^h_J(X,Y).$$

PROOF. From (1) and (2), we have

$$F_J(X,Y) = g(X,JY) = g(X,QJ'Y) = h(X,J'Y) = F^h_{J'}(X,Y).$$

The second relation follows similarly. □

In consequence, there are only two essentially different Kähler forms F_J and $F_{J'}$ (or equivalently F^h_J and $F^h_{J'}$). We will say that M is *indefinite almost Kähler* and *opposite almost Kähler* if both 2-forms F_J and $F_{J'}$ are closed: $dF_J = 0$ and $dF_{J'} = 0$.

Let us consider a local orthonormal frame $\{e_1, e_{1^*} = Je_1, \dots, e_n, e_{n^*} = Je_n\}$ in such a way that the first $2q$ vectors span the spacelike distribution τ, and the last $2(n-q)$ vectors span the timelike distribution $\tau^{\perp}$. Then the Kähler 2-forms F_J and $F_{J'}$ are written respectively by

$$F_J = -\sum_{i=1}^{q} e^i \wedge e^{i^*} + \sum_{j=q+1}^{n} e^j \wedge e^{j^*}$$

$$F_{J'} = -\sum_{i=1}^{q} e^i \wedge e^{i^*} - \sum_{j=q+1}^{n} e^j \wedge e^{j^*},$$

where $\{e^1, \dots, e^{n^*}\}$ is the dual basis of 1-forms. In this way, spacelike and timelike distributions are characterized by

$$\tau = \{X \in \mathfrak{X}(M) | i_X \Omega_+ = 0\}$$

$$\tau^\perp = \{X \in \mathfrak{X}(M) | i_X \Omega_- = 0\},$$

where

$$\Omega_+ = \frac{1}{2}\left(F_J - F_{J'}\right), \qquad \Omega_- = \frac{1}{2}\left(F_J + F_{J'}\right),$$

and i_X denotes the usual inner product of forms and vector fields, i.e., $(i_X\Omega_\pm)(Z) = \Omega_\pm(X, Z)$.

The following lemma shows the basic formulas for the differentials of $\Omega_\pm$.

Lemma 3.2. *The differentials $d\Omega_\pm$ of $\Omega_\pm$ can be obtained in terms of the covariant derivatives of J and J' by metric connection ∇ of g as follows:*

$$d\Omega_\pm(A, B, C) = \frac{1}{2}\sigma\left\{g((\nabla_A J)B, C) \pm g((\nabla_A J')B, C)\right\},$$

where σ denotes the cyclic sum over $A, B, C \in \mathfrak{X}(M)$.

On the basis of the above formulas, we have some useful relations for $d\Omega_\pm$. Associated with the holomorphic distribution τ, we can decompose any tangent vector X into two parts $X = X_+ + X_-$ such that $X_+ \in \tau$ and $X_- \in \tau^\perp$.

Lemma 3.3. *Let X, Y, Z be vector fields on M, with decompositions $X = X_+ + X_-$, $Y = Y_+ + Y_-$, $Z = Z_+ + Z_-$ such that $X_+, Y_+, Z_+ \in \tau$*

and $X_-, Y_-, Z_- \in \tau^\perp$. *Then we have the following relations*

(a) $d\Omega_+(X_+, Y_+, Z_+) = 0,$
$d\Omega_-(X_-, Y_-, Z_-) = 0$

(b) $d\Omega_+(X_-, Y_-, Z_-) = dF_J(X_-, Y_-, Z_-),$
$d\Omega_-(X_+, Y_+, Z_+) = dF_{J'}(X_+, Y_+, Z_+)$

(c) $d\Omega_+(X_+, Y_+, Z_-) = \left(i_{[X_+,Y_+]}\Omega_+\right)(Z_-),$
$d\Omega_-(X_-, Y_-, Z_+) = \left(i_{[X_-,Y_-]}\Omega_-\right)(Z_+)$

(d) $d\Omega_+(X_-, Y_-, Z_+) = dF_J(X_-, Y_-, Z_+) - \left(i_{[X_-,Y_-]}\Omega_-\right)(Z_+)$
$d\Omega_-(X_+, Y_+, Z_-) = dF_{J'}(X_+, Y_+, Z_-) - \left(i_{[X_+,Y_+]}\Omega_+\right)(Z_-).$

