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On the Navier-Stokes equation in $\mathbb{R}^2$

D. BENEDETTO

Riassunto: *Si costruiscono le soluzioni dell'equazione di Navier-Stokes in* $\mathbb{R}^2$ *per una vasta classe di dati iniziali, nella quale le velocità possono essere divergenti all'infinito. Il lavoro è un'estensione di un analogo risultato ottenuto per i flussi di Eulero [1].*

Abstract: *We construct the solution of Navier-Stokes equation in* $\mathbb{R}^2$ *for suitable initial velocity fields, possibly diverging at infinity, extending an analogous result for the Euler flow (see [1]).*

1 – Position of the Problem

The two dimensional Navier-Stokes equation, for an incompressible viscous fluid in $\mathbb{R}^2$ is:

$$(1.1)_a \qquad \partial_t\, \omega(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\,\omega(\mathbf{x},t) = \nu\triangle\omega(\mathbf{x},t)$$

$$(1.1)_b \qquad \mathbf{u}(\mathbf{x},t) = \left(\mathbf{K} * \omega(\,\cdot\,,t)\right)(\mathbf{x}) \equiv \int d\mathbf{y}\,\mathbf{K}(\mathbf{x}-\mathbf{y})\,\omega(\mathbf{y},t)$$

where:

$$\mathbf{u}(\,\cdot\,,t) : \mathbb{R}^2 \to \mathbb{R}^2$$

is the velocity field,

$$\omega(\,\cdot\,,t) : \mathbb{R}^2 \to \mathbb{R}$$

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is the third component of the vorticity field, i.e. $\omega = \nabla^\perp \cdot \mathbf{u}$, $\nabla^\perp = \binom{-\partial_y}{\partial_x}$, t is the time, ν the viscosity coefficient, and

$$\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi}\frac{\mathbf{x}^\perp}{|\mathbf{x}|^2} \tag{1.2}$$

where $\mathbf{x}^\perp = (x_2, -x_1)$ if $\mathbf{x} = (x_1, x_2)$.

Equation $(1.1)_a$ means that ω is convected by the velocity field $\mathbf{u}$ and diffused. We can rewrite eq.s (1.1), in terms of the stocastic flow associated to $\mathbf{u}$:

$$(1.3)_a \qquad \boldsymbol{\phi}_{t,\tau}(\mathbf{x}) = \mathbf{x} + \int_\tau^t \mathbf{u}(\boldsymbol{\phi}_{s,\tau}(\mathbf{x}), s)\, ds \;\; + (2\nu)^{\frac{1}{2}}(\mathbf{b}_t - \mathbf{b}_\tau)$$

$$(1.3)_b \qquad \mathbf{u}(\mathbf{x},t) = \int d\mathbf{y}\, \mathbf{K}(\mathbf{x}-\mathbf{y})\, \omega(\mathbf{y},t)$$

$$(1.3)_c \qquad \int f(\mathbf{x})\, \omega(\mathbf{x},t) = \mathbb{E}\int \mathrm{f}(\boldsymbol{\phi}_{\mathrm{t},\tau}(\mathbf{x}))\, \omega(\mathbf{x},\tau)$$

where $\mathbf{b}_t$ is a standard brownian motion, $\mathbb{E}$ is the expectation, and $(1.3)_c$ holds for all $f \in \mathbf{C}_0^\infty(\mathbb{R}^2)$. If the velocity field $\mathbf{u}$ is known, the eq. $(1.3)_a$ defines a stocastic process, for which the eq (1.1) is exactely the forward Kolmogorov equation. We obtain the solution at time t of this equation, in a dual form, in terms of the solution at time $\tau < t$ and of the trajectories of the process; this is the meaning of the equation $(1.3)_c$. To obtain the Navier-Stokes equation, we must impose $(1.3)_b$, which is the self-consistence condition of the velocity field with the process defined in $(1.3)_a$, i.e. the velocity field is the expected value of the velocity field generated by the vorticity moved by the stocastic flow $(1.3)_a$ (see also eq. $(1.19)_b$).

We will consider eq.s (1.3) for $\boldsymbol{\phi}_t = \boldsymbol{\phi}_{t,0}$: in this case $\mathbf{b}_0 = 0$, $\omega(\mathbf{x}, 0) = \omega_0$ is the initial datum of eq. $(1.1)_a$:

$$(1.4)_a \qquad \boldsymbol{\phi}_t(\mathbf{x}) = \mathbf{x} + \int_0^t \mathbf{u}(\boldsymbol{\phi}_s(\mathbf{x}), s)\, ds \;\; + (2\nu)^{\frac{1}{2}}\mathbf{b}_t$$

$$(1.4)_b \qquad \mathbf{u}(\mathbf{x},t) = \int d\mathbf{y}\, \mathbf{K}(\mathbf{x}-\mathbf{y})\, \omega(\mathbf{y},t)$$

$$(1.4)_c \qquad \int f(\mathbf{x})\, \omega(\mathbf{x},t) = \mathbb{E}\int \mathrm{f}(\boldsymbol{\phi}_{\mathrm{t}}(\mathbf{x}))\, \omega_0(\mathbf{x})$$

Given $\mathbf{u}(\,\cdot\,,t)$ uniformly Lipschitz and linearly bounded, the eq. $(1.4)_a$ admits a unique stocastic process as solution, which is almost surely continuous (see [2]). In our case the velocity field will satisfies:

$$|\mathbf{u}(\mathbf{x},t)| \le c\,(1+|\mathbf{x}|) \tag{1.5}$$

$$|\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)| \le c\,\varphi(|\mathbf{x}-\mathbf{y}|) \tag{1.6}$$

where c is a positive constant, and

$$\varphi(r) = \begin{cases} r(1-\log r) & \text{if} \quad r < \dfrac{1}{e} \\ r + \dfrac{1}{e} & \text{if} \quad r \ge \dfrac{1}{e} \end{cases} \tag{1.7}$$

The ordinary differential equation associated to such $\mathbf{u}$ has unique solution (see e.g. [4], [5]). The stocastic differential eq. $(1.4)_a$, under hypotesis (1.5) and (1.6), admits unique solution in the strong sense, since φ is continuous and φ^{-1} is not summable in the right neighborhoods of 0 (see [7]).