At this stage, we recall the definition of the *mean curvature vector* of a distribution on a pseudo-Riemannian manifold. Let D be a p-dimensional distribution on a pseudo-Riemannian manifold (M, g), with metric connection ∇. Then, the mean curvature vector of D is defined by [3, p.7],

$$H_D = \frac{1}{p}\sum_{i=1}^{p} \epsilon_i nor(\nabla_{e_i} e_i), \tag{3}$$

where $\{e_i\}$ is an orthonormal frame on D, $\epsilon_i = g(e_i, e_i)$, $(i = 1, \ldots, p)$ and $nor(\nabla_X Y)$ denotes the normal component of $\nabla_X Y$ only for $X, Y \in D$. The distribution D is said to define a *minimal foliation* if D is involutive and $H_D = 0$.

As a consequence, we establish the following.

Theorem 3.4. *If an almost pseudo-Hermitian manifold* (M, g, J, J') *is indefinite almost Kähler and opposite almost Kähler, then the spacelike and timelike distributions* τ, $\tau^\perp$ *define transversal minimal foliations on* M.

Proof. From Lemma 3.3, we see that the distributions τ, $\tau^\perp$ are integrable if and only if

$$d\Omega_{(\pm)}\left(X_{(\pm)}, Y_{(\pm)}, -\right) = 0 \qquad \forall X_+, Y_+ \in \tau, X_-, Y_- \in \tau^\perp.$$

Moreover, if $dF_J = 0$ and $dF_{J'} = 0$, it follows that $\Omega_\pm$ are closed forms, and so, τ and $\tau^\perp$ define complementary foliations on M.

From (3), for the mean curvature vector of the distribution τ we get

$$H_\tau = -\frac{1}{2p}\sum_{i=1}^{p}\left\{nor(\nabla_{U_i}U_i) + nor(\nabla_{JU_i}JU_i)\right\}, \tag{4}$$

where $\{U_i, JU_i\}$ is an orthonormal frame on τ with $(U_i)_- = 0$ and $(JU_i)_- = 0$. Then, considering the relation between the almost complex structures J and J', we have for $X_+ \in \tau$ and $Y_- \in \tau^\perp$,

$$d\Omega_-(X_+, JX_+, Y_-) = -g(\nabla_{X_+}X_+ + \nabla_{JX_+}JX_+, Y_-). \tag{5}$$

Therefore, if the 2-form Ω_- is closed, then the vector $\nabla_{X_+}X_+ + \nabla_{JX_+}JX_+$ lies in τ (its normal components are zero), and hence $H_\tau = 0$. This implies that τ is minimal. A similar result for $\tau^\perp$ follows from the same argument for Ω_+. □

COROLLARY 3.5. *Let (M, g, J) be an indefinite almost Kähler manifold. Then the distributions τ, $\tau^\perp$ are integrable if and only if the accompanied opposite almost pseudo-Hermitian structure (g, J') is opposite almost Kähler.*

PROOF. From Lemma 3.2 it follows that since $dF_J = 0$, the distributions τ, $\tau^\perp$ are integrable if and only if $d\Omega_+ = 0$, and so $dF_{J'} = 0$, $d\Omega_- = 0$. □

We restrict our attention to 4-dimensional manifolds for a while (cf. [12], [11]).

THEOREM 3.6. *Let M be a 4-dimensional manifold with an indefinite metric of signature $(+,+,-,-)$. Then, the spacelike and timelike distributions define minimal foliations on M if and only if the associated almost Hermitian structures (g, J), (g, J') are indefinite almost Kähler and opposite almost Kähler.*

PROOF. We shall prove that the 2-forms $\Omega_\pm$ are closed if and only if the distributions τ, $\tau^\perp$ are integrable and minimal.