To clarify how to construct Φ, $\mathbf{u}$, ω which satisfy (1.4), we need some remarks and propositions.

Remark 1. For all continuous realizations of the brownian motion, eq. $(1.3)_a$ defines a continuous two parameters flow on $\mathbb{R}^2$ (i.e. $\boldsymbol{\phi}_{t,\tau}$ is bijective and $\boldsymbol{\phi}_{t,\tau}(\boldsymbol{\phi}_{\tau,s}(\mathbf{x})) = \boldsymbol{\phi}_{t,s}(\mathbf{x})$) which preserves the Lebesgue measure. Namely $\boldsymbol{\phi}_{t,\tau}(\mathbf{x}) = \boldsymbol{\psi}_{t,\tau}(\mathbf{x}) + (2\nu)^{\frac{1}{2}}(\mathbf{b}_t - \mathbf{b}_\tau)$ where $\boldsymbol{\psi}_{t,\tau}(\mathbf{x})$ solves

$$\boldsymbol{\psi}_{t,\tau}(\mathbf{x}) = \mathbf{x} + \int_\tau^t \overline{\mathbf{u}}(\boldsymbol{\psi}_{s,\tau}(\mathbf{x}), s) \tag{1.8}$$

and $\overline{\mathbf{u}}(\mathbf{x},s) = \mathbf{u}(\mathbf{x} + (2\nu)^{\frac{1}{2}}(\mathbf{b}_s - \mathbf{b}_\tau), s)$ is a divergenceless field (since $\nabla \cdot \mathbf{u}(\mathbf{x},s) = 0$ as follow from $(1.3)_b$ and (1.2)).
If $\omega_0 \in \mathbf{L}_p$, $1 < p \le \infty$, eq. $(1.4)_c$ and the above remark imply, via Hölder inequality, that $\omega(\,\cdot\,,t) \in \mathbf{L}_p$, and

$$\|\omega_t\|_p \le \|\omega_0\|_p \tag{1.9}$$

We can recover the same inequality for $p = 1$, using the probability transition associated to the stocastic process $\boldsymbol{\phi}_t$.

The initial value problem (1.4) is uniquely solvable if $\omega_0 \in \mathbf{L}_p \cap \mathbf{L}_\infty$ with $p < 2$ (see [4], [6] for $p = 1$; the extention for $1 < p < 2$ is a trivial remark as in the Euelr case, see [1]), essentially because the velocity field generated by a vorticity in $\mathbf{L}_p \cap \mathbf{L}_\infty$ is bounded by $c(\|\omega\|_p + \|\omega\|_\infty)$ (and decays at infinity), and the following inequality holds (see [3], [4], [5] if $p = 1$, and [1] for the general case):

Proposition 1. *Suppose $\omega \in \mathbf{L}_p \cap \mathbf{L}_\infty$, $1 \le p < +\infty$. Then*

$$\int |\mathbf{K}(\mathbf{x}-\mathbf{z}) - \mathbf{K}(\mathbf{y}-\mathbf{z})|\ |\omega(\mathbf{z})|\, d\mathbf{z} \le c(\|\omega\|_p + \|\omega\|_\infty)\, \varphi(|\mathbf{x}-\mathbf{y}|) \tag{1.10}$$

where φ is definded by (1.7).

If $\omega_0 \in \mathbf{L}_p \cap \mathbf{L}_\infty$ with $p \ge 2$, the velocity field does not even make sense in general, but there are distributions of vorticity for which $\mathbf{u}_0 = \mathbf{K} * \omega_0$ exists as Cauchy Principal Value. More precisely, if the velocity fied exists in $\mathbf{x}$ as a limit of integrals on increasing balls of centre $\mathbf{x}$, i.e.

$$\mathbf{u}_0(\mathbf{x}) = \lim_{M\to\infty} \int_{|\mathbf{x}-\mathbf{z}|<M} \mathbf{K}(\mathbf{x}-\mathbf{z})\, \omega_0(\mathbf{z})\, d\mathbf{z} \tag{1.11}$$

then this value is independent of the centre of the balls:

$$\mathbf{u}_0(\mathbf{x}) = \lim_{M\to\infty} \int_{|\mathbf{y}-\mathbf{z}|<M} \mathbf{K}(\mathbf{x}-\mathbf{z})\, \omega_0(\mathbf{z})\, d\mathbf{z} \tag{1.12}$$

and, using Proposition 1, the velocity exists everywere in the same sense:

$$\begin{aligned} \mathbf{u}_0(\mathbf{w}) =& \mathbf{u}_0(\mathbf{x}) + \int (\mathbf{K}(\mathbf{w}-\mathbf{z}) - \mathbf{K}(\mathbf{x}-\mathbf{z}))\omega_0(\mathbf{z})\, d\mathbf{z} \\ =& \lim_{M\to\infty} \int_{|\mathbf{w}-\mathbf{z}|<M} \mathbf{K}(\mathbf{w}-\mathbf{z})\, \omega_0(\mathbf{z})\, d\mathbf{z} \end{aligned} \tag{1.13}$$

In this case the $\mathbf{L}_p$ norms of ω_0 does not give us enough control on $\mathbf{u}_0$. However a natural question arises. Do the stocastic flow $\boldsymbol{\phi}_t$ evolve the vorticity field, according $(1.4)_c$, in such a way $\mathbf{u}(\mathbf{x},t)$ exists too? As in the case of Euler flow (see [1]), the answer to this problem is positive if the velocity field, and then the flow, diverges suitably at infinity, and the vorticity is sufficientely summable.

The key of the proof is the following modification of Proposition 1, which permits to define the velocity field at any time, given the flow:

PROPOSITION 2. *Suppose that* $\omega_0 \in \mathbf{L}_p \cap \mathbf{L}_\infty$, $1 \le p < +\infty$, *and let* $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a continuous flow, measure preserving, such that*

$$r = \sup_{\mathbf{z}} \frac{|\boldsymbol{\phi}(\mathbf{z}) - \mathbf{z}|}{1 + |\mathbf{z}|^\alpha} < +\infty \tag{1.14}$$

with $\alpha < 1$ *and* $p < \dfrac{2}{\alpha}$.