The 'only if' part is a previous result. In order to prove the 'if' part, we consider the differential $d\Omega_+$, which vanishes when we apply it to three vectors on τ because both distributions are of dimension 2. According to Lemma 3.3, $d\Omega_+(X_+, Y_+, Z_-) = 0$ if and only if the distribution τ is integrable. From (4) and (5), it follows that $d\Omega_+(X_-, JX_-, Y_+) = 0$ if and only if $\tau^\perp$ is minimal. As a consequence of $\dim\tau^\perp = 2$, $d\Omega_+(X_-, Y_-, Z_+) = 0$ if and only if $\tau^\perp$ is minimal. This completes the proof. $\square$

EXAMPLE 3.7. *4–dimensional indefinite almost Kähler and opposite almost Kähler manifolds.*

Let $G(k)$ be a 3-dimensional Lie group of matrices

$$A = \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$ and $e^k + e^{-k}$ is an integer different from 2. There exists a discrete subgroup $\Gamma(k)$ of $G(k)$ such that the quotient space $M^3(k) = \Gamma(k)\backslash G(k)$ is compact, and we can consider the 4-dimensional compact parallelizable manifold $M^4(k) = M^3(k) \times S^1 \equiv (\Gamma(k) \times \mathbb{Z})\backslash(G(k) \times \mathbb{R})$.

Let $\{\tilde\alpha, \tilde\beta, \tilde\gamma\}$ be the basis of left invariant 1-forms on $G(k)$; they satisfy the equations

$$d\tilde\alpha = -k\tilde\alpha \wedge \tilde\gamma, \qquad d\tilde\beta = k\tilde\beta \wedge \tilde\gamma, \qquad d\tilde\gamma = 0.$$

Let $\{\tilde X, \tilde Y, \tilde Z\}$ be the dual basis of left invariant vector fields on $G(k)$. Let $\{\alpha, \beta, \gamma\}$ and $\{X, Y, Z\}$ denote their projections on $M^3(k)$, and let T be the coordinate vector field on S^1, $T = \partial/\partial t$, t being the usual coordinate on S^1, and $\eta = dt$. Then on $M^4(k)$ we have

$$d\alpha = -k\alpha \wedge \gamma, \quad d\beta = k\beta \wedge \gamma, \quad d\gamma = d\eta = 0.$$

Thus we can construct on $M^4(k)$: an indefinite metric of signature $(+, +, -, -)$ given by

$$g = \alpha \otimes \alpha + \beta \otimes \beta - \gamma \otimes \gamma - \eta \otimes \eta,$$

an almost complex structure J and an opposite almost complex structure J' given by

$$J(X) = Y, \quad J(Z) = T, \qquad J'(X) = Y, \quad J'(Z) = -T.$$

It is easy to check that the pair (g, J) is an almost pseudo-Hermitian structure on $M^4(k)$, and similarly that the pair (g, J') is an opposite almost pseudo-Hermitian structure on $M^4(k)$. Moreover, both structures are almost Kähler ($dF_J = 0$, $dF_{J'} = 0$), with timelike distribution $\tau^\perp =< Z, T >$ and spacelike distribution $\tau =< X, Y >$, and neither of them is Kähler because $M^4(k)$ does not admit complex structures [7]. $\square$

The following Theorem may be viewed as a certain converse of Theorem 3.4.

THEOREM 3.8. *Let (M, g, J) be a $2n$-dimensional almost pseudo-Hermitian manifold with metric of signature $(2p, 2q)$, $n-1 > p, q > 1$ such that the Kähler forms satisfy*

$$dF_J = \omega_J \wedge F_J, \quad dF_{J'} = \omega_{J'} \wedge F_{J'} \tag{6}$$

for certain 1-forms ω_J and $\omega_{J'}$ defined on M. Then the distributions τ and $\tau^\perp$ are involutive if and only if M is almost Kähler and opposite almost Kähler.

PROOF. Put

$$\omega^+ = \omega_J - \omega_{J'}, \quad \omega^- = \omega_J + \omega_{J'}. \tag{7}$$

Then, taking account of the relations among the 2-forms $\Omega_\pm$ and the Kähler forms F_J and $F_{J'}$, we have the following:

$$\begin{aligned} d\Omega_+ &= \frac{1}{2}(dF_J - dF_{J'}) = \frac{1}{2}(\omega_J \wedge F_J - \omega_{J'} \wedge F_{J'}) \\ &= (\omega^+ + \omega^-) \wedge (\Omega_+ + \Omega_-) - (\omega^- - \omega^+) \wedge (\Omega_- - \Omega_+) \\ &= 2\omega^+ \wedge \Omega_- + 2\omega^- \wedge \Omega_+. \end{aligned}$$

Since the spacelike distribution τ is characterized by $\tau = \{X \in \mathfrak{X}(M) | i_X\Omega_+ = 0\}$, it follows that

$$(\omega^+ \wedge \Omega_-)(X_+, Y_+, Z_+) = 0, \tag{8}$$

for $X_+, Y_+, Z_+ \in \tau$.

In an analogous way, from (c) in Lemma 3.3, we get

$$2\omega^+(Z_-)\Omega_-(X_+, Y_+) = \left(i_{[X_+,Y_+]}\Omega_+\right)(Z_-) \tag{9}$$

for $X_+, Y_+ \in \tau$, and $Z_- \in \tau^\perp$. From the local expressions for the Kähler forms F_J and $F_{J'}$, it follows that the spacelike (resp. timelike) distribution is contained in the kernel of the 1-form ω^+ (resp. ω^-), as observed in (8).

Now, if the distribution τ is integrable, it follows from (9) that $\omega^+(Z_-)\Omega_-(X_+, Y_+) = 0$ for $Z_- \in \tau^\perp$, $X_+, Y_+ \in \tau$, and hence, it must be $\omega^+(Z_-) = 0$, which shows that the timelike distribution $\tau^\perp$ is contained in the kernel of the 1-form ω^+.

In an analogous way for the 1-form ω^-, together with Lemma 3.3, we have that such a form is zero, and hence, from (7) the 1-forms ω_J and $\omega_{J'}$ are also zero. Consequently, from (6), M is almost Kähler and opposite almost Kähler. □

4 – Complex and opposite complex manifolds

The mutual relations (1) among J, J' and Q are similar to those of an almost quaternionic structure (of either the first or the second kind [18], [19]). See also [16]. Then, an interesting problem may be the following: Does the integrability of any two of J, J' and Q imply the integrability of the third one?

We begin with a review of an example: The manifold of Kodaira-Thurston.

Example 4.1. *Complex and opposite complex manifold, which are not locally diffeomorphic to a product.*

The manifold of Kodaira-Thurston is determined by the structure equations

$$d\omega_1 = 0, \qquad d\omega_2 = \omega_1 \wedge \overline{\omega}_1$$

and can be realized as a quotient space $K = \Gamma\backslash G$, where G is the real Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & \overline{z}_1 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix},$$

and Γ is the subgroup of G consisting of those matrices whose entries are Gaussian integers.

The Lie algebra of the manifold of Kodaira-Thurston may be described as the 4-dimensional Lie algebra with basis $\{E_1, E_2, E_3, E_4\}$ such that $[E_1, E_2] = E_3$ and the other brackets are trivial. A left invariant almost complex and opposite almost complex structures J, J' on such a manifold are given by

$$\begin{array}{llll} JE_1 = E_2, & JE_2 = -E_1, & JE_3 = E_4, & JE_4 = -E_3 \\ J'E_1 = E_2, & J'E_2 = -E_1, & J'E_3 = -E_4, & J'E_4 = E_3 \end{array}$$

Their integrability follows from the Lie algebra equation. The associated almost product structure Q acts on the bases $\{E_i\}$ as follows:

$$QE_1 = E_1, \quad QE_2 = E_2, \qquad QE_3 = -E_3, \quad QE_4 = -E_4,$$

which defines a totally geodesic foliation $\{E_3, E_4\}$. However, the complementary distribution $\{E_1, E_2\}$ is not involutive, and so the almost product structure is not integrable. □

In light of the above example, we will concentrate our attention to the conditions under which integrability of two of J, J' and Q implies that of the third one.

Let (M, g, J) be an almost pseudo-Hermitian manifold, and Q, J' be the accompanying an almost product-complex structure and an opposite almost complex structure, respectively. An almost product structure Q is said to be *integrable* if it defines two complementary foliations on the manifold. Such a condition is expressed analytically by $[Q,Q]=0$, where the Lie bracket of two $(1,1)$ tensor fields L,N is defined by [8]:

$$\begin{aligned}[L,N](X,Y) &= [LX,NY]-L[NX,Y]-N[X,LY]+[NX,LY]\\ &\quad -N[LX,Y]-L[X,NY]+(LN+NL)[X,Y].\end{aligned}$$

We also recall two operations defined in [8]

$$\begin{aligned}([L,T]\bar{\wedge}N)(X,Y) &= [L,T](NX,Y)+[L,T](X,NY)\\ (N\bar{\wedge}[L,T])(X,Y) &= N([L,T](X,Y))\end{aligned}$$

for $X,Y\in\mathfrak{X}(M)$, where L,T,N are $(1,1)$-tensor fields. The following identity holds for such operation ([8], [18]):

$$[L,TN]+[T,LN]=[L,T]\bar{\wedge}N+L\bar{\wedge}[T,N]+T\bar{\wedge}[L,N] \tag{10}$$

Let us consider the $(1,1)$ tensor fields J,J',Q on the manifold M with a holomorphic distribution τ.