Then

$$\begin{aligned} &\int |\mathbf{K}(\mathbf{x} - \boldsymbol{\phi}(\mathbf{z})) - \mathbf{K}(\mathbf{x} - \mathbf{z})|\, |\omega_0(\mathbf{z})|\, d\mathbf{z} - \le \\ &\le c(1 + |\mathbf{x}|^\alpha)(\|\omega_0\|_p + |\omega_0\|_\infty)\, \varphi(r) \end{aligned} \tag{1.15}$$

(the proof is in Appendix; see also [1]).

So, if $\mathbf{u}_0 = \mathbf{K} * \omega_0$, in the sense of eq. (1.11), we can construct the velocity field genereted by $\omega_0(\,\boldsymbol{\phi}^{-1}\,)$ as

$$\mathbf{u}_0(\mathbf{x}) + \int (\mathbf{K}(\mathbf{x} - \boldsymbol{\phi}(\mathbf{z})) - \mathbf{K}(\mathbf{x} - \mathbf{z}))\, \omega_0(\mathbf{z})\, d\mathbf{z}$$

Now we can formulate the main result of this paper.

THEOREM 1. *Let* $\omega_0 \in \mathbf{L}_p \cap \mathbf{L}_\infty$; *suppose that*

$$\mathbf{u}_0 = \mathbf{K} * \omega_0 \tag{1.16}$$

exists in the sense of Cauchy Principal Value as in (1.11) *and satisfies*

$$\sup_{\mathbf{z}} \frac{|\mathbf{u}_0(\mathbf{z})|}{1 + |\mathbf{z}|^\alpha} < +\infty \tag{1.17}$$

with $\alpha < 1$ *and* $p < \dfrac{2}{\alpha}$.

Then there exist unique a velocity field $\mathbf{u}(\,\cdot\,,t)$ *and a stocastic flow* $\boldsymbol{\phi}_t : \mathbb{R}^2 \to \mathbb{R}^2$, *with*

$$(1.18)_a \qquad \sup_{\mathbf{z}} \frac{|\mathbf{u}(\mathbf{z},t)|}{1+|\mathbf{z}|^\alpha} < +\infty$$

$$(1.18)_b \qquad \mathbb{E}\left(\sup_{\mathbf{z}} \frac{|\boldsymbol{\phi}_t(\mathbf{z})-\mathbf{z}|}{1+|\mathbf{z}|^\alpha}\right) < +\infty$$

satisfying

$$(1.19)_a \qquad \boldsymbol{\phi}_t(\mathbf{x}) = \mathbf{x} + \int_0^t \mathbf{u}(\boldsymbol{\phi}_s(\mathbf{x}),s)\,ds \;\; + (2\nu)^{\frac{1}{2}}\mathbf{b}_t$$

$$(1.19)_b \qquad \mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}) + \mathbb{E}\int \mathrm{d}\mathbf{z}\;(\mathbf{K}(\mathbf{x}-\boldsymbol{\phi}_t(\mathbf{z})) - \mathbf{K}(\mathbf{x}-\mathbf{z}))\;\omega_0(\mathbf{z})$$

The flow $\boldsymbol{\phi}_t$ *is unique in the sense that if* $\boldsymbol{\psi}_t$ *also satisfies* $(1.19)_a$, *then* $\mathbf{P}(\sup_t \sup_{\mathbf{x}} |\boldsymbol{\phi}_t(\mathbf{x}) - \boldsymbol{\psi}_t(\mathbf{x})| > 0) = 0$.

REMARK 2. Let $\omega_t \in \mathbf{L}_p \cap \mathbf{L}_\infty$ be defined by

$$(1.20) \qquad \int f(\mathbf{x})\,\omega_t(\mathbf{x}) = \mathbb{E}\int \mathrm{f}(\boldsymbol{\phi}_t(\mathbf{x}))\omega_0(\mathbf{x})$$

where $f \in \mathbf{C}_0^\infty(\mathbb{R}^2)$. From $(1.19)_b$ and (1.20):

$$(1.21) \qquad \mathbf{u}(\mathbf{x},t) = \mathbb{E}\lim_{\mathrm{M}\to\infty} \int_{|\mathbf{x}-\mathbf{z}|<M} \mathbf{K}(\mathbf{x}-\boldsymbol{\phi}_t(\mathbf{z}))\,\omega_0(\mathbf{z})\,\mathrm{d}\mathbf{z}$$

and, for all test funcions f:

$$(1.22) \qquad \int \nabla^\perp f \cdot \mathbf{u}_t = -\int f\,\omega_t$$

(i.e. ω_t is the curl of $\mathbf{u}_t$), where $\mathbf{u}_t(\,\cdot\,) = \mathbf{u}(\,\cdot\,,t)$.
From $(1.19)_a$ and (1.20) follows that ω_t is a solution of Navier-Stokes equation in weak form:

$$(1.23) \qquad \frac{d}{dt}\int f\,\omega_t = \int \mathbf{u}_t \cdot \nabla f\,\omega_t + \nu\int \triangle f\,\omega_t$$

Moreover, from (1.22) and (1.23) and using the fact that $\mathbf{u}(\mathbf{x}, t)$ is locally bounded, follows that it is a weak solution of the Navier-Stokes equation in the velocity formalism:
for all $\mathbf{w}(\mathbf{x}) \in \mathbf{C}_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, with $\nabla \cdot \mathbf{w} = 0$

$$\frac{d}{dt} \int w_i u_i = \int \partial_j w_i \, u_i u_j + \nu \int \triangle w_i \, u_i \tag{1.24}$$

where u_i and w_i, $i = 1, 2$ are the components of vectors $\mathbf{u}$ and $\mathbf{w}$, and in (1.24) the index are summed.
It is well known that the diffusion makes regular $\mathbf{u}_t$ and ω_t, for $t > 0$, so they are classical solutions of Navier-Stokes equation in the formalism of velocity and vorticity respectively.

Remark 3. We observe that we can recover the velocity field by the vorticity via (1.21) modulo a potential field, which can only be a constant since $\mathbf{u}$ increase at infinity less than linearly, and this constant is zero at time zero from the hypotesis. We construct the solution imposing (1.21) via $(1.19)_b$. If we add some known vectorial function $\mathbf{c}(t)$, with $\mathbf{c}(0) = \mathbf{0}$, in the right hand side of $(1.19)_b$, we obtain an other solution of Navier-Stokes equation with the same initial datum $\mathbf{u}_0$. This solution, at time t, is exactely the solution of problem (1.19) translated by the vector $\int_0^t \mathbf{c}(t)\, dt$. This seeming trivial loss of uniqueness is due to the fact that we do not impose the boundary conditions, while we obtain an unique solution specifying how we do the Cauchy Principal Value of the integral $(1.4)_b$; namely the unique solution of problem (1.19) is the unique solution of Navier-Stokes equation which satisfies (1.21).