Theorem 4.2. *Let (M,g,J) be a pseudo-Hermitian manifold with a holomorphic distribution τ, which satisfies $[Q,J]=0$. Then, the almost product structure Q is integrable if and only if the opposite almost complex structure J' is integrable.*

Proof. If we consider the particular choice in (10) such that $L=T=J, N=J'$, then

$$[J,Q]=-J\bar{\wedge}[J,J']-\frac{1}{2}[J,J]\bar{\wedge}J'$$

and thus $[J,J']=0$. Now, if we take $L=T=J', N=J$ in (10), we get

$$[J',Q]=-\frac{1}{2}[J',J']\bar{\wedge}J$$

and so, if the opposite almost complex structure is integrable it follows $[J',Q]=0$. Taking $L=J, T=Q, N=J'$ in (10), we have

$$[Q,Q]=-J\bar{\wedge}[Q,J'],$$

and hence $[Q, Q] = 0$.

Conversely, if we supposse that the almost product structure Q is integrable, taking $L = T = Q, N = J$ in (10) we obtain $[Q, J'] = 0$. Specifying $L = J', T = Q, N = J$ in (10), we get the integrability of the opposite almost complex structure J'. ∎

In an analogous way the following Theorem and Corollary can be obtained.

THEOREM 4.3. *Let (M, g, J) be an almost pseudo-Hermitian manifold with an integrable holomorphic distribution τ, which satisfies $[Q, J] = 0$. Then, the almost complex structure J is integrable if and only if the opposite almost complex structure J' is integrable.*

COROLLARY 4.4.

1. *If M is a complex and opposite complex manifold, then the almost product structure Q is integrable if and only if $[Q, J] = 0$.*
2. *If M is a complex manifold with integrable almost product structure Q, then the opposite almost complex structure is integrable if and only if $[Q, J] = 0$.*

It should be noted that $[Q, J] = 0$ occurs, for instance, if M is a complex manifold and the decomposition is compatible with the complex structure, or if the almost product structure Q is integrable and the metric g is fiber-like for both timelike and spacelike foliations.

At this stage, we give some examples, which will be helpful for illustration.

EXAMPLE 4.5. *Complex and opposite complex manifolds, which are locally diffeomorphic to a product.*

We consider the product manifold $M = M_1 \times_{(f_1, f_2)} M_2$ of two Hermitian manifolds $(M_i, h_i,\ J_i), (i = 1, 2)$, endowed with a double warped product structure $g = {f_2}^2 h_1 - {f_1}^2 h_2$ and the complex structure $J = J_1 + J_2$, where $f_i : M_i \mapsto \mathbb{R}$ are differentiable functions $(i = 1, 2)$.Then (M, g, J) is a pseudo-Hermitian manifold with spacelike distribution consisting of vectors tangent to M_1 and with timelike distribution consisting of vectors tangent to M_2. In consequence, both distributions define complementary foliations on M. Now, a simple calculation shows that $[Q, J] = 0$, and hence, by Theorem 4.2 the opposite almost complex structure $J' = J_1 - J_2$ defines a complex structure on M. Moreover, spacelike and timelike distributions are totally umbilical with shape operator depending on the functions f_2 and f_1, respectively [15]. In consequence, M is neither indefinite almost Kähler nor opposite almost Kähler unless it is a product. This follows from the umbilicity of the distributions τ, $\tau^\perp$, since any holomorphic submanifold in an indefinite almost Kähler manifold must be minimal. □

Example 4.6. *Complex and opposite almost complex (not opposite complex) manifold.*

Let $(M, <,>)$ be a 4-dimensional Riemannian manifold, and h be the induced Riemannian metric on the vector bundle $\pi : \wedge^2 M \mapsto M$ defined by

$$h(A_1 \wedge A_2, A_3 \wedge A_4) = \frac{1}{2} det(< A_i, A_j >).$$

The Hodge operator $\star$ gives rise to a decomposition $\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$, where $\wedge^2_\pm TM$ are the subbundles corresponding to the eigenvalues ± 1 of the $\star$-operator. In this way, the twistor space Z over M is defined as the submanifold of $\wedge^2_- TM$ determined by those vectors of norm 1. We recall that each point $\sigma \in Z$ determines a complex structure K on the tangent space T_pM as follows: $< KA, B >= 2h(\sigma, A \wedge B)$, where $A, B \in T_pM, p = \pi(\sigma)$.