Remark 4. In the case $\nu = 0$ (see [1]), we prove the estimate $(1.18)_a$ for the growth of the velocity field, given a bound of the type $(1.18)_b$ on the growth of the displacement $\boldsymbol{\phi}_t(\mathbf{z}) - \mathbf{z}$, essentially using the fact that the flow preserves the measure (see Proposition 2). For our case, the flow $\boldsymbol{\phi}_t$ preseves the measure almost surely (see Remark 1). Then, we can reproduce the same argument of the inviscous case if we control the expectation of the moments of the displacement.

2 – Costruction of the solution

We construct the solution of eq. (1.19) by iterations.
Let

$$(2.1) \qquad \mathbf{u}^0(\mathbf{x},t) = \mathbf{u}_0$$

$$(2.2) \qquad \phi_t^n(\mathbf{x}) = \mathbf{x} + \int_0^t \mathbf{u}^n(\phi_s^n(\mathbf{x}), s)\, ds \quad + (2\nu)^{\frac{1}{2}} \mathbf{b}_t$$

$$(2.3) \qquad \mathbf{u}^{n+1}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}) + \mathbb{E} \int d\mathbf{z}\ (\mathbf{K}(\mathbf{x} - \phi_t^n(\mathbf{z})) - \mathbf{K}(\mathbf{x} - \mathbf{z}))\ \omega_0(\mathbf{z})$$

Let

$$(2.4) \qquad r_t^n = \sup \frac{|\phi_t^n(\mathbf{x}) - \mathbf{x}|}{1 + |\mathbf{x}|^\alpha}, \quad R_t^n = \mathbb{E} r_t^n$$

If r_t^{n-1} is finite, by Proposition 2 it follows that the right hand side of (2.3) is bounded by

$$(2.5) \qquad |\mathbf{u}_0(\mathbf{x})| + c\,(1 + |\mathbf{x}|^\alpha) \mathbb{E} \varphi(r_t^{n-1})$$

where, from now on, $c = \text{constant} \cdot (\|\omega_0\|_p + \|\omega_0\|_\infty)$. If R_t^{n-1} is finite, by Jensen inequality (φ is convex) it follows that $\mathbf{u}^n(\,\cdot\,,t)$ exists and

$$(2.6) \qquad U_t^n = \sup \frac{|\mathbf{u}^n(\mathbf{x},t)|}{1 + |\mathbf{x}|^\alpha} \le U_0 + c\varphi(R_t^{n-1})$$

where $U_0 = \sup \dfrac{|\mathbf{u}_0(\mathbf{x})|}{1 + |\mathbf{x}|^\alpha}$. From (2.2), using Proposition 1:

$$(2.7) \quad |\phi_t^n(\mathbf{x}) - \mathbf{x}| \le (2\nu)^{1/2} |\mathbf{b}_t| + \int_0^t ds\, U_s^n (1 + |\mathbf{x}|^\alpha) + c \int_0^t ds\, \varphi(|\phi_s^n(\mathbf{x}) - \mathbf{x}|)$$

Notice that if $b \ge 1$ then $\dfrac{1}{b}\varphi(r) \le \varphi(\dfrac{r}{b})$ we have:

$$(2.8)_a \qquad r_t^n \le (2\nu)^{1/2} |\mathbf{b}_t| + \int_0^t ds\, U_s^n + c \int_0^t ds\, \varphi(r_s^n)$$

$$(2.8)_b \qquad R_t^n \le (2\nu t)^{1/2} + \int_0^t ds\, U_s^n + \int_0^t ds\, \varphi(R_s^n)$$

Hence, the definitions (2.2) and (2.3) are well posed.
Collecting (2.6) and $(2.8)_b$, notice that $\varphi(r) \leq \frac{1}{e} + r$, and fixing $T > 0$, for $t < T$:

$$R_t^n \leq \left(C(T) + c\int_0^t ds\, R_s^{n-1}\right) + c\int_0^t ds\, R_s^n \leq C(T) + C(T)\int_0^t ds\, R_s^{n-1} \tag{2.9}$$

where we have used the Gronwall Lemma, and C(T), from now on, is any constant depending only on T, $\|\omega_0\|_p$, $\|\omega_0\|_\infty$, U_0. Iterating (2.9), and using (2.8), we obtain for $t < T$ and $a > 0$:

$$(2.10)_a \qquad R_t^n \leq C(T)$$

$$(2.10)_b \qquad U_t^n \leq C(T)$$

$$(2.10)_c \qquad \mathbb{E}\left((\mathrm{r_t^n})^{\mathrm{a}}\right) \leq \mathrm{C(T)}$$

uniformly on n and on $t < T$.
Given the field $\mathbf{u}^n(\,\cdot\,, t)$, we can construct the two paremeters stocastic flow

$$\boldsymbol{\phi}_{t,\tau}^n(\mathbf{x}) = \mathbf{x} + \int_\tau^t \mathbf{u}^n(\boldsymbol{\phi}_{s,\tau}^n(\mathbf{x}), s)\, ds \quad + (2\nu)^{1/2}(\mathbf{b}_t - \mathbf{b}_\tau) \tag{2.11}$$

with $\boldsymbol{\phi}_{t,0}^n = \boldsymbol{\phi}_t^n$. Let us define:

$$r_{t,\tau}^n = \sup \frac{|\boldsymbol{\phi}_{t,\tau}^n(\mathbf{x}) - \mathbf{x}|}{1 + |\mathbf{x}|^\alpha}, \qquad R_{t,\tau}^n = \mathbb{E}\mathrm{r}_{\mathrm{t},\tau}^{\mathrm{n}} \tag{2.12}$$

then, as in (2.10), we obtain:

$$R_{t,\tau}^n \leq C(T), \qquad \mathbb{E}(\mathrm{r}_{\mathrm{t},\tau}^{\mathrm{n}})^{\mathrm{a}} \leq \mathrm{C(T)} \tag{2.13}$$

We proof the convergence of $\boldsymbol{\phi}_t^n$. Let us define

$$\begin{aligned} \delta_t^{n+1} &= \sup \frac{|\boldsymbol{\phi}_t^{n+1}(\mathbf{x}) - \boldsymbol{\phi}_t^n(\mathbf{x})|}{1 + |\boldsymbol{\phi}_t^n(\mathbf{x})|^\alpha}, \\ \delta_t^0 &= \sup \frac{|\boldsymbol{\phi}_t^0(\mathbf{x}) - \mathbf{x}|}{1 + |\mathbf{x}|^\alpha}, \qquad \Delta_t^n = \mathbb{E}\delta_\mathrm{t}^\mathrm{n} \end{aligned} \tag{2.14}$$

We have:

$$
(2.15)\qquad
\begin{aligned}
|\boldsymbol{\phi}_t^{n+1}(\mathbf{x}) - \boldsymbol{\phi}_t^{n}(\mathbf{x})| \le & \int_0^t ds\, |\mathbf{u}^{n+1}(\boldsymbol{\phi}_s^{n+1}(\mathbf{x}), s) - \mathbf{u}^{n+1}(\boldsymbol{\phi}_s^{n}(\mathbf{x}), s)| \\
& + \int_0^t ds\, |\mathbf{u}^{n+1}(\boldsymbol{\phi}_s^{n}(\mathbf{x}), s) - \mathbf{u}^{n}(\boldsymbol{\phi}_s^{n}(\mathbf{x}), s)|
\end{aligned}
$$

Using Proposition 1, the first integral is bounded by

$$
(2.16)\qquad c\int_0^t ds\, \varphi(|\boldsymbol{\phi}_s^{n+1}(\mathbf{x}) - \boldsymbol{\phi}_s^{n}(\mathbf{x})|)
$$

For the second, we need to estimate

$$
(2.17)\qquad
\begin{aligned}
&|\mathbf{u}^{n+1}(\mathbf{y}, s) - \mathbf{u}^{n}(\mathbf{y}, s)| \le \\
&\qquad \le \mathbb{E}\int \left|\mathbf{K}(\mathbf{y} - \boldsymbol{\phi}_{\mathrm{s}}^{\mathrm{n}}(\mathbf{z})) - \mathbf{K}(\mathbf{y} - \boldsymbol{\phi}_{\mathrm{s}}^{\mathrm{n}-1}(\mathbf{z}))\right| \, |\omega_0(\mathbf{z})|\, \mathrm{d}\mathbf{z}
\end{aligned}
$$

The flow $\boldsymbol{\phi}_s^{n-1}$ preserve the Lebesgue measure, hence

$$
(2.18)\qquad
\begin{aligned}
&|\mathbf{u}^{n+1}(\mathbf{y}, s) - \mathbf{u}^{n}(\mathbf{y}, s)| \le \\
&\le \int \left|\mathbf{K}(\mathbf{y} - \boldsymbol{\phi}_s^{n}(\, (\boldsymbol{\phi}_s^{n-1})^{-1}(\mathbf{z}))) - \mathbf{K}(\mathbf{y} - \mathbf{z})\right| \, |\omega_0((\boldsymbol{\phi}_s^{n-1})^{-1}(\mathbf{z}))|\, d\mathbf{z} \\
&\le c(1 + |\mathbf{y}|^{\alpha})\mathbb{E}\varphi\left(\sup \frac{|\boldsymbol{\phi}_{\mathrm{s}}^{\mathrm{n}}(\, (\boldsymbol{\phi}_{\mathrm{s}}^{\mathrm{n}-1})^{-1}(\mathbf{z})) - \mathbf{z}|}{1 + |\mathbf{z}|^{\alpha}}\right) \\
&= c(1 + |\mathbf{y}|^{\alpha})\mathbb{E}\varphi\left(\sup \frac{|\boldsymbol{\phi}_{\mathrm{s}}^{\mathrm{n}}(\mathbf{z}) - \boldsymbol{\phi}_{\mathrm{s}}^{\mathrm{n}-1}(\mathbf{z})|}{1 + |\boldsymbol{\phi}_{\mathrm{s}}^{\mathrm{n}-1}(\mathbf{z})|^{\alpha}}\right) \le \mathrm{c}(1 + |\mathbf{y}|^{\alpha})\varphi(\Delta_{\mathrm{s}}^{\mathrm{n}})
\end{aligned}
$$

where the second inequality is consequence of Proposition 2. Collecting (2.16) and (2.18) we have that:

$$
(2.19)\qquad
\begin{aligned}
|\boldsymbol{\phi}_t^{n+1}(\mathbf{x}) - \boldsymbol{\phi}_t^{n}(\mathbf{x})| \le c & \int_0^t ds\, \varphi(|\boldsymbol{\phi}_s^{n+1}(\mathbf{x}) - \boldsymbol{\phi}_s^{n}(\mathbf{x})|) + \\
& + c\int_0^t ds\, (1 + |\boldsymbol{\phi}_s^{n}(\mathbf{x})|^{\alpha})\varphi(\Delta_s^{n})
\end{aligned}
$$

Notice that if $b \geq 1$ then $\varphi(r) = \varphi(b\,\frac{r}{b}) \leq b\varphi(\frac{r}{b})$, we have that:

$$\delta_t^{n+1} \leq c \int_0^t ds \, \sup \left(\frac{1 + |\boldsymbol{\phi}_s^n(\mathbf{x})|^\alpha}{1 + |\boldsymbol{\phi}_t^n(\mathbf{x})|^\alpha} \right) (\varphi(\delta_s^{n+1}) + \varphi(\Delta_s^n)) \tag{2.20}$$

Using $\boldsymbol{\phi}_t = \boldsymbol{\phi}_{t,s}(\boldsymbol{\phi}_s)$

$$\sup \left(\frac{1 + |\boldsymbol{\phi}_s^n(\mathbf{x})|^\alpha}{1 + |\boldsymbol{\phi}_t^n(\mathbf{x})|^\alpha} \right) = \sup \left(\frac{1 + |\mathbf{x}|^\alpha}{1 + |\boldsymbol{\phi}_{t,s}^n(\mathbf{x})|^\alpha} \right) \tag{2.21}$$