For an arbitrary tangent vector $A \in T_pM$, let A^h denote its horizontal lift, and consider the almost complex structures J and J' defined by

$$J(A^h) = (KA)^h, \qquad JV = \sigma \times V$$

$$J'(A^h) = (KA)^h, \qquad J'V = -\sigma \times V,$$

where V is a tangent vector to the fiber $\pi^{-1}(p)$ and $\times$ is the standard vector product on the oriented 3-dimensional vector space $\wedge^2_- TM$ [14]. The different classes of (positive definite) almost Hermitian structures are also studied in [14].

In a particular case of M being self-dual, J defines a complex structure on Z, which is Kähler if and only if M is a self-dual Einstein manifold of positive constant scalar curvature. However, the opposite almost complex structure can never be integrable. See also [5].

The almost product structure determined by J and J' is equivalently defined by the distributions generated by vertical and horizontal vectors on the bundle Z. So, if we consider *a particular case of M being a flat torus, its twistor space is a complex manifold and locally a product, but it never be opposite complex.* ∎

In [2], BEAUVILLE considered a problem: When does a surface admit an integrable opposite almost complex structure? We now consider a broader situation: When does an almost complex 4-manifold admit an integrable opposite almost complex structure? Therefore, we concentrate our attention to 4-manifolds with almost complex structures and opposite almost complex structures (equivalently, the existence of fields of 2-planes [11, Fact.7]). Then we have the following.

THEOREM 4.7. *Let M be a 4-manifold which admits an almost complex structure and an opposite almost complex structure. If $\chi[M] \leq 0$ and $\tau[M] < 0$, or if $\chi[M] > 0$ and $\chi[M] + 3\tau[M] < 0$, then the opposite almost complex structure is not integrable.*

PROOF. Let $c_1^2(-M)$, $c_2(-M)$ denote the opposite Chern numbers determined by an opposite almost complex structure. We already know [12, Th. 13] that

$$c_1^2(-M) = -3\tau[M] + 2\chi[M], \quad c_2(-M) = \chi[M]$$

From MIYAOKA [13, p.226], we see that a 4-manifold of non positive Euler characteristic whose opposite Chern numbers satisfy

$$c_1^2(-M) > 2c_2(-M)$$

does not admit an integrable opposite almost complex structure. Therefore, we have that $c_1^2(-M) > 2c_2(-M) \Rightarrow \tau[M] < 0$.

In the case of positive Euler characteristic, it is necessary for M to admit an integrable opposite almost complex structure that $c_1^2(-M) \leq 3c_2(-M)$, and hence that $\chi[M] + 3\tau[M] \geq 0$.
This proves the assertion. ∎

Remark 4.8 Let $c_1^2(M)$ and $c_2(M)$ be the Chern numbers of M determined by an almost complex structure on M. Then, the inequalities $c_1^2(-M) > 2c_2(-M)$ and $c_1^2(-M) \leq 3c_2(-M)$ can be written in terms of $c_1^2(M)$, $c_2(M)$, respectively, as follows

$$c_1^2(M) < 2c_2(M), \quad c_1^2(M) \geq c_2(M).$$

5 – Indefinite Kähler and opposite Kähler manifolds

This section is an extended application of our analysis in sections 3 and 4, and includes some results and examples concerning indefinite and opposite Kähler structures (cf. [1]). Our first result is the following

Theorem 5.1. *Let (M, g, J) be an indefinite Kähler manifold with $[Q, J] = 0$. Then, the opposite almost complex structure is complex and almost Kähler if and only if M is opposite Kähler. In such a case M is locally a pseudo-Riemannian product manifold.*

Proof. If M is indefinite Kähler and opposite almost Kähler, by Theorem 3.4, the almost product structure is integrable, and so, using Theorem 4.2, the opposite almost complex structure is integrable. Now, a complex and almost Kähler manifold is Kähler, which proves that M is indefinite Kähler and opposite Kähler.