Splitting the regions $|\mathbf{x}| \leq (4r_{t,s}^n)^{\frac{1}{1-\alpha}}$ and $|\mathbf{x}| > (4r_{t,s}^n)^{\frac{1}{1-\alpha}}$, where $r_{t,s}^n$ is given by (2.12), we obtain:

$$\sup \left(\frac{1 + |\mathbf{x}|^\alpha}{1 + |\boldsymbol{\phi}_{t,s}^n(\mathbf{x})|^\alpha} \right) \leq c((r_{t,s}^n)^{\frac{1}{1-\alpha}} + c) \tag{2.22}$$

Observing that $r_{t,s}^n$ is independent of the $\sigma-$algebra genereted by $\boldsymbol{\phi}_s$, and putting (2.22) in (2.20), using the Jensen inequality and (2.13), finally we have:

$$\Delta_t^{n+1} \leq C(T) \int_0^t ds \, (\varphi(\Delta_s^{n+1}) + \varphi(\Delta_s^n)) \tag{2.23}$$

Let

$$\rho_t^n = \sup_{m \geq n} \Delta_t^n \tag{2.24}$$

then

$$\rho_t^n \leq C(T) \int_0^t ds \, \varphi(\rho_s^{n-1}) \tag{2.25}$$

Notice that if $\varepsilon \leq \dfrac{1}{e}$

$$\varphi(r) \leq \varepsilon + |\log(\varepsilon)|\, r \tag{2.26}$$

iterating (2.25), we obtain

$$\rho_t^n \leq \frac{\varepsilon}{|\log(\varepsilon)|} e^{C(T)|\log(\varepsilon)|t} + \frac{(|\log(\varepsilon)|C(T)t)^n}{n!} \sup_{[0,T]} \rho_s^0 \tag{2.27}$$

Choosing

$$\varepsilon = e^{-n} \tag{2.28}$$

and using the Stirling inequality we conclude that

$$\rho_t^n \leq C(T) e^{-c(1-C(T)t)n} \tag{2.29}$$

We need also to control

$$\sigma_t^n = \mathbb{E}((\delta_{\mathrm{t}}^{\mathrm{n}})^2) \tag{2.30}$$

From (2.20)

$$(\delta_t^{n+1})^2 \leq ct \int_0^t ds\, A_{t,s}^n ((\varphi(\delta_s^{n+1}))^2 + (\varphi(\Delta_s^n))^2) \tag{2.31}$$

where, following (2.21) and (2.22), $A_{t,s}^n = c((r_{t,s}^n)^{\frac{1}{1-\alpha}} + c)^2$. Notice that $(\varphi(r))^2 \leq \varphi(r) + 2\varphi(r^2)$, and using definition (2.24), we have:

$$(\delta_t^{n+1})^2 \leq ct \int_0^t ds\, A_{t,s}^n (\varphi((\delta_s^{n+1})^2) + \varphi(\delta_s^{n+1}) + \varphi(\rho_s^n) + (\varphi(\rho_s^n)^2)) \tag{2.32}$$

and then

$$\sigma_t^{n+1} \leq C(T) \int_0^t ds\, (\varphi(\sigma_s^{n+1}) + \varphi(\rho_s^n) + \varphi((\rho_s^n)^2)) \tag{2.33}$$

Using (2.26), (2.28) and the Gronwell lemma:

$$\sigma_t^{n+1} \leq C(T) e^{ntC(T)} \int_0^t ds\, (e^{-n} + n\rho_s^n + n(\rho_s^n)^2) \tag{2.34}$$

Using (2.29)

$$\sigma_t^n \le C(T) e^{-c(1-C(T)t)n} \tag{2.35}$$

Now we can proof the convergence of $\boldsymbol{\phi}_t^n$.

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}\left(\sup_{\mathbf{x}} \frac{|\boldsymbol{\phi}_{\mathrm{t}}^{\mathrm{n}+1}(\mathbf{x}) - \boldsymbol{\phi}_{\mathrm{t}}^{\mathrm{n}}(\mathbf{x})|}{1+|\mathbf{x}|^\alpha}\right) &\le \sum_{n=0}^{\infty} \mathbb{E}\left(\delta_{\mathrm{t}}^{\mathrm{n}+1} \sup_{\mathbf{x}} \frac{1+|\boldsymbol{\phi}_{\mathrm{t}}^{\mathrm{n}}(\mathbf{x})|^\alpha}{1+|\mathbf{x}|^\alpha}\right) \\ &\le c \sum_{n=0}^{\infty} \mathbb{E}(\delta_{\mathrm{t}}^{\mathrm{n}+1}\,(\mathrm{r}_{\mathrm{t}}^{\mathrm{n}} + \mathrm{c})) \\ &\le C(T) \sum_{n=0}^{\infty} (\sigma_t^{n+1})^{1/2}\end{aligned} \tag{2.36}$$

Using (2.35) we proof that the series converges if $t \le T_0 < 1/C(T)$; T_0 is depending only on T, ω_0 and U_0.
From (2.36) follows that the process

$$\phi_t(\mathbf{x}) = \sum_{0}^{\infty} \left(\phi_t^{n+1}(\mathbf{x}) - \phi_t^n(\mathbf{x})\right) + \phi_t^0(\mathbf{x}) \tag{2.37}$$

exists and satisfies

$$\mathbb{E}\left(\sup_{\mathbf{x}} \frac{|\phi_{\mathrm{t}}(\mathbf{x}) - \mathbf{x}|}{1+|\mathbf{x}|^\alpha}\right) \le \mathrm{C}(\mathrm{T}_0) \tag{2.38}$$

We can define the velocity field $\mathbf{u}(\,\cdot\,,t)$ via equation $(1.19)_b$. Proceeding as in (2.17), (2.18) we obtain:

$$\sup_{\mathbf{x}} \frac{|\mathbf{u}(\mathbf{x},t) - \mathbf{u}^{n+1}(\mathbf{x},t)|}{1+|\mathbf{x}|^\alpha} \le \varphi\left(\mathbb{E}\left(\sup_{\mathbf{x}} \frac{|\boldsymbol{\phi}_{\mathrm{t}}(\mathbf{x}) - \boldsymbol{\phi}_{\mathrm{t}}^{\mathrm{n}}(\mathbf{x})|}{1+|\boldsymbol{\phi}_{\mathrm{t}}^{\mathrm{n}}(\mathbf{x})|^\alpha}\right)\right) \tag{2.39}$$

then $\mathbf{u}^n \to \mathbf{u}$, uniformly on the compact sets of $\mathbb{R}^2$. It is not hard to prove that $\boldsymbol{\phi}_t$ solves eq. $(1.19)_a$.