Suppose that M is indefinite Kähler and opposite complex. Then both almost complex structures are integrable, and so is the almost product structure. Then, from Corollary 3.5 it follows that M is opposite almost Kähler, which proves that both (M, g, J) and (M, g, J') are indefinite Kähler manifolds.

On the other hand, (M, g, J, J') is indefinite Kähler and opposite Kähler if and only if the metric connection ∇ makes parallel both complex structures: $\nabla J = 0$ and $\nabla J' = 0$. Hence, it makes parallel the almost product structure, $\nabla Q = 0$, which shows that the distributions $\tau, \tau^{\perp}$ are integrable and their leaves are totally geodesic submanifolds. Consequently, M is locally isomorphic to a direct product. $\square$

It should be noted that if M is locally a product of two 2-dimensional manifolds M_1 and M_2, both of which can be endowed with a structure of Kähler manifold, then M is a product.

Example 5.2. *Indefinite Kähler and opposite Kähler manifold.*

Complex tori $\mathbf{T}^n$ can be considered as a quotient space of an abelian group of n-tuples of complex numbers $\mathbb{C}^n$ by discrete subgroups, and therefore they admit n-linearly independent left invariant 1-forms $d\omega_i = o$ $(i = 1, \ldots, n)$.

Theorem 5.3. *(M, g, J) is a complex and opposite almost Kähler manifold if and only if the associated almost Hermitian manifold (M, h, J) is Kähler.*

Proof. The Kähler form F_J^h of the Hermitian manifold (M, h, J) which is associated with (M, g, J) coincides with $F_{J'}$ (Proposition 3.1). Consequently, (M, h, J) is a positive definite Kähler manifold. Converseley, if (M, h, J) is a Kähler manifold admitting a field of complex q-planes, then the associted almost pseudo-Hermitian manifold (M, g, J) is complex and opposite almost Kähler. ∎

There are many examples concerned with the above theorem. Some of them are as follows.

Example 5.4. *Complex and opposite almost Kähler manifolds, but not indefinite Kähler and opposite Kähler.*

In [6] the existence of submersions with complex totally geodesic fibers S^2 from the complex projective space $\mathbb{C}P^{2n+1}$ into the quaternionic projective space $\mathbb{Q}P^n$ is shown. If we denote by π such a submersion, it follows that $Ker\pi_*$ and $(Ker\pi_*)^\perp$ are orthogonal complex distributions in $\mathbb{C}P^{2n+1}$. Let Q be the almost product structure induced by that decomposition in such a way that its restriction to the tangent space to the fibers is minus the identity. Then it satisfies $JQ = QJ$, where J denotes the complex structure on $\mathbb{C}P^{2n+1}$. From section 2, we see the existence of an almost pseudo-Hermitian metric on $\mathbb{C}P^{2n+1}$ of signature $(4n, 2)$ with closed opposite Kähler form. However, such a structure can never be indefinite Kähler and opposite almost Kähler, because of the indecomposibility of $\mathbb{C}P^{2n+1}$. □

We end this paper with some remarks.

Remark 5.5

1. In Example 5.4, an indefinite metric of signature $(4n, 2)$ on $\mathbb{C}P^{2n+1}$ is treated. In general, the existence of an indefinite metric on a projective space is restrictive. For example, one of the authors [10] shows non existence of almost pseudo-Hermitian metrics of signature $(2p, 8m-2p-4), 2p \equiv 2 \bmod 4$ in the complex projective space $\mathbb{C}P^{2n+1}$.
2. Examples of Kähler and opposite complex manifolds but neither indefinite Kähler nor opposite Kähler are not known by the authors. In the spirit of Theorem 5.3, they must come from Kähler manifolds admitting fields of complex planes, such that the opposite almost complex structure is integrable, but not the almost product structure.
3. The manifold of Kodaira-Thurston, in Example 4.1, admits many indefinite Kähler metrics. However, the opposite almost complex structures are neither integrable nor almost Kähler with respect to any of those metrics. If any one of these cases occurs, such a manifold might admit positive definite Kähler metrics with respect to the complex structure J or the opposite one J', according to Theorem 5.3. But, the first Betti number of the manifold of Kodaira-Thurston is odd, and hence, it does not admit any positive definite Kähler metric.

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