To prove the existence of the solution for all time, we construct the flow $\boldsymbol{\phi}_{t,s}$ for $0 \le s \le t \le T_0$ as solution of eq. $(1.3)_a$. We have

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},s) + \mathbb{E}\int d\mathbf{z}\ \left(\mathbf{K}(\mathbf{x} - \boldsymbol{\phi}_{\mathrm{t,s}}(\mathbf{z})) - \mathbf{K}(\mathbf{x}-\mathbf{z})\right)\,\omega(\mathbf{z},\mathrm{s}) \tag{2.40}$$

As before, we extend the solution for $t < s + T_0$, and then for $t < T$, and finally, for all t, since T is any real value.

We proof the uniqueness. Let $(\mathbf{u}_t, \boldsymbol{\phi}_t)$ and $(\mathbf{v}_t, \boldsymbol{\psi}_t)$ are two solutions, and let us define

$$\Delta_t = \mathbb{E}\left(\sup_{\mathbf{x}} \frac{|\boldsymbol{\phi}_{\mathrm{t}}(\mathbf{x}) - \boldsymbol{\psi}_{\mathrm{t}}(\mathbf{x})|}{1 + |\boldsymbol{\psi}_{\mathrm{t}}(\mathbf{x})|^\alpha}\right) \tag{2.41}$$

then, proceeding as in (2.15) – (2.23)

$$\sup_{\mathbf{x}} \frac{|\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)|}{1 + |\mathbf{x}|^\alpha} \le c\varphi(\Delta_t) \ \text{ and } \ \Delta_t \le C(T) \int_0^t ds\, \varphi(\Delta_s) \tag{2.42}$$

Hence

$$\Delta_t = 0 \tag{2.43}$$

and then $\mathbf{u} = \mathbf{v}$, and

$$\mathbf{P}(\sup_{\mathbf{x}} |\boldsymbol{\phi}_t(\mathbf{x}) - \boldsymbol{\psi}_t(\mathbf{x})| > 0) = 0 \tag{2.44}$$

From the fact that $\boldsymbol{\phi}_t$ and $\boldsymbol{\psi}_t$ are continuous almost surely, follows that:

$$\mathbf{P}(\sup_t \sup_{\mathbf{x}} |\boldsymbol{\phi}_t(\mathbf{x}) - \boldsymbol{\psi}_t(\mathbf{x})| > 0) = 0 \tag{2.45}$$

3 – Appendix

We proof Proposition 2. We estimate the integral in the right hand side of eq. (1.15) by splitting the integration domain into two parts:

$$A = \{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| < 2r(1 + |\mathbf{y}|^\alpha)\} \tag{A.1}$$

and its complement A^c.

$$\begin{aligned} &\int_A |\mathbf{K}(\mathbf{x} - \boldsymbol{\phi}(\mathbf{y})) - \mathbf{K}(\mathbf{x} - \mathbf{y})|\, |\omega_0(\mathbf{y})|\, d\mathbf{y} \le \\ &\qquad \le c\Big(\int_A \frac{|\omega_0(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|}\, d\mathbf{y} + \int_A \frac{|\omega_0(\mathbf{y})|}{|\mathbf{x} - \boldsymbol{\phi}(\mathbf{y})|}\, d\mathbf{y} \Big) \end{aligned} \tag{A.2}$$

To estimate the first integral observe that

$$|\mathbf{y}| \leq 2\,|\mathbf{x}| \quad \Rightarrow \quad |\mathbf{y}|^\alpha < 2\,|\mathbf{x}|^\alpha \tag{A.3}$$

and

$$|\mathbf{y}| > 2\,|\mathbf{x}| \tag{A.4}$$

implies

$$|\mathbf{y}|^\alpha < |\mathbf{x}-\mathbf{y}|^\alpha + |\mathbf{x}|^\alpha < |\mathbf{x}-\mathbf{y}|^\alpha + \frac{1}{2^\alpha}\,|\mathbf{y}|^\alpha \tag{A.5}$$

from which

$$|\mathbf{y}|^\alpha < c_\alpha\,|\mathbf{x}-\mathbf{y}|^\alpha, \quad c_\alpha = \frac{2^\alpha}{2^\alpha - 1} > 1 \tag{A.6}$$

Therefore

$$\int_A \frac{|\omega_0(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} \leq \int_{|\mathbf{x}-\mathbf{y}|\leq 4r(1+|\mathbf{x}|^\alpha)} \frac{|\omega_0(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} \quad + \int_{|\mathbf{x}-\mathbf{y}|\leq 2c_\alpha r(1+|\mathbf{x}-\mathbf{y}|^\alpha)} \frac{|\omega_0(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} \tag{A.7}$$

The first integral is bounded by

$$c\,\|\omega_0\|_\infty\, r\,(1+|\mathbf{x}|^\alpha) \tag{A.8}$$

To estimate the second one, we split it into two parts

$$|\mathbf{x}-\mathbf{y}| < 1 \quad \Rightarrow \quad |\mathbf{x}-\mathbf{y}| \leq 4c_\alpha r \tag{A.9}$$

$$|\mathbf{x}-\mathbf{y}| > 1 \quad \Rightarrow \quad |\mathbf{x}-\mathbf{y}| \leq (4c_\alpha r)^{\frac{1}{1-\alpha}} \tag{A.10}$$

The first contribution yields

$$c\,\|\omega_0\|_\infty r$$

while the second can be bounded by Hölder inequality:

$$\|\omega_0\|_{\frac{2}{\alpha}} \left(\int_{|\mathbf{x}-\mathbf{y}|\leq(4c_\alpha\, r)^{1/(1-\alpha)}} \left[|\mathbf{x}-\mathbf{y}|^{\frac{2}{2-\alpha}}\right]^{-1} d\mathbf{y} \right)^{\frac{2-\alpha}{2}} \leq c\,(\|\omega_0\|_p + \|\omega_0\|_\infty) r \tag{A.11}$$

where we have estimate the $\frac{2}{\alpha}$−norm of ω_0 by interpolation, with $p \leq \frac{2}{\alpha} \leq +\infty$.

Collecting the above estimates, the first integral in the right hand side of (A.2) is bounded by:

$$c(\|\omega_0\|_p + \|\omega_0\|_\infty)(1 + |\mathbf{x}|^\alpha)\, r \tag{A.12}$$

The second integral in the right hand side of (A.2) is, by the conservation of the Lebesgue measure:

$$\int_B \frac{|\omega_0(\boldsymbol{\phi}^{-1}(\mathbf{z}))|}{|\mathbf{x} - \mathbf{z}|}\, d\mathbf{z} \tag{A.13}$$

where

$$B = \{\mathbf{z} \mid |\mathbf{x} - \boldsymbol{\phi}^{-1}(\mathbf{z})| < 2r(1 + |\boldsymbol{\phi}^{-1}(\mathbf{z})|^\alpha)\,\} \tag{A.14}$$

We proceed as before replacing $\mathbf{y}$ by $\boldsymbol{\phi}^{-1}(\mathbf{z})$. Thus we are lead to evaluate integrals of the type (after having realized that $\|\omega_0(\boldsymbol{\phi}^{-1})\|_p = \|\omega_0\|_p$):

$$\int_{|\mathbf{x}-\boldsymbol{\phi}^{-1}(\mathbf{z})|<a} \frac{1}{|\mathbf{x} - \mathbf{z}|^\beta}\, d\mathbf{z} \tag{A.15}$$

where $a = 4r(1 + |\mathbf{x}|^\alpha)$, $\beta = 1$ (see formulas $(A.7)$ and $(A.9)$) and $a = (4c_\alpha r)^{\frac{1}{1-\alpha}}$, $\beta = 2/(2 - \alpha)$ (see $(A.11)$).
The domain $|\mathbf{x} - \boldsymbol{\phi}^{-1}(\mathbf{z})| < a$ has the same measure of the disk $|\mathbf{x} - \mathbf{z}| < a$, and the symmetric arrangement of this domain around $\mathbf{x}$ maximizes the integral (A.15). In conculsion we can bound (A.13) by the expression (A.12). Denoting by K_i, $i = 1, 2$, the two components of $\mathbf{K}$, we estimate

$$\begin{aligned} &\int_{A^c} |K_i(\mathbf{x} - \boldsymbol{\phi}(\mathbf{y})) - K_i(\mathbf{x} - \mathbf{y})|\, |\omega_0(\mathbf{y})|\, d\mathbf{y} \leq \\ &\leq c \int_{A^c} \frac{|\mathbf{y} - \boldsymbol{\phi}(\mathbf{y})|}{|\mathbf{x} - \boldsymbol{\xi}_i|^2} |\omega_0(\mathbf{y})|\, d\mathbf{y} \end{aligned} \tag{A.16}$$

where $\boldsymbol{\xi}_i$, $i = 1, 2$, belong to the segment $(\mathbf{y}, \boldsymbol{\phi}(\mathbf{y}))$.
For $\mathbf{y} \in A^c$, $|\mathbf{x}-\mathbf{y}| \geq 2r(1+|\mathbf{y}|^\alpha)$, then

$$\begin{aligned} |\mathbf{x}-\boldsymbol{\xi}_i| &\geq |\mathbf{x}-\mathbf{y}| - |\mathbf{y}-\boldsymbol{\xi}_i| \\ &\geq |\mathbf{x}-\mathbf{y}| - |\mathbf{y}-\boldsymbol{\phi}(\mathbf{y})| \\ &\geq |\mathbf{x}-\mathbf{y}| - r(1+|\mathbf{y}|^\alpha) \geq \frac{1}{2}|\mathbf{x}-\mathbf{y}| \end{aligned} \tag{A.17}$$

from which the right hand side of (A.16) is bounded by

$$cr\int_{A^c} \frac{1+|\mathbf{y}|^\alpha}{|\mathbf{x}-\mathbf{y}|^2}|\omega_0(\mathbf{y})|\,d\mathbf{y} \tag{A.18}$$

If $r \geq 1/e$, the integral $(A.18)$ is bounded by

$$\begin{aligned} &cr\ (1+|\mathbf{x}|^\alpha)\int_{|\mathbf{x}-\mathbf{y}|>2/e} \frac{1}{|\mathbf{x}-\mathbf{y}|^2}|\omega_0(\mathbf{y})|\,d\mathbf{y} + cr\int_{|\mathbf{x}-\mathbf{y}|>2/e} \frac{1}{|\mathbf{x}-\mathbf{y}|^{2-\alpha}}|\omega_0(\mathbf{y})|\,d\mathbf{y} \leq \\ &\leq cr\ (1+|\mathbf{x}|^\alpha)\|\omega_0\|_p \end{aligned} \tag{A.19}$$

where $p < 2/\alpha$ (using Hölder inequality in the last step).
If $r < 1/e$ we have to add to $(A.19)$ the term:

$$\begin{aligned} &cr\int_{2r(1+|\mathbf{y}|^\alpha \leq|\mathbf{x}-\mathbf{y}|<2/e} \frac{1+|\mathbf{y}|^\alpha}{|\mathbf{x}-\mathbf{y}|^2}|\omega_0(\mathbf{y})|\,d\mathbf{y} \leq cr\ \|\omega_0\|_\infty(1+|\mathbf{x}|^\alpha)\int_{2r\leq|\mathbf{x}-\mathbf{y}|\leq 1} \frac{1}{|\mathbf{x}-\mathbf{y}|^2}\,d\mathbf{y} \\ &\leq cr\ |\log r|\ \|\omega_0\|_\infty(1+|\mathbf{x}|^\alpha) \end{aligned} \tag{A.20}$$

(we have used that $|\mathbf{x}-\mathbf{y}| < 2/e \Rightarrow |\mathbf{y}|^\alpha \leq c(1+|\mathbf{x}|^\alpha)$).

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Indirizzo dell'Autore:

Dario Benedetto – Dipartimento di Matematica Pura ed Applicata – Università de L'Aquila – 67100 L'Aquila, Italia